ON THE LOCAL DISTRIBUTION OF ANALYTIC CONTRACTIONS IN $\mathbb{C}^3$ WITH RESPECT TO EMBEDDABILITY

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1. Introduction. Applications of general results

In this paper we want to continue our previous investigations of the local distribution of power series transformations with respect to embeddability (see [4] and [5]). By embeddability of a (formal) power series transformation F we mean the fact that F can be embedded into an analytic 1-parameter group $(F_t)_{t\in\mathbb{C}}$ of such transformations so that $F_1 = F$. As we emphasized in [5], all information about distribution of embeddable (iterable) or noniterable power series transformations depends on how much we know about the so called normal (i.e., semicanonical) forms of these transformations. In dimension $n > 1$ the first case we may think of are the so called analytic contractions whose normal forms are of polynomial type (see [2] and [3]). In particular, the semicanonical forms for $n = 2$ and $n = 3$ are rather well understood, and they were used in [1] to give the first examples of analytic contractions in $\mathbb{C}^3$ which are not embeddable. In [6] J. Schwaiger used these normal forms for contractions in $\mathbb{C}^3$ to characterize contractions which are not iterable, and we will heavily depend on his results. Let me also note that if a contraction respresents an analytic mapping (i.e., the series are convergent) then the results mentioned above and those presented below make also sense for analytic mappings instead only for mere automorphisms of the ring $\mathbb{C}[[x_1, x_2, x_3]]$ of formal series. This stems from the fact that both the normal forms and the transformations to these forms converge in the case of biholomorphic contractions (see [7]). Let me also mention that the problem of the distribution of iterable and noniterable power series transformations was the first time raised by Sternberg in [7].

For details about the notions and the basic results we refer the reader to [5]. Here we shall only recall the most important facts.

An automorphism $F : z \longmapsto Az + \mathcal{P}(z)$ of the ring of formal series $\mathbb{C}[\![z_1, \dots, z_n]\!]$ (which is continuous in the weak topology) is called a contraction if the eigenvalues $\rho_1, \dots, \rho_n$ of the matrix of its linear part $z \longmapsto Az$ fulfill the condition

$$0 < |\rho_i| < 1, \quad i = 1, \dots, n. \tag{1}$$

By a suitable linear change of coordinates we may arrange the eigenvalues so that

$$0 < |\rho_n| \le |\rho_{n-1}| \le \cdots \le |\rho_1| \tag{2}$$

holds, which we shall assume in the sequel. In [2] it was shown that the normal (rather semicanonical) forms of contractions with respect to conjugation can be described as follows: They have the structure

$$N(z) = Jz + \mathcal{N}(z), \quad \mathcal{N}(z) = (\mathcal{N}_1(z), \dots, \mathcal{N}_n(z))$$

where J is the Jordan canonical form of A while $\mathcal{N}_k(z)$ is a polynomial in $z = {}^t(z_1, \dots, z_n)$ such that a monomial $z_1^{\nu_1} \cdots z_n^{\nu_n}$ ($\nu_1 + \nu_2 + \cdots + \nu_n \ge 2$) may have a nonzero coefficient in $\mathcal{N}_k(z)$ only if the relation

$$\rho_k = \rho_1^{\nu_1} \cdots \rho_{k-1}^{\nu_{k-1}} \tag{3}$$

together with $\nu_k = \ldots = \nu_n = 0$ holds. There is only a finite set of relations of type (3) for $k = 1, \dots, n$, this set clearly may be empty. As a consequence, the power series $\mathcal{N}_k(z)$ reduce to polynomials, and as a consequence of (2) and (3), to polynomials in $z_1, z_2, \cdots, z_{k-1}$ ($k \ge 2$, for $k = 1$ there is no relation (3)).

It is well known that all contractions in $\mathbb{C}^1$ and $\mathbb{C}^2$ are iterable. The first examples of noniterable contractions were constructed in [1] by E. Peschl and the author, and later on J. Schwaiger proved in [6] that these examples covered already the general case of noniterable contractions in $\mathbb{C}^3$. For what follows we need to describe the Schwaiger's result in some detail.

It was shown in [6] that the noniterable contractions in $\mathbb{C}^3$, when assumed in the normal form, can be characterized in the following way. They have a representation

$$\begin{aligned} y_1 &= \rho_1 x_1 \\ y_2 &= \qquad \rho_2 x_2 \\ y_3 &= \qquad\qquad \rho_3 x_3 + \sum_{\nu=0}^{[\beta_0/\beta_1]} a_\nu x_1^{\alpha_0+\nu\alpha_1} x_2^{\beta_0-\nu\beta_1} \end{aligned} \tag{4}$$

with three distinct eigenvalues ρ_1, ρ_2, ρ_3; there is no relation $\rho_2 = \rho_1^l$ ($l \ge 2$) for ρ_2, but at least one relation of type

$$\rho_2^{\bar{\beta}_1} = \rho_1^{\bar{\alpha}_1}, \quad 1 < \bar{\beta}_1 < \bar{\alpha}_1.$$

Among these relations there is exactly one with minimal $\bar{\beta}_1 = \beta_1$, we shall denote it by

$$\rho_2^{\beta_1} = \rho_1^{\alpha_1} \tag{5}$$

Furthermore, let us assume that there are relations for ρ_3, i.e., of type

$$\rho_3 = \rho_1^{\bar{\alpha}_o} \rho_2^{\bar{\beta}_o}, \quad \bar{\alpha}_0, \bar{\beta}_0 \geq 0, \quad \bar{\alpha}_0 + \bar{\beta}_0 \geq 2.$$

The set of all relations for ρ_3 is then given by

$$\rho_3 = \rho_1^{\alpha_0 + \nu\alpha_1} \rho_2^{\beta_0 + \nu\beta_1}, \quad \nu = 0, 1, \ldots, [\beta_0/\beta_1], \tag{6}$$

if we denote by $\rho_3 = \rho_1^{\alpha_0} \rho_2^{\beta_0}$ the unique relation of type (6) for which $\bar{\alpha}_0$ $(= \alpha_0)$ is minimal.

The necessary and sufficient conditions on an analytic contraction in $\mathbb{C}^3$ of the form (4) (all other Jordan normal forms of linear parts lead to iterable mappings) include two more properties. It is easy to see that under the assumptions made so far there exist at least two different relations (6), which means relations where $(\alpha_0 + k\alpha_1, \beta_0 - k\beta_1) \neq (\alpha_0 + l\alpha_1, \beta_0)$, and we have as a further property that there are $k, l, k \neq l, 0 \leq k, l \leq [\beta_0/\beta_1]$ for which the coefficients a_k, a_l in (4) are not zero. In addition one has to require conditions (A) of arithmetic type for ρ_1, ρ_2 which we do not need in this paper (see [6] for the complete description).

In this paper we shall investigate the local distribution of iterable and noniterable power series transformations in the neighbourhood of a fixed analytic contraction F in $\mathbb{C}^3$ which we shall assume in its normal form. This makes sense since our problem is invariant under conjugation (see [4] or [5]). By a neighbourhood we mean any neighbourhood of F with respect to the weak (i.e., coefficientwise) topology which is introduced in [4] and [5]. Let us recall that in this topology a sequence $(G^{(k)})_{k\in\mathbb{N}}$ of power series transformations

$$G^{(k)}(x) = A^{(k)}x + \sum_{\substack{\nu\in\mathbb{N}_0^3 \\ \nu_1+\nu_2+\nu_3\geq 2}} g_\nu^{(k)} x^\nu \quad \text{tends to} \quad G(x) = Ax + \sum_{\substack{\nu\in\mathbb{N}_0^3 \\ \nu_1+\nu_2+\nu_3\geq 2}} g_\nu x^\nu$$

if for each ν

$$\lim_{k\to\infty} g_\nu^{(k)} = g_\nu,$$

the coefficients $g_\nu^{(k)}$, g_r being vectors in $\mathbb{C}^3$.

Eventually we recall the contents of Theorem 1 and Theorem 2 in [5]. Theorem 1 says that in each neighbourhood of each power series transformation F (here in $\mathbb{C}[\![x_1, x_2, x_3]\!]$) there are iterable transformations, i.e., for each F we can find a sequence $(G^{(k)})_{k\in N}$ such that $G^{(k)}$ is iterable, $G^{(k)} \neq G^{(l)}$ if $k \neq l$, and such that

$$\lim_{k\to\infty} G^{(k)} = F.$$

In order to apply Theorem 2 we have to assume that the set of relations

$$\rho_k = \rho_1^{\nu_1}\rho_2^{\nu_2}\rho_3^{\nu_3}, \quad \nu_1 + \cdots + \nu_n \geq 2, \quad k = 1, 2, 3 \tag{7}$$

is finite (or empty) for the eigenvalues ρ_1, ρ_2, ρ_3 of the given F. This is indeed true if F is a contraction. Furthermore we have to assume that F is noniterable which means in our case of a contraction that the conditions of J. Schwaiger are fulfilled. Then Theorem 2 says that all formal power series transformations with the same linear part as F and lying in a sufficiently small neighbourhood of F are also noniterable. We shall now summarize the implications of the general results of [5].

Theorem 1. (a) *In each neighbourhood of an analytic contraction F of $\mathbb{C}^3$ there are power series transformations G, $G \neq F$, which are iterable. These G's are analytic contractions if the neighbourhood is small enough.*

(b) *If F is a noniterable analytic contraction in $\mathbb{C}^3$ then in each neigbourhood of F there are noniterable analytic contractions G with $G \neq F$.*

(c) *Let F be an analytic contraction in $\mathbb{C}^3$, let $F(x)$ have the normal form* (4) *and assume that the conditions of* [6] (*concerning noniterability*) *on the eigenvalues ρ_1, ρ_2, ρ_3 of F are fulfilled, but that F is iterable* (*i.e., a_k may be different from zero only for one k*). *Then in every neighbourhood of F there is an analytic contraction which is noniterable.*

Proof. (a) We may assume that F is already in normal form. Then we know from Theorem 1 in [5] that in each neighbourhood of F there is an iterable G, $G \neq F$. It follows from the proof of this theorem that the eigenvalues $\sigma_1, \sigma_2, \sigma_3$ of G may be assumed to be algebraically indepent over $\mathbb{Q}$. Now, if the neighbourhood of F is small enough and we put 0 in the linear part of G at each place where 0 stands in the linear part of F (which is a Jordan canonical form) then necessarily also $0 < |\sigma_i| < 1$, $i = 1, 2, 3$, and there are no relations (7) for $\sigma_1, \sigma_2, \sigma_3$ since they are algebraically independent over $\mathbb{Q}$. So G is a contraction and iterable.

(b) As we have already seen this is an immediate consequence of Theorem 2 in [5]. However, it follows also from Schwaiger's characterization of noniterable contractions in $\mathbb{C}^3$. To show this we assume that F is in its normal form (4), so that Schwaiger's conditions on the eigenvalues ρ_1, ρ_2, ρ_3 and on the coefficients a_k are fulfilled. So let $a_k \neq 0$, $a_l \neq 0$ for $0 \leq k < l \leq [\beta_0/\beta_1]$, and consider the sequence $(G^{(k)})_{k\in N}$ of contractions in their normal form

$$\begin{aligned}
y_1 &= \rho_1 x_1 \\
y_2 &= \qquad \rho_2 x_2 \\
y_3 &= \qquad\qquad \rho_3 x_3 + \sum_{\nu=0}^{[\beta_o/\beta_1]} a_\nu^{(m)} x_1^{\alpha_0+\nu\alpha_1} x_2^{\beta_0-\nu\beta_1}
\end{aligned}$$

such that $\lim_{m\to\infty} a_\nu^{(m)} = a_\nu$. Hence $a_k^{(m)} \neq 0$, $a_l^{(m)} \neq 0$ if m is large and we assume this for all $m \geq 1$. All $G^{(m)}$ have the same linear part as F and are contractions in normal form. Since each $G^{(m)}$ fulfills all of Schwaiger's conditions it is noniterable. Since way may also suppose $a_k^{(m)} \neq a_k$ for all m we have $G^{(m)} \neq F$, $m \geq 1$. Obviously $\lim_{m\to\infty} G^{(m)} = F$, so that (b) is proved.

(c) We may assume that F is in normal form. Then this normal form is of type (4) and fulfils Schwaiger's conditions on ρ_1, ρ_2, ρ_3 but in the normal form at most one coefficient a_{k_0} is not 0. Consider the following sequence of analytic contractions $(G^{(m)})_{m\geq 1}$:

$$\begin{aligned} y_1 &= \rho_1 x \\ y_2 &= \rho_2 x \\ y_3 &= \rho_3 x + \sum_{\nu=0}^{[\beta_0/\beta_1]} a_\nu^{(m)} x_1^{\alpha_0+\nu\alpha_1} x_2^{\beta_0-\nu\beta_1} \end{aligned}$$

where $\lim_{m\to\infty} a_\nu^{(m)} = a_\nu$, $0 \leq \nu \leq [\beta_0/\beta_1]$, but $a_l^{(m)} \neq 0$ for a fixed $l \neq k_0$ (so that $a_l = 0$). Then according to [6] each transformation $G^{(m)}$ is noniterable but $\lim_{m\to\infty} G^{(m)} = G$ is iterable.

2. Further properties of analytic contractions in $\mathbb{C}^3$ with respect to the local distribution of iterable transformations

If we summarize briefly the results of Theorem 1 then we may say:

1. There are noniterable analytic contractions F in $\mathbb{C}^3$ such that in each neighbourhood of F there is also a noniterable contraction G (with $G \neq F$).
2. There are iterable analytic contraction in $\mathbb{C}^3$ such that in each neighbourhood of F there is an noniterable contraction (with $G \neq F$).
3. In each neighbourhood of each analytic contraction F there is an iterable analytic contraction G (with $G \neq F$).

We shall now investigate one more case which gives us examples of (iterable) analytic contractions F in $\mathbb{C}^3$ such that each sufficiently small neighbourhood of F contains only iterable contractions (in fact only iterable power series transformations). We have to say that according to this result (Theorem 2) the investigation is not yet complete since the contractions considered do not cover all possible conjugacy classes (normal forms) of contractions. The proof of Theorem 2 again depends very much on Schwaiger's characterization in [6] but does not involve the more general results of [5].

Theorem 2. *Let F be a contraction*

$$\begin{aligned} y_1 &= \rho_1 x_1 &&+\cdots \\ y_2 &= \qquad \rho_2 x_2 &&+\cdots \quad (\textit{terms of degree at least } 2) \\ y_3 &= \qquad\qquad \rho_3 x_3 &&+\cdots \end{aligned}$$

where $0 < |\rho_3| < |\rho_2| < |\rho_1| < 1$. *Let* $\rho_1 = r_1 e^{l\pi i\varphi_1}$, $\rho_2 = r_2 e^{l\pi i \varphi_2}$, *with* $r_j > 0$, $0 \leq \varphi_j < 1$ *for* $j = 1, 2$ *and assume that*

$$\frac{\ln r_2}{\ln r_1} \notin \mathbb{Q}. \tag{8}$$

Then there is a neighbourhood of F such that each contraction in this neighbourhood is iterable.

Proof. Firstly, we show that F itself is iterable. Otherwise, according to [6], there would exist a relation $\rho_1^{\alpha_1} = \rho_2^{\beta_1}$, $1 < \alpha_1 < \beta_1, \alpha_1, \beta_1 \in N$, and hence $\alpha_1 \ln r_1 = \beta_1 \ln r_2$, which is impossible since $\ln r_2 / \ln r_1 \notin \mathbb{Q}$.

Assume, on the contrary, that there is a sequence $(G_m)_{m\in N}$ of contractions such that $\lim_{m\to\infty} G_m = F$ and each G_m is noniterable. Then clearly $G_m \neq F$, since F is iterable. Denote by $\rho_1^{(m)}, \rho_2^{(m)}, \rho_3^{(m)}$ the eigenvalues of the linear part $A^{(m)}$ of $G^{(m)}$. which tends to the linear part of F which is $A = \operatorname{diag}(\rho_1, \rho_2, \rho_3)$. By assumption, the zeros ρ_1, ρ_2, ρ_3 of det $(A - \rho E)$ are distinct. Since $\lim_{m\to\infty} A^{(k)} = A$, the discriminant of each polynomial $\det(A^{(m)} - \rho E)$ is different from zero if we substitute the eigenvalues $\rho_1^{(m)}, \rho_2^{(m)}, \rho_3^{(m)}$ for $m \geq m_0$, and, moreover, we may arrange the eigenvalues in such a way that $\lim \rho_j^{(m)} = \rho_j$, $j = 1, 2, 3$, and that $0 < |\rho_3^{(m)}| < |\rho_2^{(m)}| < |\rho_1^{(m)}| < 1$ for $m \geq m_0$.

This is a consequence of the continuous dependence of the roots of $\det(B - \rho E)$ on the matrix B if B is close enough to A since the discriminant of $\det(A - \rho E)$ is not zero at ρ_1, ρ_2, ρ_3 (see [9], p. 148 or [8], p. 48). In particular we may assume that each $G^{(m)}$ is a contraction if we neglect a finite number of transformations $G^{(\nu)}$, $\nu < m_0$. Now, according to the characterization of noniterable contractions in $\mathbb{C}^3$ in [6] for each m there is a relation

$$\rho_1^{(m)\alpha_1^{(m)}} = \rho_2^{(m)\beta_1^{(m)}} \tag{9}$$

with $1 < \alpha_1^{(m)} < \beta_1^{(m)}$, where $\alpha_1^{(m)}$ is minimal, and there is a finite set of relations

$$\rho_3^{(m)} = (\rho_1^{(m)})^{\alpha_0^{(m)} + k\alpha_1^{(m)}} (\rho_2^{(m)})^{\beta_0^{(m)} - k\beta_1^{(m)}}, \quad 0 \leq k \leq \Big[\frac{\beta_0^{(m)}}{\beta_1^{(m)}}\Big], \tag{10}$$

with cardinality at least 2, where

$$0 \le \alpha_0^{(m)} + k\alpha_1^{(m)},\ \beta_0^{(m)} - k\beta_1^{(m)},$$
$$2 \le (\alpha_0^{(m)} + k\alpha_1^{(m)}) + (\beta_0^{(m)} - k\beta_1^{(m)})$$

for $0 \le k \le [\beta_0^{(m)}/\beta_1^{(m)}]$. From (9) we get, if $|\rho_j^{(m)}| = r_j^{(m)}$, $0 < r_j^{(m)} < 1$, that $\alpha_1^{(m)} \ln r_1^{(m)} = \beta_1^{(m)} \ln r_2^{(m)}$ for all m. Since $\rho_j^{(m)} \xrightarrow{m\to\infty} \rho_j$, we find

$$\lim_{m\to\infty} \frac{\ln r_1^{(m)}}{\ln r_2^{(m)}} = \lim_{m\to\infty} \frac{\beta_1^{(m)}}{\alpha_1^{(m)}} = \frac{\ln r_1}{\ln r_2} \tag{11}$$

or, for each $\epsilon > 0$,

$$\left|\frac{\ln r_1}{\ln r_2} - \frac{\beta_1^{(m)}}{\alpha_2^{(m)}}\right| < \epsilon \tag{12}$$

if $m > M_0(\epsilon)$. By assumption $\ln r_1/\ln r_2 \notin \mathbb{Q}$, and $\beta_1^{(m)}/\alpha_1^{(m)} \in \mathbb{Q}$.

From (10) we get

$$\ln r_3^{(m)} = (\alpha_0^{(m)} + k\alpha_1^{(m)}) \ln r_1^{(m)} + (\beta_0^{(m)} - k\beta_1^{(m)}) \ln r_2^{(m)} \tag{13}$$

for all m.

Since $\lim_{m\to\infty} r_j^{(m)} = r_j$, $j = 1, 2, 3$, $\ln r_j^{(m)} < 0$, $\ln r_j < 0$, there exist $\Theta_1, \Theta_2, \Theta_3 > 0$ such that

$$\Theta_3 > (\alpha_0^{(m)} + k\alpha_1^{(m)})\Theta_1 + (\beta_0^{(m)} - k\beta_1^{(m)})\Theta_2$$

from $m > M_1$. Hence there exists $C > 0$ such that

$$0 \le \alpha_0^{(m)} + k\alpha_1^{(m)},\ \beta_0^{(m)} - k\beta_1^{(m)} < C$$

for all m, $0 \le k \le [\beta_0^{(m)}/\beta_1^{(m)}]$. This shows that there are altogether only finitely many possibilities for $\alpha_0^{(m)}, \alpha_1^{(m)}, \alpha_0^{(m)} + k\alpha_1^{(m)}$. From (11) we deduce that there is a $D > 0$ such that $\beta_1^{(m)}/\alpha_1^{(m)} < D$, for all m, and since we have only finitely many possibilities for $\alpha_1^{(m)}$ $(\le D_1)$ we get $0 < \beta_1^{(m)} \le DD_1 = D_2$ for all m. So (12) can be fulfilled for each $\epsilon > 0$ and all $m > M_0(\epsilon)$, but $\ln r_1/\ln r_2 \notin \mathbb{Q}$, and the rational numbers $\beta_1^{(m)}/\alpha_1^{(m)}$ form a finite set. This is a contradiction. Therefore, a sequence $(G^{(m)})_{m\in\mathbb{N}}$ of noniterable contractions such that $\lim_{m\to\infty} F^{(m)} = F$ cannot exist, and Theorem 2 is proved.

We notice that only part of the characterizing conditions of [6] was needed in the proof. Examples of the same type are also given by the following results.

Theorem 3. *Let F be an contraction of the form*

$$\begin{aligned} y_1 &= \rho_1 x_1 \qquad\qquad +\cdots \\ y_2 &= \qquad \rho_2 x_2 \qquad +\cdots \quad (\textit{terms of degree at least } 2) \\ y_3 &= \qquad\qquad \rho_3 x_3 +\cdots \end{aligned}$$

where $0 < |\rho_1| < |\rho_2| < |\rho_3| < 1$. *Suppose that* $\ln r_3$ *is not contained in the subgroup generated by* $\ln r_1$ *and* $\ln r_2$ *in* $(\mathbb{R}, +)$. *Then there exists a neighbourhood of F such that each transformation in this neighbourhood is iterable.*

Proof. Again, it is easy to see that F is iterable. Assume as in the proof of Theorem 2 the existence of sequence $(G^{(m)})_{m\in\mathbb{N}}$ of contrtactions which are noniterable and which converge to F. Using exactly the same notations as above we see that $\alpha_1^{(m)}$, $\beta_1^{(m)}$, $\alpha_0^{(m)}$, $\beta_0^{(m)}$, $\alpha_0^{(m)} + k\alpha_1^{(m)}$, $\beta_0^{(m)} - k\beta_1^{(m)}$ with $0 < k \leq [\beta_0^{(m)}/\beta_1^{(m)}]$ are bounded from above. Then (13) yields for $k = 0$

$$\ln r_3^{(m)} = \alpha_0^{(m)} \ln r_1^{(m)} + \beta_0^{(m)} \ln r_2^{(m)}$$

for all m. Since $\lim_{m\to\infty} r_j^{(m)} = r_j$, $j = 1, \ldots, m$, and since there are only finitely many distinct possibilities for $(\alpha_0^{(m)}, \beta_0^{(m)})$, we may select a subsequence $(m_l)_{l\in N}$, for which $\alpha_0^{(m_l)} = A$, $\beta_0^{(m_l)} = B (\in N)$ independently of $l \in$ N, and therefore $\ln r_3 = A \ \ln r_1 + B \ln r_2$ which is a contradition to our assumption that $\ln r_3$ does not belong to the subgroup of R generalted by $\ln r_1$, $\ln r_2$. This finishes the proof of Theorem 3.

In Theorem 4 which refers to a quite similar situation we make assumptions on the arguments of the eigenvalues ρ_1, ρ_2, ρ_3 rather than on the absolute values.

Theorem 4. *Let again F be a contraction*

$$\begin{aligned} y_1 &= \rho_1 x_1 \qquad\qquad +\cdots \\ y_2 &= \qquad \rho_2 x_2 \qquad +\cdots \quad (\textit{terms of degree at least } 2) \\ y_3 &= \qquad\qquad \rho_3 x_3 +\cdots \end{aligned}$$

where $0 < |\rho_3| < |\rho_2| < |\rho_1| < 1$. *Let* $\rho_j = r_j e^{2\pi i\varphi_j}$, $0 < r_j < 1, 0 \leq \varphi_j < 1$, *for* $j = 1, 2, 3$, *and assume that*

(i) $\varphi_1, \varphi_2, 1$ *are linearly indepent over* $\mathbb{Q}$ *or*

(ii) φ_3 *does not belong to the subgroup of* $\mathbb{R}$ *generated by* φ_1, φ_2 *and* 1.

Then there is a neighbourhood of F which is free of noniterable contractions.

Proof. As in previous cases we see that F is iterable. Using the same technique and notations as above we find in addition to relations for the logarithms of $r_j^{(m)}$,

$j = 1, 2, 3$, like (13) also relations for the arguments $\varphi_j^{(m)}$, namely

$$\alpha_1^{(m)}\varphi_1^{(m)} = \beta_1^{(m)}\varphi_2^{(m)} + R_m \tag{14}$$

and

$$\varphi_3^{(m)} = (\alpha_0^{(m)} + k\alpha_1^{(m)})\varphi_2^{(m)} + (\beta_0^{(m)} - k\beta_1^{(m)})\varphi_1^{(m)} + S_{m,k} \tag{15}$$

for all m, where $\rho_j^{(m)} = r_j e^{2\pi i \rho_j^{(m)}}$, $j = 1, 2, 3$, R_m and S_m are integers, and $0 \leq k \leq [\beta_0^{(m)}/\beta_1^{(m)}]$. Since $\rho_j^{(m)}$ tends to ρ_j for $m \to \infty$, we have also $\lim_{m\to\infty} \varphi_j^{(m)} = \varphi_j$. As in the proofs of Theorem 2 and Theorem 3 we get that the sequences $(\alpha_1^{(m)})_{m\in N}$, $(\beta_1^{(m)})_{m\in N}$, $(\alpha_0^{(m)})_{m\in N}$, $(\beta_0^{(m)})_{m\in N}$ are bounded above. From (14) we get,

$$|R_m| \leq \alpha_1^{(m)} + \beta_1^{(m)} \leq C_1$$

for all m, and from (15) (putting $k = 0$) $|S_{m,0}| \leq 1 + \alpha_0^{(m)} + \beta_0^{(m)} \leq C_2$ for all m, so there are only finitely many possibilities for R_m *and* $S_{m,0}$, and also for $(\alpha_1^{(m)}, \beta_1^{(m)}, R_m)$ and for $(\alpha_0^{(m)}, \beta_0^{(m)}, S_{m,0})$. Hence we may select a subsequence $(m_l)_{l\in N}$ of indices for which $\alpha_1^{(m_l)} = a$, $\beta_1^{(m_l)} = b$, $R_m = R$, and $\alpha_0^{(m_l)} = A$, $\beta_0^{(m_l)} = B$, $S_{m_l,0} = S$ for all l. If we go to the limit $l \to \infty$ we find from (14)

$$a\varphi_1 = b\varphi_2 + R \quad (a, b, R \in \mathbb{Z},\ a \neq 0) \tag{16}$$

and from (15)

$$\varphi_3 = A\varphi_1 + B\varphi_2 + S \quad (A, B, S \in \mathbb{Z}) \tag{17}$$

Under the assumption (i) of Theorem 4 we have a contradiction from (16), and under assumption (ii) we have a contradiction from (17).

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