SIMPLE TRIANGULAR MAPS OF THE UNIT SQUARE*

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ABSTRACT

We study triangular maps of the unit square into itself trying to classify simple maps according to their topological behaviour. We give some results for the class of maps which verify that Per(f) is closed, where f is the base map. We conjecture possible generalizations of these results.

1. Introduction

We are interested in determining when a continuous triangular map $F : I^2 \to I^2$ can be said to be simple. For a triangular map we mean that F can be written as $F(x, y) = (f(x), g(x, y))$.

For the case of continuous maps of the interval into itself, the following criteria could be used to call a continuous map, $f : I \to I$, simple.

(i) the topological entropy of the map, $h(f)$, is zero.

(ii) the set of recurrent points of f, Rec(f), and the set of uniformly recurrent points of f, UR(f), coincide.
Recall that Rec(f) = $\{x \in I \mid$ there exists a sequence $(n_k) : f^{n_k}(x) \to x\}$ and that $x \in$ UR(f) iff for any neighborhood $U(x)$ there exists a natural number N such that given any i, we can find a j, $i < j \leq i + N$ with $f^j(x) \in U(x)$.

(iii) f is of periodic type equal or less than 2^∞, that is, all the periodic points of f have periods that are a power of 2.

(iv) the set of chain recurrent points of f, CR(f), is the union of all the closed invariant simple sets M_α of f: CR(f) $= \bigcup_{\alpha\in A} M_\alpha$.

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A point x is called chain recurrent for f if for all $\varepsilon > 0$ there exists a set of points $\{x_0, x_1, \dots, x_n\}$, with $x_0 = x = x_n$ and $|x_{k+1} - f(x_k)| < \varepsilon$, for all $k < n$. For a definition of what a simple set is, refer to section 4.

All these conditions provide reasonable definitions for what we could call a simple map of the interval into itself. In fact, it is known that

Theorem 1.1. *If $f : I \to I$ is continuous, then all conditions above are equivalent.*

The first idea would then be to use any of the previous conditions as a definition for simplicity of a triangular map of the unit square I^2 into itself. However, this can not be done due to the following known results:

(a) if F is not of type equal or less than 2^∞, then F has positive topological entropy. Obviously, positive topological entropy implies complexity in the dynamical behaviour of F.
(b) there exists a triangular map F of type 2^∞ which has positive topological entropy.
(c) there exists a triangular map F with zero topological entropy such that $\mathrm{Rec}(F) \neq \mathrm{UR}(F)$.

These results discard some of the properties above as sufficient conditions for simplicity. Our working hypothesis is to call a triangular map F, simple if and only if $\mathrm{Rec}(F) = \mathrm{UR}(F)$. In what follows we show some results that endorse this definition. We shall consider that $F(x, y) = (f(x), g(x, y))$ and denote by $\mathrm{Per}(f)$ the set of periodic points of the base map f.

2. Results about entropy

In this section we are using the notion of topological entropy given in [1].

Theorem 2.1. *In the case that* $\mathrm{Per}(f)$ *is a closed set, the following conditions are equivalent*:
(*i*) $\mathrm{Rec}(F) = \mathrm{UR}(F)$,
(*ii*) $h(F) = 0$,
(*iii*) *F is of type equal or less than 2^∞.*

It can also be shown that, even if $\mathrm{Per}(f)$ is not a closed set, $\mathrm{Rec}(F) = \mathrm{UR}(F)$ always implies that $h(F|_{\mathrm{Per}(F)}) = 0$.

Theorem 2.2. *In the case that the center of F, $C(F)$, (defined as the closure of* $\mathrm{Rec}(F)$) *equals the closure of* $\mathrm{Per}(F)$, *then* $\mathrm{Rec}(F) = \mathrm{UR}(F)$ *implies* $h(F)=0$ (*and hence F is of type equal or less than* 2^∞).

Also, there exists a triangular map F such that $\mathrm{Rec}(F) = \mathrm{UR}(F)$, $h(F) = 0$, but $C(F) \neq \overline{\mathrm{Per}(F)}$. With respect to this theorem, we conjecture that for every triangular map F, $\mathrm{Rec}(F) = \mathrm{UR}(F)$ implies that the topological entropy of F must be zero.

3. Results about chaoticity

Theorem 3.1. *If* $\mathrm{Per}(f)$ *is a closed set, then* $\mathrm{Rec}(F) \neq \mathrm{UR}(F)$ *if and only if* $F|_{\mathrm{Rec}(F)}$ *is chaotic in the sense of Li and Yorke.*

Recall that the second condition means that there exist two points $x, y \in \mathrm{Rec}(F)$ such that $\limsup |F^n(x) - F^n(y)| > 0$ and $\liminf |F^n(x) - F^n(y)| = 0$. For a detailed description and properties, see [4].

Definition 3.1. Let M be an invariant set for F. M is said to be F-separated or $(1, F)$-separated, if there exist M_1 and M_2 such that

(i) $M_1 \cup M_2 = M$, $M_1 \cap M_2 = \emptyset$,

(ii) there exist closed intervals I_i and J_i, for $i = 1, 2$ such that $M_i \subset I_i \times J_i$, for $i = 1, 2$ and $I_1 \times J_1 < I_2 \times J_2$ (which means that $\sup I_1 < \inf I_2$ or $\sup J_1 < \inf J_2$),

(iii) $F(M_1) = M_2$ and $F(M_2) = M_1$.

Definition 3.2. Let M be an invariant set for F. M is said to be (n, F)-separated if M is F-separated in M_1 and M_2 and both sets are $(n-1, F^2)$-separated. If M is an invariant set for F we say that M is $(0, F)$-separated.

This last definition implies the existence of 2^n sets $\{M_i\}_{i=1,\ldots,2^n}$ such that

(i) $\bigcup_{i=1}^{2^n} M_i = M$, $M_i \cap M_j = \emptyset$ for all $i \neq j$,

(ii) for every i there exist closed intervals I_i and J_i such that $M_i \subset I_i \times J_i$ and $I_1 \times J_1 < I_2 \times J_2 < \cdots < I_{2^n} \times J_{2^n}$,

(iii) for every j, $F^{2^{n-1}}(M_{2j-1}) = M_{2j}$ and $F^{2^{n-1}}(M_{2j}) = M_{2j-1}$.

Such $\{M_i\}_{i=1}^{2^n}$ is called a simple decomposition of order n for M.

Definition 3.3. Let M be an invariant set for F. M is said to be simple for F if either M consists of a fixed point of F or, for all $n \geq \log_2(\#M)$, there exists a simple decomposition of order n for M.

With these definitions we can now study the condition $\mathrm{CR}(F) = \bigcup_{\alpha \in A} M_\alpha$, where $\{M_\alpha\}_{\alpha \in A}$ are all the invariant simple sets for F. It can be proven that, in general, $\mathrm{Rec}(F) = \mathrm{UR}(F)$ does not imply $\mathrm{CR}(F) \subset \bigcup_{\alpha \in A} M_\alpha$, but if $\mathrm{CR}(F) = \bigcup_{\alpha \in A} M_\alpha$ then is $\mathrm{Rec}(F) = \mathrm{UR}(F)$.

If we consider the set

$$\mathrm{CR}(F)_* = \{(x, y) \in \mathrm{CR}(F) \mid x \notin \mathrm{Per}(f)\} \cup A$$

where $A = \{(x, y) \mid x \text{ is a periodic point of period } n \text{ and } y \in CR(g_x^n)\}$ and $g_x^n = g_{f^{n-1}(x)} \circ \ldots. \circ g_x$ with $g_x \in C(I, I)$ is given by $g_x(y) = g(x, y)$ (for more details see [2]), then we obtain the following result:

Theorem 3.2. *If* $\mathrm{Per}(f)$ *is a closed set, then* $\mathrm{Rec}(F) = \mathrm{UR}(F)$ *if and only if* $\mathrm{CR}(F)_* = \bigcup_{\alpha\in A} M_\alpha$.

Finally, it is our conjecture that this result also holds in general. A complete study for maps on the interval can be seen in [3].

References

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