PROBLEMS

Problem 1. (J. DOMSTA) Let $\mathbb{R}_+$ stand for the interval $(0, \infty)$, and let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nondecreasing function satisfying the following conditions

(a) $$f(x) < x, \quad \text{for } x \in \mathbb{R}_+,$$

(b) $$d := \lim_{x \to 0^+} \frac{f(x)}{x} \quad \text{exists in } (0, 1).$$

Find all solutions $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ of the equation

$$\Phi(f(x)) = \Phi(x) \qquad \text{for all } x \in \mathbb{R}_+,$$

assuming that Φ is almost regularly varying with exponent 0, or, almost slowly varying, i.e.,

$$\lim_{x \to 0^+} \Phi(\lambda x)/\Phi(x) = 1 \qquad \forall \lambda \in \mathbb{R}_+.$$

Remark. If *at least one* of the following additional conditions is fulfilled:
(c.1) Φ is measurable,
(c.2) $f(x) = d \cdot x$ in some interval $x \in (0, \alpha) \neq \emptyset$,
then the only solutions are constant functions.

Problem 2. (G.L. FORTI) Assume Φ is a solution of the translation equation

$$\Phi[\Phi(x, s), t] = \Phi(x, s + t), \quad x \in X, \ s, t \in \mathbb{R} \tag{T}$$

and define $F(x, u, v) := \Phi(x, u - v)$. Then F is a solution of the functional equation

$$F[F(x, u, v), v, w] = F(x, u, w). \tag{$*$}$$

Thus, equation $(*)$ represents in some sense a weaker form of (T).

Having in mind the role of (T) in iteration theory, one may ask if $(*)$ can be used to define a notion of a *weak dynamical system*.

Problem 3. (S. KOLYADA and Ľ. SNOHA) Let I be a compact real interval. All maps under consideration are supposed to be continuous. A selfmap $F : I^2 \to I^2$ is called triangular if it is of the form $F(x, y) = (f(x), g(x, y))$. For a map φ, let $\mathrm{Per}(\varphi)$ be the set of periodic points of φ and $h(\varphi)$ be the topological entropy of φ. Transitivity means topological transitivity.

(a) Find

$$\inf\{h(f) : f \text{ is a transitive selfmap of } I^2 \text{ with } \mathrm{Per}(f) \text{ dense in } I^2\}.$$

(The answer is 0, if we have no restriction on the set of periodic points).

(b) Find

$$\inf\{h(F) : F \text{ is a triangular and transitive selfmap of } I^2 \text{ with } \mathrm{Per}(F) \text{ dense in } I^2\}.$$

(The answer is $\frac{1}{2}\log 2$, if we have no restriction on the set of periodic points).

Problem 4. (S. KOLYADA and Ľ. SNOHA) We consider continuous selfmaps of the unit real interval $I = [0, 1]$. If $\pi = \{x_1, x_2, \ldots, x_n\}$ is a periodic orbit of a map f then the *gravity* of π is defined by

$$\mathrm{gr}(\pi) = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

A map f is said to be *gravitational* if all periodic orbits of f have the same gravity. (Example: $f(x) = 1 - x$ or, more general, any map with $f([0, \frac{1}{2}]) \subset [\frac{1}{2}, 1]$ and $f|_{[\frac{1}{2},1]}(x) = 1 - x$.) A map is said to be *antigravitational* if any two different periodic orbits have different gravities. (Example: $f(x) = x$.)

Question: Is the standard tent map $f(x) = 1 - |2x - 1|$ antigravitational?

Problem 5. (J. SMÍTAL) Sequence topological entropy of continuous mappings as developed, e.g., in [3], is a useful tool in topological dynamics [1]. However, in application, the proof that a sequence topological entropy of a particular map is zero, is complicated, and the methods developed in [3] are not applicable even for quite simple mappings, cf., e.g. [2]. The problem is to develop further the theory.

[1] N. Franzová and J. Smítal, Positive sequence topological entropy characterizes chaotic maps, *Proc. Amer. Math. Soc.* 112 (1991), 1083–1086.

[2] G.L. Forti, L. Paganoni and J. Smítal, Strange triangular maps of the square, 20 pp., Preprint 20/1993, Università degli Studi di Milano.

[3] T.N.T. Goodman, Topological sequence entropy, *Proc. London Math. Soc.* (3) 29 (1974), 331–350.

Problem 6. (GY. TARGONSKI) At this conference a lecture by D. Gronau discussed the rôle of G. Frege who found and published fundamental results in iteration theory.

It appears that, independently, Heinrich Eggers found fundamental results in iteration theory; he communicated these in a correspondence with E. Schröder, whose work in iteration theory is, of course, well known.

It appears that Eggers later gave up his position of "Gymnasiallehrer" (preparatory school teacher) in Germany and emigrated to America. There he disappeared from the mathematical scene and his trace was lost.

I strongly suggest a doctoral dissertation in the history of mathematics, clearing up the rôle of Frege and of Eggers (and possibly others) in laying the foundations of iteration theory and, if possible, the later life of Eggers.

Problem 7. (M. HMISSI) Let S be the unit circle of $\mathbb{R}^2$ and $T = S \times S$ the two-dimensional torus. Characterize all homeomorphisms $f : T \to T$ which are iterable, i.e., such that there exists a family of homeomorphisms $(f_t)_{t\in\mathbb{R}}$ on T such that $f_1 = f$ and, for all $x \in T$, the map

$$\begin{array}{rcl} \mathbb{R} & \to & T, \\ t & \mapsto & f_t(x) \end{array}$$

is continuous.

Remark: In [SLNM 1163, ECIT'84] M.C. Zdun characterized all homeomorphisms $g : S \to S$ which are iterable.