ON AN ITERATIVE FUNCTIONAL EQUATION

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ABSTRACT

Motivated by the equation $(*)_m$ considered by D. Gronau and the author in their joint paper [3], we study a problem of determining iteration semigroups of functions conjugate to a given monotonic mapping φ through multiplication by the iteration index. We also make a remark on an equation slightly more general than $(*)_m$, namely with a homeomorphism h replacing the linear mapping $x \mapsto mx$.

In the paper [3] written jointly with D. Gronau we dealt with the functional equation

$$\varphi^m(x) = \frac{1}{m}\varphi(mx) \tag{$(*)_m$}$$

where φ was supposed to be real, continuous (in some theorems more regular) and defined in a real interval containing 0, with $\varphi(0) = 0$. The equation $(*)_m$ arises from an asymptotic formula for iterates of a function, more details on its origin can be found in [1], [2] and [3]. As it turns out (cf. Proposition 2.1, Theorems 2.1 and 2.2 in [3]), if a solution φ of $(*)_m$ is continuous and two times differentiable at 0 then it is either the zero function, or $\varphi(x) = -x$ for all x in a neighbourhood of 0, or φ is given in a neighbourhood of 0 by

$$\varphi(x) = \frac{x}{1 - bx} \tag{$(**)$}$$

where $b \in \mathbb{R}$ is a constant.

1. Since actually all the functions enumerated above are solutions of $(*)_m$ for arbitrary positive integer m, it is interesting to ask whether we can get a similar result without

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assumming differentiability of φ but supposing that $(*)$ holds for, say, two different integers m and p. A very partial result in this direction is our Corollary from [3] stating that the identity is the only continuous and nondecreasing solution of $(*)_m$ and $(*)_p$ which is defined in $[0, \infty)$ and has a positive fixed point.

On the other hand let us observe that if φ is given by $(**)$ with a positive b in a neighbourhood of 0 and we define f_s, $s \in [0, \infty)$, by

$$f_s(x) = \frac{x}{1+sbx} = \frac{1}{s}\varphi(sx)$$

then the family $(f_s)_{s\geq 0}$ satisfies

$$f_{s+t}(x) = f_s\left(f_t(x)\right) \tag{T}$$

whenever all the functions are defined and the composition on the right hand side is performable. Thus $(f_s)_{s\geq 0}$ is a *local iteration semigroup*, as we will call it, referring to the fact that for every s and t equality (T) holds in a sufficiently small neighbourhood of 0. We will show that the above property characterizes in a way the *Möbius transformation* (or *homographic function*). Let us start with the following.

Theorem 1.1. *Let $\varphi : (0, \infty) \to (0, \infty)$ be a monotonic function satisfying $\varphi(x) \leq x$, $x > 0$, and assume that the family $(f_s)_{s>0}$ defined by*

$$f_s(x) = \frac{1}{s}\varphi(sx), \quad x > 0,\ s > 0, \tag{1}$$

is an iteration semigroup, i.e.,

$$f_{s+t} = f_s \circ f_t, \quad s, t > 0. \tag{2}$$

Then there exists a $b \geq 0$ such that

$$f_s(x) = \frac{x}{1+sbx}, \quad s > 0,\ x > 0. \tag{3}$$

Proof. Let us observe first that for every $x > 0$ the function $s \to \varphi(sx)$ is monotonic and hence (cf. [5], Theorem 1.1) the function $s \to f_s(x)$ has to be continuous (i.e., $(f_s)_{s>0}$ is a *continuous iteration semigroup*). Suppose that $\varphi(x) = f_1(x) = x$ for some $x > 0$. Then Lemma 2.1 in [5] implies that $f_s(x) = x$, $s > 0$, or

$$\frac{1}{s}\varphi(sx) = x, \quad s > 0,$$

whence it follows that $\varphi = \mathrm{id}_{(0,\infty)}$. Consequently, we get (3) with $b = 0$.

Assume now that φ has no fixed points, i.e., $\varphi(x) < x$, $x > 0$. In particular, $d := \varphi(1) \in (0, 1)$. Observe that (2) implies

$$\frac{1}{s+t}\varphi((s+t)x) = \frac{1}{s}\varphi\left(\frac{s}{t}\varphi(tx)\right), \quad s, t > 0,\ x > 0. \tag{4}$$

Put $u = s/t$, and $x = 1/t$. We get from (4) (taking $x = 1$)

$$\frac{u}{u+1}\varphi(u+1) = \varphi(du), \quad u > 0. \tag{5}$$

Define $\sigma : (0, \infty) \to (0, \infty)$ by

$$\sigma(y) = y\varphi\left(\frac{1}{y}\right).$$

Then it is easy to check that σ satisfies

$$\sigma\left(\frac{dy}{1+dy}\right) = d\sigma(y), \quad y > 0. \tag{6}$$

Since the function $y \to \sigma(y)/y = \varphi(1/y)$ is monotonic, we get by Theorem 2.3.12 from [4] (p. 68) that there exists a constant $c \in \mathbb{R}$ such that

$$\sigma(y) = \frac{cy}{1-d+dy}, \quad y > 0,$$

whence

$$\varphi(y) = \frac{cy}{d+(1-d)y}, \quad y > 0.$$

Taking into account that $d = \varphi(1) = c$ we obtain the formula (3) with $b = (1-d)/d$. This ends the proof.

In an analogous way, but using Theorem 2.4.4 from [4] (p. 74), resp. Theorem 2.5.1 ([4], p. 77) one can prove the following two results.

Theorem 1.2. *Let $\varphi : (0, \infty) \to (0, \infty)$ be a function satisfying $\varphi(x) \le x$, $x > 0$, and such that the function $x \to \varphi(1/x)$ is convex or concave in $(0, \infty)$. If the family $(f_s)_{s>0}$ defined by (1) is an iteration semigroup then there exists a $b \ge 0$ such that (3) holds.*

Theorem 1.3. *Let $\varphi : (0, \infty) \to (0, \infty)$ be a function satisfying $\varphi(x) \le x$, $x > 0$, and regularly varying at the infinity, i.e., φ is continuous and there is a $\delta \in \mathbb{R}$ such that for every $\lambda > 0$*

$$\lim_{x\to\infty} \frac{\varphi(\lambda x)}{\varphi(x)} = \lambda^{\delta}.$$

If the family $(f_s)_{s>0}$ defined by (1) is an iteration semigroup then there exists a $b \ge 0$ such that (3) holds.

2. The equation $(*)_m$ reflects the fact that the m-th iterate of φ is conjugated to φ via the homeomorphism h, defined by $h(x) = mx$, $x > 0$. It would be therefore interesting to study a slightly more general version of $(*)_m$, namely

$$\varphi^m = h^{-1} \circ \varphi \circ h. \tag{E}$$

with an arbitrary homeomorphism $h : (0, \infty) \to (0, \infty)$. Suppose that $\alpha : (0, \infty) \to (0, \infty)$ is an invertible solution of the *Schröder functional equation*

$$\alpha(h(x)) = \frac{1}{m}\alpha(x). \tag{S}$$

Then it is easy to check that $\varphi : (0, \infty) \to (0, \infty)$ defined by

$$\varphi(x) = \alpha^{-1}(\alpha(x) + 1) \tag{7}$$

is a solution of (E). Usually however, (S) has many solutions, even in classes of rather regular functions, and even if $h(x) = mx$, $x > 0$ (cf. [4]) and thus (E) would also have many solutions.

Let us denote by T_r the translation with increment r, i.e., $T_r(y) = y + r$ for all y in $(0, \infty)$. Below we shall restrict ourselves to the solutions of (E) which are of the form (7), i.e., $\varphi = \alpha^{-1} \circ T_1 \circ \alpha$ with some decreasing bijection $\alpha : (0, \infty) \to (0, \infty)$.

As an example of a uniqueness theorem we shall prove the following.

Theorem 2.1. *Let $m \in \mathbb{N}$ be fixed and assume that $h : (0, \infty) \to (0, \infty)$ is an increasing homeomorphism such that*

$$\lim_{z\to\infty} \frac{h(z)}{z} = m \qquad \text{and} \qquad z \mapsto \frac{h(z)}{z} \text{ is monotonic.}$$

Then there exists at most one strictly increasing bijection $\alpha : (0, \infty) \to (0, \infty)$ such that

(*i*) *$\varphi = \alpha^{-1} \circ T_1 \circ \alpha$ is a solution of (E);*
(*ii*) *$\alpha \circ h \circ \alpha^{-1}$ is convex or concave;*
(*iii*) *the function $x \mapsto x\alpha(x)$ is monotonic.*

The function α is given by

$$\alpha(u) = c \lim_{n\to\infty} \frac{h^n(u_o)}{h^n(u)}, \tag{8}$$

where $u_o \in (0, \infty)$ is arbitrarily fixed and $c > 0$ is any constant.

Proof. Suppose that $\alpha : (0, \infty) \to (0, \infty)$ is a strictly increasing bijection satisfying (i)–(iii). From (i), taking into account that $\varphi^m = \alpha^{-1} \circ T_m \circ \alpha$, we infer that

$$\alpha^{-1} \circ T_m \circ \alpha = h^{-1} \circ \alpha^{-1} \circ T_1 \circ \alpha \circ h$$

whence putting $\gamma = \alpha \circ h \circ \alpha^{-1}$ we get

$$\gamma \circ T_m = T_1 \circ \gamma,$$

or

$$\gamma(y + m) = \gamma(y) + 1$$

for every $y \in (0, \infty)$. By a Theorem of Krull (cf. [4], Theorem 4.2.4.) we get in view of (ii) and because $\lim_{y\to 0} \gamma(y) = 0$, the equality $\gamma(y) = y/m,\ y > 0$.

Thus α satisfies the following functional equation

$$\alpha(h(x)) = \frac{1}{m}\alpha(x), \tag{9}$$

for all $x > 0$. Define functions $\widehat{\alpha}, \widehat{h} : (0, \infty) \to (0, \infty)$ by

$$\widehat{\alpha}(z) = \alpha\left(\frac{1}{z}\right) \qquad \text{and} \qquad \widehat{h}(z) = \frac{1}{h\left(\frac{1}{z}\right)}.$$

The equation (9) is equivalent to

$$\widehat{\alpha}(\widehat{h}(z)) = \frac{1}{m}\widehat{\alpha}(z), \tag{10}$$

for all $z > 0$. From assumptions on h we easily see that

$$z \mapsto \frac{\widehat{h}(z)}{z} \text{ is monotonic and } \lim_{z\to 0} \frac{\widehat{h}(z)}{z} = \frac{1}{m}.$$

Moreover, by (iii) we have monotonicity of $x \mapsto \widehat{\alpha}(x)/x$. Thus by Theorem 2.3.12. from [4] $\widehat{\alpha}$ is given by

$$\widehat{\alpha}(x) = c \lim_{n\to\infty} \frac{\widehat{h}^n(x)}{\widehat{h}^n(x_o)},$$

where $c \in \mathbb{R}$ is any constant and $x_o \in (0, \infty)$ is fixed arbitrarily. Therefore it is straightforward that α is given by (8) with $c > 0$ in view of our assumptions, and u_o is chosen arbitrarily.

Let us note that if $h(z) = mz,\ z \in (0, \infty)$, then formula (8) yields the function α defined by $\alpha(u) = d/u$, where $d > 0$ is a constant. The resulting solution φ of (E) is given by

$$\varphi(x) = \frac{x}{1 + \frac{x}{d}}, \quad x > 0$$

and thus it is the only solution of (E) which can be written in the form (7) with a generator α satisfying property (iii) of the above theorem and such that $x \mapsto \alpha(m\alpha^{-1}(x))$ is convex or concave.

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