REMARK ON THE COEFFICIENTS OF THE NATURAL ITERATES OF A FORMAL POWER SERIES IN ONE INDETERMINATE

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Let $\hat{F}(x) = C_1x + C_2x^2 + \cdots$ be a formal series with coefficients $(C_\nu)_{\nu\geq 1}$ which are independent indeterminates over $\mathbb{C}$ (the "general" series over $\mathbb{C}$). Denote by $\hat{F}^m$ the mth iterate of $\hat{F}$ ($m \in \mathbb{N}, m \geq 2$), $\hat{F}^m(x) = \sum_{\nu\geq 1} C_\nu^{[m]}x^\nu$. Then $C_\nu^{[m]}$ is a polynomial over $\mathbb{Z}$ in $C_1, \ldots, C_\nu$. Let $j \geq 1$, $2 \leq \mu \leq m$ be integers. Then the following polynomial identities hold.

Theorem 1. Let $Z_m(C_1)$ be the mth cyclotomic polynomial, (m, j, μ) as above. Then there exist integers $\alpha_0, \alpha_1, \ldots, \alpha_j \geq 0$ and polynomials $\Phi^{(\nu_1\cdots\nu_\gamma)}_{(m,j,\mu)}$ over $\mathbb{Z}$ in C_k, $1 \leq k \leq jm$, $k \not\equiv 1 (\operatorname{mod} m)$ for $k > 1$, and a polynomial $\tilde{\Phi}^{(0\ldots0)}_{(m,j,\mu)}$ over $\mathbb{Z}$ in C_k, $1 \leq k \leq jm + \mu$, $k \not\equiv 1 (\operatorname{mod} m)$ for $k > 1$, such that

$$(C_1^{m-1})^{\alpha_0}(1 + C_1^m + \cdots + C_1^{(m-1)m})^{\alpha_1} \cdots (1 + C^{jm} + \cdots + C_1^{(m-1)jm})^{\alpha_j} \cdot C^{[m]}_{jm+\mu}$$
$$= \sum_{\nu_1+\cdots+\nu_j\geq 1} \Phi^{(\nu_1,\ldots,\nu_j)}_{(m,j,\mu)} \cdot (C^{[m]}_{m+1})^{\nu_1} \cdots C^{[m]}_{jm+1})^{\nu_j} + Z_m(C_1) \cdot \tilde{\Phi}^{(0\ldots0)}_{(m,j,\mu)}$$

Corollary 1. If C_1 is specialized to a primitive root ρ of unity of order m (i.e. $Z_m(\rho) = 0$) then we obtain an identity in $C_2, \ldots, C_{jm+\mu}$ over $\mathbb{Z}[\rho, 1/m]$, namely, if we write $\tilde{C}_l^{[m]} = C_l^{[m]}(\rho, C_2, \ldots, C_l)$:

$$\tilde{C}^{[m]}_{jm+\mu} = \sum_{\nu_1+\cdots+\nu_j\geq 1} \tilde{\Phi}^{(\nu_1\cdots\nu_j)}_{(m,j,\mu)}(\tilde{C}^{[m]}_{m+1})^{\nu_1} \cdots (\tilde{C}^{[m]}_{jm+1})^{\nu_j}$$

Furthermore:

(i) If $Z_m(\rho) = 0$ then the ideal generated by $\tilde{C}^{[m]}_{m+1}, \ldots, \tilde{C}^{[m]}_{jm+1}$ is prime ($\tilde{C}^{[m]}_{jm+1}$ belongs to this ideal).

(ii) In $\tilde{C}^{[m]}_{jm+\mu}$ the indeterminates $C_{jm+2}, \ldots, C_{jm+\mu}$ do not appear.

(iii) $\tilde{C}^{[m]}_{m+1} = 0, \ldots, \tilde{C}^{[m]}_{jm+1} = 0$ (for special values $C_2, \ldots, C_{jm+1}$) implies $\tilde{C}_b^{[m]} = 0$ for all b with $2 \leq b \leq (j + 1)m$.