THRON'S THEOREMS IN BANACH SPACES

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ABSTRACT

We propose a generalization of some theorems of W.J. Thron [4] on the speed of convergence of iteration sequences for functions acting on cones in Banach spaces.

For functions $f : [0, a] \longrightarrow \mathbb{R}$ of the form

$$f(x) = \rho x + x^{1+\alpha} F(x)$$

where $\rho \in [0, 1]$, $\alpha > 0$, and $F : [0, a] \longrightarrow \mathbb{R}$ is bounded, we have very useful theorems of W.J. Thron [4; Theorems 2.1, 2.2 and 3.1] which say how fast the sequence of iterates converges to zero, the unique fix-point, depending on whether $\rho \in (0, 1)$, $\rho = 0$, $\rho = 1$ (see also [3; §1.3] where some generalizations of these Thron's results are presented, and [1] where a stochastic version of Thron's theorems are obtained). In the present paper we prove some versions of W.J. Thron's theorems for functions acting on cones in Banach spaces.

Fix a normed space $(X, \|\cdot\|)$. We shall consider the asymptotic behavior of the sequence of iterates $(f^n(x_0))_{n\in\mathbb{N}}$ of f starting from a point $x_0 \neq \theta$ such that

$$\lim_{n\to\infty} f^n(x_0) = \theta. \tag{1}$$

1. We begin with the following simple fact being a version of a theorem of Thron for $\rho = 0$.

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Theorem 1. *Let $D \subset X\backslash\{\theta\}$, $f : D \longrightarrow D$, $x_0 \in D$ and assume that* (1) *holds. If*

$$0 < \liminf_{x\to\theta} \frac{\|f(x)\|}{\|x\|^{1+\alpha}}, \quad \limsup_{x\to\theta} \frac{\|f(x)\|}{\|x\|^{1+\alpha}} < \infty$$

for some $\alpha \in (0, \infty)$, then the sequence

$$\left(\|f^n(x_0)\|^{(1+\alpha)^{-n}}\right)_{n\in\mathbb{N}} \tag{2}$$

converges to a positive real number.

Proof. We may (and we do) assume that there exist positive real numbers m and M such that

$$m \le \frac{\|f(x)\|}{\|x\|^{1+\alpha}} \le M \quad \text{ for } x \in D.$$

Define $\beta : D \longrightarrow [m, M]$ by

$$\beta(x) = \frac{\|f(x)\|}{\|x\|^{1+\alpha}}.$$

Then, for $n \in \mathbb{N}$,

$$\|f^n(x_0)\| = \|x_0\|^{(1+\alpha)^n} \prod_{k=0}^{n-1} \beta(f^k(x_0))^{(1+\alpha)^{n-k-1}},$$

whence

$$\|f^n(x_0)\|^{(1+\alpha)^{-n}} = \|x_0\| \prod_{k=0}^{n-1} \beta(f^k(x_0))^{(1+\alpha)^{-(k+1)}},$$

i.e.,

$$\log \|f^n(x_0)\|^{(1+\alpha)^{-n}} = \log \|x_0\| + \sum_{k=0}^{n-1} \left(\frac{1}{1+\alpha}\right)^{k+1} \log \beta(f^k(x_0)).$$

This shows that the sequence

$$\left(\log \|f^n(x_0)\|^{(1+\alpha)^{-n}}\right)_{n\in\mathbb{N}}$$

tends to a real number and, consequently, the sequence (2) converges to a positive real number.

2. Fix a non-degenerate Banach space $(X, \|\cdot\|)$ and a closed cone $K \subset X$ with non empty interior, i.e., (cf. [2; p. 217, Definition 2.1]), K is a closed subset of X such that $K + K \subset K$, $tK \subset K$ for every $t \ge 0$, $K \cap (-K) = \{\theta\}$ and $\operatorname{Int} K \ne \emptyset$. We define a (partial) order $\le$ on X by $x \le y$ iff $y - x \in K$, and we assume that the norm $\|\cdot\|$ is an increasing function on K, i.e., $\theta \le x \le y$ implies $\|x\| \le \|y\|$.

Let $A : X \to X$ be a completely continuous linear operator such that $AK \subset K$ and for every $x \in K \setminus \{\theta\}$ there exists a positive integer n such that $A^n x \in \operatorname{Int} K$.

By the Krein–Rutman theorem [2; p. 267] the spectral radius ρ of A is positive and there exists exactly one vector $u \in \operatorname{Int} K$ and exactly one continuous linear functional $g : X \to \mathbb{R}$ such that $Au = \rho u$,

$$g(Ax) = \rho g(x) \quad \text{for } x \in X, \tag{3}$$

$g(x) > 0$ for every $x \in K \setminus \{\theta\}$, $\|u\| = 1$ and $g(u) = 1$.

We consider a function $f : K \to K$ such that

$$f(x) \neq \theta \quad \text{for } x \in K \setminus \{\theta\},$$
$$\lim_{x \to \theta} \frac{f(x) - Ax}{\|x\|} = \theta \tag{4}$$

and there exists a positive c such that

$$g(x) \geq c\|g\|\|x\| \quad \text{for } x \in f(K). \tag{5}$$

The following is proved in [5] (as Theorem 1).

Proposition. *Assume that either $\rho < 1$, or*

$$\rho = 1 \ \ \textit{and} \ \ f(x) \leq Ax \quad \textit{for } x \in K. \tag{6}$$

If $x_0 \in K \setminus \{\theta\}$ and (1) *holds, then*

$$\lim_{n \to \infty} \frac{f^n(x_0)}{g(f^n(x_0))} = u.$$

Using this Proposition we shall prove what follows.

Theorem 2. *Assume $\rho \in (0, 1)$, $x_0 \in K \setminus \{\theta\}$ and* (1) *holds. If there exists a real number $\alpha > 0$ such that*

$$\limsup_{x \to \theta} \frac{\|f(x) - Ax\|}{\|x\|^{1+\alpha}} < \infty, \tag{7}$$

then

$$\lim_{n \to \infty} \frac{f^n(x_0)}{\rho^n} = \eta u \tag{8}$$

with some $\eta \in (0, \infty)$.

Proof. Define $\beta : K \setminus \{\theta\} \longrightarrow \mathbb{R}$ by

$$\beta(x) = \frac{g(f(x) - Ax)}{g(x)^{1+\alpha}} \tag{9}$$

and observe that

$$|\beta(f^n(x_0))| \leq \|g\| \frac{\|f^{n+1}(x_0) - Af^n(x_0)\|}{\|f^n(x_0)\|^{1+\alpha}} \left(\frac{\|f^n(x_0)\|}{g(f^n(x_0))} \right)^{1+\alpha} \quad \text{for } n \in \mathbb{N}.$$

Hence, from eqs. (1), (7) and from the Proposition it follows that the sequence $(\beta(f^n(x_0)))_{n\in\mathbb{N}}$ is bounded. Let

$$\beta_n := \beta(f^n(x_0)), \quad \gamma_n := g(f^n(x_0)) \quad \text{for } n \in \mathbb{N}\cup\{0\}. \tag{10}$$

Taking into account (3) and the definition of β we get

$$\frac{\gamma_{n+1}}{\gamma_n} = \rho + \gamma_n^\alpha \beta_n \tag{11}$$

whence

$$\frac{\gamma_n}{\rho^n} = \gamma_0 \prod_{k=0}^{n-1} \frac{\gamma_{k+1}}{\rho\gamma_k} = \gamma_0 \prod_{k=0}^{n-1} \left(1 + \rho^{-1}\gamma_k^\alpha \beta_k\right) \quad \text{for } n \in \mathbb{N}. \tag{12}$$

Since $(\beta_n)_{n\in\mathbb{N}}$ is bounded and $(\gamma_n)_{n\in\mathbb{N}}$ converges to zero, there are positive reals M_1 and M_2 such that

$$|\log(1+\rho^{-1}\gamma_n^\alpha\beta_n)| \le M_1|\rho^{-1}\gamma_n^\alpha\beta_n)| \le M_2|\gamma_n^\alpha| \quad \text{for } n \in \mathbb{N}.$$

Moreover, on account of (11),

$$\lim_{n\to\infty} \frac{\gamma_{n+1}^\alpha}{\gamma_n^\alpha} = \rho^\alpha < 1.$$

Consequently the series

$$\sum_{n=0}^{\infty} \log(1+\rho^{-1}\gamma_n^\alpha\beta_n)$$

(absolutely) converges. Hence and from (12) we infer that $(\log(\gamma_n/\rho^n)_{n\in\mathbb{N}})$ tends to a real number, and so $(\gamma_n/\rho^n)_{n\in\mathbb{N}}$ has a positive and finite limit η:

$$\lim_{n\to\infty} \frac{g(f^n(x_0))}{\rho^n} = \eta. \tag{13}$$

This jointly with the Proposition gives (8).

Now we pas to the case $\rho = 1$.

Theorem 3. *Assume $x_0 \in K\backslash\{\theta\}$ and suppose that* (1) *and* (6) *hold. If*

$$\lim_{x\to\theta} \frac{f(x) - Ax}{\|x\|^{1+\alpha}} = -a \tag{14}$$

for some $\alpha > 0$ and $a \in K \setminus \{\theta\}$, then

$$\lim_{n\to\infty} n^{\frac{1}{\alpha}} f^n(x_0) = (\alpha g(a))^{-\frac{1}{\alpha}} u.$$

Proof. Define a function $\beta : K\setminus\{\theta\} \longrightarrow \mathbb{R}$ by (9) and sequences $(\beta_n)_{n\in\mathbb{N}}$ and $(\gamma_n)_{n\in\mathbb{N}}$ by (10). Then, according to (14) and to the Proposition,

$$\begin{aligned}\lim_{n\to\infty}\beta_n &= \lim_{n\to\infty}\frac{g(f^{n+1}(x_0)-Af^n(x_0))}{g(f^n(x_0))^{1+\alpha}}\\ &= \lim_{n\to\infty} g\left(\frac{f^{n+1}(x_0)-Af^n(x_0)}{\|f^n(x_0)\|^{1+\alpha}}\right)\left(\frac{\|f^n(x_0)\|}{g(f^n(x_0))}\right)^{1+\alpha} = -g(a).\end{aligned}$$

Moreover,

$$\gamma_{n+1}=\gamma_n(1+\gamma_n^\alpha\beta_n)$$

whence

$$\frac{1}{\gamma_{n+1}^\alpha}=\frac{1}{\gamma_n^\alpha}\left(1-\frac{\gamma_n^\alpha\beta_n}{1+\gamma_n^\alpha\beta_n}\right)^\alpha \qquad \text{for } n\in\mathbb{N}. \tag{15}$$

Let $M>0$ be a real number such that

$$1+\alpha t-Mt^2\le(1+t)^\alpha\le 1+\alpha t+Mt^2 \qquad \text{for } t\in[0,1]. \tag{16}$$

Since

$$\left(-\frac{\gamma_n^\alpha\beta_n}{1+\gamma_n^\alpha\beta_n}\right)_{n\in\mathbb{N}}$$

is a sequence of non-negative reals converging to zero, it follows from (16) that for n large enough, $n>N_1$ say, we have

$$\begin{aligned}1+\alpha\left(-\frac{\gamma_n^\alpha\beta_n}{1+\gamma_n^\alpha\beta_n}\right)-M\left(\frac{\gamma_n^\alpha\beta_n}{1+\gamma_n^\alpha\beta_n}\right)^2 &\le \left(1-\frac{\gamma_n^\alpha\beta_n}{1+\gamma_n^\alpha\beta_n}\right)^\alpha\\ &\le 1+\alpha\left(-\frac{\gamma_n^\alpha\beta_n}{1+\gamma_n^\alpha\beta_n}\right)+M\left(\frac{\gamma_n^\alpha\beta_n}{1+\gamma_n^\alpha\beta_n}\right)^2.\end{aligned}$$

Hence, taking also into account (15),

$$\begin{aligned}\frac{1}{\gamma_n^\alpha}+\alpha(-\beta_n)\frac{1}{1+\gamma_n^\alpha\beta_n}-M\frac{\gamma_n^\alpha\beta_n^2}{(1+\gamma_n^\alpha\beta_n)^2} &\le \frac{1}{\gamma_{n+1}^\alpha}\\ \le \frac{1}{\gamma_n^\alpha}+\alpha(-\beta_n)\frac{1}{1+\gamma_n^\alpha\beta_n}+M\frac{\gamma_n^\alpha\beta_n^2}{(1+\gamma_n^\alpha\beta_n)^2} &\qquad \text{for } n>N_1.\end{aligned} \tag{17}$$

Fix $\varepsilon>0$ and a positive integer $N_2>N_1$ such that for $n>N_2$ we have

$$M\frac{\gamma_n^\alpha\beta_n^2}{(1+\gamma_n^\alpha\beta_n)^2}\le\alpha\varepsilon,$$

$$g(a)-\varepsilon\le(-\beta_n)\frac{1}{1+\gamma_n^\alpha\beta_n}\le g(a)+\varepsilon.$$

Then, according to (17),

$$\frac{1}{\gamma_n^\alpha} + \alpha(g(a) - 2\varepsilon) \le \frac{1}{\gamma_{n+1}^\alpha} \le \frac{1}{\gamma_n^\alpha} + \alpha(g(a) + 2\varepsilon) \quad \text{for } n > N_2.$$

Hence we infer that

$$\frac{1}{\gamma_{N_2}^\alpha} + (n - N_2)\alpha(g(a) - 2\varepsilon) \le \frac{1}{\gamma_n^\alpha} \le \frac{1}{\gamma_{N_2}^\alpha} + (n - N_2)\alpha(g(a) + 2\varepsilon) \quad \text{for } n > N_2,$$

that is

$$\frac{1}{n\gamma_{N_2}^\alpha} + \left(1 - \frac{N_2}{n}\right)\alpha(g(a) - 2\varepsilon) \le \frac{1}{n\gamma_n^\alpha} \le \frac{1}{n\gamma_{N_2}^\alpha} + \left(1 - \frac{N_2}{n}\right)\alpha(g(a) + 2\varepsilon) \quad \text{for } n > N_2,$$

whence

$$\alpha(g(a) - 2\varepsilon) \le \liminf_{n\to\infty} \frac{1}{n\gamma_n^\alpha} \le \limsup_{n\to\infty} \frac{1}{n\gamma_n^\alpha} \le \alpha(g(a) + 2\varepsilon).$$

Consequently

$$\lim_{n\to\infty} \frac{1}{n\gamma_n^\alpha} = \alpha g(a)$$

i.e.,

$$\lim_{n\to\infty} n^{\frac{1}{\alpha}} g(f^n(x_0)) = (\alpha g(a))^{-\frac{1}{\alpha}}.$$

An application of the Proposition ends the proof.

At the end we return to the case $\rho < 1$. Let $A_0 := A|_K$.

Theorem 4. *Assume $\rho \in (0, 1)$, $x_0 \in K\setminus\{\theta\}$ and (1) holds. If $f - A_0$ is an increasing function and for some $a \in \operatorname{Int} K$ the function*

$$''t \longrightarrow g(f(ta)), \quad t \in [0, \infty)''$$

is twice differentiable at zero then (8) holds with some $\eta \in (0, \infty)$.

Proof. Since $f - A_0$ and A_0 are increasing functions, so is also f. In particular,

$$f(\theta) \le f(f^n(x_0)) = f^{n+1}(x_0) \quad \text{for } n \in \mathbb{N}$$

and, consequently, $f(\theta) = \theta$. Define $F : K \to K$ and $\alpha : [0, \infty) \to [0, \infty)$ by

$$F = f - A_0, \quad \alpha(t) = g(F(ta)).$$

On account of Taylor's Theorem there exist positive real constants M and t_0 such that

$$\alpha(t) \le Mt^2 \quad \text{for } t \in [0, t_0]. \tag{18}$$

Fix $\varepsilon > 0$ with

$$(1+\varepsilon)^2\rho < 1 \tag{19}$$

and let

$$\beta := (1+\varepsilon)\rho.$$

Since (3), (4) and (5) give

$$\lim_{x\to 0, x\in f(K)} \frac{g(f(x))}{g(x)} = \rho,$$

it follows that there exists a $\delta > 0$ such that

$$g(f(x)) \le \beta g(x) \quad \text{for } x \in f(K) \text{ with } \|x\| \le \delta. \tag{20}$$

Fix $x \in K$. Applying in turn: (1), the Proposition and (19) we infer that there exist positive integers N and $N_1 > N$ such that

$$\|f^n(x_0)\| \le \delta \quad \text{for } n \ge N, \tag{21}$$

$$f^n(x_0) \le (1+\varepsilon)g(f^n(x_0))u \quad \text{for } n > N, \tag{22}$$

$$(1+\varepsilon)g(f^N(x_0))u \le a, \tag{23}$$

$$\beta^{n-N} \le t_0 \quad \text{for } n > N_1. \tag{24}$$

It follows from (20) and (21) that

$$g(f^n(x_0)) \le \beta^{n-N} g(f^N(x_0)) \quad \text{for } n > N. \tag{25}$$

Consequently, making use in turn (22), (25), (23), we get

$$f^n(x_0) \le \beta^{n-N}a \quad \text{for } n > N.$$

Hence, taking also into account monotonicity of F, definition of α, (24) and (18) we have

$$g(F(f^n(x_0))) \le \alpha(\beta^{n-N}) \le M\beta^{2(n-N)} \quad \text{for } n > N_1.$$

Therefore,

$$\frac{g(F(f^n(x_0)))}{\rho^{n+1}} \le \frac{M}{\rho\beta^{2N}}((1+\varepsilon)^2\rho)^n \quad \text{for } n > N_1,$$

and according to (19) the series

$$\sum_{n=0}^{\infty} \frac{g(F(f^n(x_0)))}{\rho^{n+1}}$$

converges. Since

$$\frac{g(f^n(x))}{\rho^n} = g(x) + \sum_{k=0}^{n-1} \frac{g(F(f^k(x)))}{\rho^{k+1}} \quad \text{for } x \in K, n \in \mathbb{N},$$

we infer that (13) holds with some $\eta \in (0, \infty)$ which jointly with the Proposition gives (8).

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