ITERATION GROUPS WITH GENERALIZED CONVEX AND CONCAVE ELEMENTS

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ABSTRACT

The continuous multiplicative iteration groups of homemorphisms f^t : $(0,\infty) \to (0,\infty)$, $f^s \circ f^t = f^{st}$ $(s,t > 0)$ such that for every $t > 0$, the function f^t is M-convex or M-concave with respect to a given continuous function $M : (0,\infty)^2 \to (0,\infty)$ are considered. In the case $M(x, y) = x + y$ we show in particular that the iteration group with a geometrically convex generator consists of only subadditive or superadditive elements. In the case $M(x, y) = \frac{1}{2}(x + y)$ we show that if the derivative of the generator group is geometrically convex (or geometrically concave) then it consists only of convex and concave elements. Some applications to the converse of the Minkowski inequality are given.

Introduction

We examine the continuous multiplicative iteration groups of homemorphisms f^t : $(0,\infty) \to (0,\infty)$, $f^s \circ f^t = f^{st}$ $(s,t > 0)$, having the following property: for every $t > 0$, the function f^t is M-convex or M-concave (f^t is M-convex if $f^t(M(x, y)) \le M(f^t(x), f^t(y))$, where M is a continuous function on $(0,\infty)^2$. In Section 1 we prove that actually this basic property is a consequence of a considerably weaker condition (cf. Theorem 2). The main result of this section (Theorem 1) says that if there are two elements f^s and f^r, $s < 1 < r$, which are both M-convex or both M-concave, then

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all members of the group are M-affine, i.e., $f^t(M(x,y)) = M(f^t(x), f^t(y))$ for all $x, y, t > 0$. Some other results characterizing the considered iteration group $(f^t)_{t>0}$ in term of its generator ϕ such that $f^t = \phi \circ (t\phi^{-1})$ are also given.

In Section 2 we assume that $M(x,y) = x + y$. Now every M-convex (M-concave) function is subadditive (superadditive). Applying Theorem 1 we prove its more detailed counterpart (Theorem 3). The generator ϕ of the iteration group is a power function, and f^t is linear for every $t > 0$. This result is an essential generalization of the main result of [6]. Moreover, we show that each iteration group with geometrically convex generator ϕ has the property that its elements f^t are subadditive or superadditive.

In Section 3 we apply Theorem 3 to prove a new converse of the Minkowski's inequality.

Section 4 is devoted to the case $M(x,y) = (x+y)/2$. Thus f^t are convex or concave in the classical sense. Theorem 7, the counterpart of Theorem 1, is an obvious consequence of Theorem 3. The main result of this section, Theorem 8, asserts that every iteration group with a generator ϕ such that the derivative ϕ' is geometrically convex or geometrically concave has only convex and concave elements.

In the last section we discuss the possible generalizations of main results for the iteration groups $(f^t)_{t>0}$ with (M,N)-convex elements (i.e., such that $f^t(M(x,y)) \le N(f^t(x), f^t(y))$).

1. Iteration groups with M-convex and M-concave elements

Let $(f^t)_{t>0}$ be a multiplicative iteration group of homeomorphisms f^t of $(0,\infty)$, i.e., $f^s \circ f^t = f^{st}$ for all $s, t > 0$. The iteration group is continuous if for every $x > 0$ the function

$$(0,\infty) \ni t \to f^t(x)$$

is continuous.

Let $M : (0,\infty) \times (0,\infty) \to (0,\infty)$ be an arbitrary function. A function $f : (0,\infty) \to (0,\infty)$ is *M-convex* if

$$f(M(x,y)) \le M(f(x), f(y)), \qquad x, y > 0.$$

If the inequality is reversed the function f is termed *M-concave*. The function f is *M-affine* if

$$f(M(x,y)) = M(f(x), f(y)), \qquad x, y > 0.$$

Remark 1. Note that if $f, g : (0,\infty) \to (0,\infty)$ are increasing and M-convex then $f \circ g$ is also M-convex. If f is one-to-one and onto then the inverse function of f is M-concave.

Of course, if f and g are M-affine then $f \circ g$ is M-affine. Under some conditions on f and g the converse implication holds true. Namely, we have the following:

Lemma 1. *Suppose that $f, g : (0,\infty) \to (0,\infty)$ are M-convex (or M-concave), f is strictly increasing, and g is onto. If $f \circ g$ is M-affine then the functions f and g are M-affine.*

Proof. Assume that $f \circ g$ is M-affine. If f were not M-affine then there would exist $u_0, v_0 > 0$ such that

$$f(M(u_0, v_0)) < M(f(u_0), f(v_0)).$$

Since g is onto, there are $x_0, y_0 > 0$ such that $g(x_0) = u_0$ and $g(y_0) = v_0$. By the M-convexity of f and g and the monotonicity of f we hence get

$$\begin{aligned} f \circ g(M(x_0, y_0)) &\le f\,[\,M(g(x_0), g(y_0))\,] = f(M(u_0, v_0)) \\ &< M(f(u_0), f(v_0)) = M(f \circ g(x_0), f \circ g(y_0)), \end{aligned}$$

which is a contradiction.

Similarly, if g were not affine then we would have

$$g(M(x_0, y_0)) < M(g(x_0), g(y_0))$$

for some $x_0, y_0 > 0$. Since f is strictly increasing and M-convex, we obtain

$$\begin{aligned} f \circ g(M(x_0, y_0)) = f\,[\,g(M(x_0, y_0))\,] &< f\,[\,M(g(x_0), g(y_0))\,] \\ &\le M(f \circ g(x_0), f \circ g(y_0))\,. \end{aligned}$$

This contradiction completes the proof. □

As an obvious consequence we get the following

Corollary 1. *Suppose that $f : (0,\infty) \to (0,\infty)$ is M-convex (or M-concave), strictly increasing, and onto. If for a positive integer m, the m-th iterate of f is M-affine, then f is M-affine.*

Now we prove the main result of this section.

Theorem 1. Let $M : (0,\infty) \times (0,\infty) \to (0,\infty)$ be a continuous function and suppose that $(f^t)_{t>0}$ is a continuous multiplicative iteration group such that

(i) f^t is a homeomorphism of $(0,\infty)$ for every $t > 0$;
(ii) for some $t > 0$, f^t has no fixed points;
(iii) f^t is M-convex or M-concave for every $t > 0$;
(iv) there exist s and r, $0 < s < 1 < r$, such that f^s and f^r are both M-convex (or both M-concave).

Then

$$f^t(M(x,y)) = M(f^t(x), f^t(y)), \qquad x, y, t > 0,$$

i.e., f^t is M-affine for every $t > 0$.

Proof. Since f^t for some $t > 0$ is fixed point free it follows that for every $t > 0$, $t \neq 1$, the function f^t is also fixed point free. Consequently, f^t is an increasing homeomorphism of $(0, \infty)$ for every $t > 0$.

Suppose for instance that f^s and f^r are M-convex.

If $\log r / \log s$ is irrational then the set

$$D := \{r^n s^m : n, m \in \mathbb{N}\}$$

is dense in $(0, \infty)$ and, by Remark 1, we have

$$f^t(M(x,y)) \leq M(f^t(x), f^t(y)), \qquad x, y > 0,$$

for all $t \in D$. Since the iteration group and the function M are continuous, this inequality holds true for all $t > 0$. As $f^{1/t}$ is the inverse of f^t, we hence get

$$f^{1/t}(M(x,y)) \geq M(f^{1/t}(x), f^{1/t}(y)), \qquad x, y > 0,$$

for all $t > 0$. Consequently

$$f^t(M(x,y)) = M(f^t(x), f^t(y)), \qquad x, y, t > 0.$$

Now suppose that $\log r / \log s$ is rational. Since $0 < s < 1 < r$, there exist positive integers m, n such that $r^n s^m = 1$. Hence the functions f^{r^n} and f^{s^m} are inverses of each other, and, because they are increasing, one of them is M-convex and the other M-concave. On the other hand, each of them is M-convex as the composition of increasing M-convex functions (cf. Remark 1). It proves that they both are M-affine. In particular we have shown that there exist $q > 0$, $q \neq 1$, such that

$$f^q(M(x,y)) = M(f^q(x), f^q(y)), \qquad x, y > 0,$$

and

$$f^{1/q}(M(x,y)) = M(f^{1/q}(x), f^{1/q}(y)), \qquad x, y > 0.$$

Thus, without any loss of generality we may assume that $q > 1$. Take arbitrary $m \in \mathbb{N}$. Since

$$\underbrace{\left(f^{q^{1/m}}\right) \circ \cdots \circ \left(f^{q^{1/m}}\right)}_{m \text{ times}} = f^q,$$

the Corollary 1 implies that $f^{q^{1/m}}$ is M-affine. Hence, for every positive integer k, the function

$$f^{q^{k/m}} = \underbrace{\left(f^{q^{1/m}}\right) \circ \cdots \circ \left(f^{q^{1/m}}\right)}_{k \text{ times}}$$

is also M-affine. The set $\{q^{k/m} : m, k \in \mathbb{N}\}$ is dense in $(1, \infty)$. Now the continuity of the iteration group and the function M imply that for every $t > 1$, f^t is M-affine. This implies that f^t is affine for every $t > 0$, what was to be shown.

Remark 2. In iteration theory it is a well known fact (cf. for instance M. Kuczma [3], p. 198) that for each multiplicative continuous iteration group $(f^t)_{t>0}$ satisfying the conditions (i) and (ii) of Theorem 1 there exist an open interval $I \subset \mathbb{R}$ and a homeomorphism $\phi : I \to (0, \infty)$, called a *generator* of the iteration group, such that $f^t = \phi \circ (t\phi^{-1})$ for every $t > 0$ (here ϕ^{-1} denotes the inverse function of ϕ). Since for all $t > 0$ and $x \in (0, \infty)$, $t\phi^{-1}(x) \in I$, we have either $I = (0, \infty)$ or $I = \mathbb{R}$. In the sequel $I = (0, \infty)$ or $I = \mathbb{R}$.

Using this fact we can write Theorem 1 in the following equivalent form:

Corollary 2. *Let* $M : (0,\infty)\times(0,\infty) \to (0,\infty)$ *be a continuous function and* $(f^t)_{t>0}$ *a continuous multiplicative iteration group such that conditions* (i)–(iv) *of Theorem* 1 *are fulfilled. Let* $\phi : I \to (0, \infty)$ *be a generator of the iteration group* $(f^t)_{t>0}$.

Then the function $M_\phi : (0, \infty) \times (0, \infty) \to (0, \infty)$ *defined by*

$$M_\phi(x, y) := \phi^{-1} [\, M(\phi(x), \phi(y)) \,], \qquad x, y > 0,$$

is positively homogeneous, *i.e.*,

$$M_\phi(tx, ty) = tM_\phi(x, y), \qquad x, y, t > 0 .$$

Theorem 1 and Corollary 2 give the general conditions under which an iteration group must be trivial in a sense that all its members are M-affine functions. The crucial role in these theorems plays the assumption (iv). As an obvious consequence of Theorem 1 we obtain

Corollary 3. *Let* $M : (0, \infty) \times (0, \infty) \to (0, \infty)$ *be a continuous function and suppose that* $(f^t)_{t>0}$ *is a continuous multiplicative iteration group such that*
(*i*) f^t *is a homeomorphism of* $(0, \infty)$ *for every* $t > 0$;
(*ii*) *for some* $t > 0$, f^t *has no fixed points*;
(*iii*) f^t *is* M*-convex or* M*-concave for every* $t > 0$.
If the iteration group $(f^t)_{t>0}$ *is not trivial*, *i.e.*, *there is a* $t > 0$ *such that* f^t *is not* M*-affine*, *then* f^t *is not affine for every* $t > 0$, $t \neq 1$. *Moreover, either*

f^t *is* M*-convex for every* $t > 1$ *and* M*-concave for every* $t \in (0, 1)$,

or

f^t *is* M*-concave for every* $t > 1$ *and* M*-convex for every* $t \in (0, 1)$.

A natural problem to determine the iteration groups such that all the elements are M-convex or M-concave seems to be difficult to decide in a such general setting. However, making use of Corollary 3, and the generator representation of a multiplicative iteration group, we obtain the following:

Proposition 1. *Let* $M : (0,\infty) \times (0,\infty) \to (0,\infty)$ *be a continuous function*, ϕ *a homeomorphism of* I *onto* $(0,\infty)$, *and suppose that* $f^t = \phi \circ (t\phi^{-1})$ for $t > 0$.

1^0 *If* ϕ *is increasing then* f^t *is* M*-convex for every* $t > 1$, *iff the function* M_ϕ *is superhomogeneous*, *i.e.*,

$$tM_\phi(x,y) \le M_\phi(tx,ty), \qquad x,y > 0;\ t > 1.$$

2^0 *If* ϕ *is decreasing then* f^t *is* M*-convex for every* $t > 1$, *iff the function* M_ϕ *is subhomogeneous*, *i.e.*,

$$M_\phi(tx,ty) \le tM_\phi(x,y), \qquad x,y > 0;\ t > 1\,.$$

Let $(f^t)_{t>1}$ be a (multiplicative) iteration semi-group of functions $f^t : (0,\infty) \to (0,\infty)$, i.e., $f^s \circ f^t = f^{st}$ for all $s,t > 1$. The iteration semi-group is continuous if for every $x > 0$ the function

$$(1,\infty) \ni t \to f^t(x)$$

is continuous.

The next result shows that condition (iii) of Theorem 1, and Corollary 3 can be replaced by a weaker one.

Proposition 2. *Let* $(f^t)_{t>1}$ *be a continuous multiplicative iteration semi-group*, *and* $M : (0,\infty) \times (0,\infty) \to (0,\infty)$ a *continuous function. If there exists a sequence* $t_k > 1$, $\lim_{k\to\infty} t_k = 1$ *such that for every* $k \in \mathbb{N}$ *the function* f^{t_k} *is increasing and* M*-convex* (*resp.* M*-concave*), *then for every* $t > 1$ *the function* f^t *is increasing and* M*-convex* (*resp.* M*-concave*).

Proof. Since $f^{t_k} : (0,\infty) \to (0,\infty)$ is increasing and M-convex, the composite function

$$f^{t_k^2} = f^{t_k} \circ f^{t_k}$$

is increasing and, in view of Remark 1, M-convex. In the same way we can show that the function $f^{t_k^m}$ is increasing and M-convex for all $k,m \in \mathbb{N}$. Put

$$S := \{t_k^m : k,m \in \mathbb{N}\},$$

and take an arbitrary $t > 1$. The density of the set S in $(1,\infty)$ implies the existence of a sequence $s_n \in S$ such that $\lim_{n\to\infty} s_n$=t. Moreover f^{s_n} is increasing and we have

$$f^{s_n}\left(M(x,y)\right) \le M\left(f^{s_n}(x), f^{s_n}(y)\right), \qquad x,\ y > 0\,.$$

By the continuity of the iteration semigroup, letting $n \to \infty$, we hence get the M-convexity of f^t.

In the same way one can prove

Theorem 2. *Let* $(f^t)_{t>0}$ *be a continuous multiplicative iteration group, and* $M : (0,\infty)\times(0,\infty) \to (0,\infty)$ *a continuous function. If there exists a sequence* $t_k > 0$, $t_k \neq 1$, $\lim_{k\to\infty} t_k = 1$ *such that for every* $k \in \mathbb{N}$ *the function* f^{t_k} *is increasing and* M*-convex or* M*-concave, then*:

1^0 *for every* $t > 0$ *the function* f^t *is increasing*;

2^0 *either* f^t *is* M*-convex for all* $t > 1$, *and* f^t *is* M*-concave for all* $t \in (0,1)$, *or* f^t *is* M*-convex for all* $t > 1$, *and* f^t *is* M*-concave for all* $t \in (0,1)$.

2. Iteration groups with subadditive and superadditive functions

Take $M(x,y) := x + y$ for $x, y > 0$. A function $f : (0,\infty) \to (0,\infty)$ is M-convex iff

$$f(x+y) \le f(x) + f(y), \qquad x,\ y > 0\,,$$

i.e., f is subadditive. Of course, f is M-concave, and M-affine if, respectively, f is superadditive, and additive. Applying Theorem 1 with $M(x,y) = x + y$ we shall prove the following

Theorem 3. *Suppose that* $(f^t)_{t>0}$ *is a continuous multiplicative iteration group such that*

(*i*) f^t *is a homeomorphism of* $(0,\infty)$ *for every* $t > 0$;

(*ii*) *for some* $t > 0$, f^t *has no fixed points*;

(*iii*) f^t *is subadditive or superadditive for every* $t > 0$;

(*iv*) *there exist* s *and* r, $0 < s < 1 < r$, *such that* f^s *and* f^r *are both subadditive* (*or both superadditive*).

Then there exists a $p \in \mathbb{R}\backslash\{0\}$ *such that*

$$f^t(x) = t^p x, \qquad x,\ t > 0\,.$$

Moreover, the generator ϕ *of the iteration group is a power function, i.e.,* $\phi(x) = \phi(1)x^p$, $x > 0$, *for some* $p \in \mathbb{R}$, $p \neq 0$, *and* $\phi(1) > 0$ *is an arbitrary constant.*

Proof. Let a homeomorphism $\phi : I \to (0,\infty)$ be a generator of $(f^t)_{t>0}$. By Theorem 1 with $M(x,y) = x + y$, $f^t = \phi \circ (t\phi^{-1})$ is additive for every $t > 0$. Consequently, for every $t > 0$, there exists a $c(t) > 0$ such that

$$\phi[t\phi^{-1}(x)] = c(t)x, \qquad x > 0.$$

Writing an analogous equation for every $s > 0$ we have

$$\phi[s\phi^{-1}(x)] = c(s)x, \qquad x > 0.$$

Composing separately the functions on the left and the right–hand sides of the above equations we obtain

$$\phi[st\phi^{-1}(x)] = c(s)\,c(t)x, \qquad x > 0\,.$$

On the other hand we have

$$\phi[st\phi^{-1}(x)] = c(st)x, \qquad x > 0.$$

The last two equations give

$$c(st) = c(s)\,c(t), \qquad s, t > 0\,,$$

which shows that $c : (0, \infty) \to (0, \infty)$ is a continuous solution of the multiplicative Cauchy equation. Since,

$$c(t) = \phi[t\phi^{-1}(1)], \qquad t > 0,$$

is continuous, there exists a $p \in \mathbb{R}$, $p \neq 0$, such that $c(t) = t^p$, $t > 0$, (cf. J.Aczél [1], p. 41). Now (6) implies that ϕ is a power function. This completes the proof. □

Theorem 3 is a generalization of Theorem 1 in [6] where only the case when $\log s/\log r$ is irrational was considered.

An application of Theorem 2 for $M(x, y) = x + y$ gives the following

Corollary 4. *Let $(f^t)_{t>0}$ be a continuous multiplicative iteration group, and $M : (0, \infty) \times (0, \infty) \to (0, \infty)$ a continuous function. If there exists a sequence $t_k > 0$, $t_k \neq 1$, $\lim_{k\to\infty} t_k = 1$, such that for every $k \in \mathbb{N}$ the function f^{t_k} is increasing and subadditive or superadditive, then:*

1^0 *for every $t > 0$ the function f^t is increasing;*

2^0 *either f^t is subadditive for all $t > 1$, and superadditive for all $t \in (0, 1)$, or f^t is superadditive for all $t > 1$, and subadditive for all $t \in (0, 1)$.*

Corollary 4 shows that assumption (iii) of Theorem 3 can be considerably weakened.

Applying Proposition 1 for $M(x, y) = x + y$ gives the following

Corollary 5. *Let $\phi : I \to (0, \infty)$ be a homeomorphism, and suppose that $f^t = \phi \circ (t\phi^{-1})$* for $t > 0$.

1^0 *If ϕ is increasing then f^t is subadditive for every $t > 1$, iff*

$$t\phi^{-1}(\phi(x) + \phi(y)) \le \phi^{-1}(\phi(tx) + \phi(ty)), \qquad x, y > 0;\; t > 1.$$

2^0 *If ϕ is decreasing then f^t is subadditive for every $t > 1$, iff*

$$\phi^{-1}(\phi(tx) + \phi(ty)) \le t\phi^{-1}(\phi(tx) + \phi(ty)), \qquad x,\; y > 0;\; t > 1\,.$$

To determine some iteration groups such that all the elements are subadditive or superadditive we need the following definition.

A function $g:(0,\infty)\to(0,\infty)$ is *geometrically convex* iff

$$g(x^{\lambda}y^{1-\lambda})\le g(x)^{\lambda}g(y)^{1-\lambda},\quad x,\ y>0;\quad \lambda\in(0,1)\,.$$

If the reversed inequality holds true, the function g is said to be *geometrically concave.* Note that g is geometrically convex iff the function $\log\circ g\circ\exp$ is convex on $\mathbb{R}$. A continuous function g is geometrically convex iff

$$g(\sqrt{xy})\le\sqrt{g(x)\,g(y)},\quad x,\ y>0\,,$$

and g is geometrically concave iff this inequality is reversed.

Lemma 2. *Let $\phi:(0,\infty)\to(0,\infty)$ be bijective. For every $t>0$ define the function $g_t:(0,\infty)\to(0,\infty)$ by the formula*

$$g_t(x):=\frac{\phi[t\phi^{-1}(x)]}{x},\quad x>0.$$

1^0 *If ϕ is increasing and geometrically convex, then g_t is decreasing for every $t\in(0,1)$ and increasing for every $t>1$.*
2^0 *If ϕ is increasing and geometrically concave, then g_t is increasing for every $t\in(0,1)$ and decreasing for every $t>1$.*
3^0 *If ϕ is decreasing and geometrically convex, then g_t is increasing for every $t\in(0,1)$ and decreasing for every $t>1$.*
4^0 *If ϕ is decreasing and geometrically concave, then g_t is decreasing for every $t\in(0,1)$ and increasing for every $t>1$.*

Proof. 1^0 Since ϕ is increasing it is enough show that the function

$$(g_t\circ\phi)(x):=\frac{\phi(tx)}{\phi(x)},\quad x>0,$$

is decreasing for $t\in(0,1)$ and increasing for $t>1$. From the convexity of $\log\circ\phi\circ\exp$ it follows that the right derivative $\phi':=\phi'_+$ exists everywere in $(0,\infty)$, and the function

$$x\to\frac{\phi'(x)}{\phi(x)}x,\quad x>0\,,$$

is increasing. Therefore for all $x>0$ the right derivative $(g_t\circ\phi)'_+(x)$ exists and we have

$$\begin{aligned}(g_t\circ\phi)'_+(x)&=\frac{\phi'(tx)t\phi(x)-\phi(tx)\phi'(x)}{[\phi(x)]^2}\\&=\frac{\phi(tx)}{x\phi(x)}\left(\frac{\phi'(tx)}{\phi(tx)}(tx)-\frac{\phi'(x)}{\phi(x)}x\right).\end{aligned}$$

Now the continuity of $g_t\circ\phi$ implies that it is decreasing for $t\in(0,1)$ and increasing for $t>1$.

The proofs of statements 2^0–4^0 are analogous. □

Remark 3. Let $f : (0,\infty) \to (0,\infty)$. If the function $g : (0,\infty) \to (0,\infty)$ given by the formula

$$g(x) := \frac{f(x)}{x}, \quad x > 0,$$

is decreasing (resp. increasing) then f is subadditive (resp. superadditive), (cf. E. Hille, R.S. Phillips [2], Theorem 7.2.4.). Note that g is decreasing (resp. increasing) if, and only if, f is *subhomogeneous*, i.e.,

$$f(rx) \geq rf(x), \quad r \leq 1;\ x > 0; \qquad f(rx) \leq rf(x), \quad r \geq 1;\ x > 0,$$

(resp. f is *superhomogeneous*, i.e.,

$$f(rx) \leq rf(x), \quad r \leq 1;\ x > 0; \qquad f(rx) \geq rf(x), \quad r \geq 1;\ x > 0.$$

From this Remark and Lemma 2 we obtain

Theorem 4. *Let $(f^t)_{t>0}$ a multiplicative iteration group with a generator ϕ : $(0,\infty) \to (0,\infty)$.*

1^0 *If ϕ is increasing and geometrically convex then*
(a) *f^t is subadditive (and subhomogeneous) for every $t \in (0,1)$;*
(b) *f^t is superadditive (and superhomogeneous) for every $t > 1$.*

2^0 *If ϕ is increasing and geometrically concave then*
(a) *f^t is superadditive (and superhomogeneous) for every $t \in (0,1)$;*
(b) *f^t is subadditive (and subhomogeneous) for every $t > 1$.*

3^0 *If ϕ is decreasing and geometrically convex then*
(a) *f^t is superadditive (and superhomogeneous) for every $t \in (0,1)$;*
(b) *f^t is subadditive (and subhomogeneous) for every $t > 1$.*

4^0 *If ϕ is decreasing and geometrically concave then*
(a) *f^t is subadditive (and subhomogeneous) for every $t \in (0,1)$;*
(b) *f^t is superadditive (and superhomogeneous) for every $t > 1$.*

Example 1. It is easy to verify that the function $\phi(x) := e^x - 1$, $(x > 0)$, is an increasing and geometrically convex bijection of $(0,\infty)$. Thus the iteration group

$$f^t(x) := \phi[t\phi^{-1}(x)] = \exp[t\ \log(x+1)] = (x+1)^t - 1, \quad t > 0;\ x > 0,$$

consists of subadditive functions for $t < 1$ and superadditive for $t > 1$.

3. An application to the converse of Minkowski's inequality

Let χ_C denote the characteristic function of a set C. The symbol $\mathbf{Lin}(\chi_C)$ stands here for the one dimensional linear space $\{u\,\chi_C : u \in \mathbb{R}\}$. By $\mathbf{Lin}_+(\chi_C)$ we denote the set of all elements of $\mathbf{Lin}(\chi_C)$ which are positive on C.

In this section we apply Theorem 3 to prove the following result which can be treated as a partial converse of the Minkowski's inequality:

Theorem 5. *Let* (Ω, Σ, μ) *be a measure space such that* 1 *is an accumulation point of the range of measure* $\mu(\Sigma)$ *and suppose that there exist two sets* $A, B \in \Sigma$ *such that*

$$0 < \mu(A) < 1 < \mu(B) < \infty.$$

If $\gamma : [0,\infty) \to [0,\infty)$ *is a continuous bijection such that*

$$\gamma^{-1}\Big(\int_\Omega \gamma \circ (x+y)\, d\mu\Big) \le \gamma^{-1}\Big(\int_\Omega \gamma \circ x\, d\mu\Big) + \gamma^{-1}\Big(\int_\Omega \gamma \circ y\, d\mu\Big) \qquad (1)$$

for all $x,\ y \in \mathbf{Lin}_+(\chi_C)$, $C \in \Sigma$, $\mu(C) < \infty$, *then there exists* a $p > 0$ *such that* $\gamma(t) = \gamma(1)t^p$, $t \ge 0$. *Moreover the both sides of inequality* (1) *are equal.*

Proof. Take an arbitrary $C \in \Sigma$ such that $\mu(C) < \infty$, and $x, y \in \mathbf{Lin}_+(\chi_C)$. There are $u, v \ge 0$ such that $x = u\chi_C$ and $y = v\chi_C$. Substituting these functions into (1) gives

$$\gamma^{-1}\Big(\int_\Omega \gamma\circ((u+v)\chi_C)\, d\mu\Big) \le \gamma^{-1}\Big(\int_\Omega \gamma\circ(u\chi_C)\, d\mu\Big) + \gamma^{-1}\Big(\int_\Omega \gamma\circ(v\chi_C)\, d\mu\Big).$$

Since $\gamma(0) = 0$, we can write this inequality in the form

$$\gamma^{-1}(\mu(C)\,\gamma(u+v)) \le \gamma^{-1}(\mu(C)\,\gamma(u)) + \gamma^{-1}(\mu(C)\gamma(v)), \quad u, v \ge 0\,. \qquad (2)$$

Thus the function $\gamma^{-1} \circ (t\gamma)$ is subadditive for every $t \in \mu(\Sigma)\backslash\{\infty)$. Put $\phi := \gamma^{-1}|_{(0,\infty)}$. Then $\phi \circ (t\phi^{-1}) = \gamma^{-1} \circ (t\gamma)$ for all $t > 0$, and $(f^t)_{t>0}$ defined by $f^t(x) := \phi \circ (t\phi^{-1})$ is a continuous multiplicative iteration group satisfying the conditions (i) and (ii) of Theorem 3. Since 1 is an accumulation point of $\mu(\Sigma)$, there exists a sequence $t_n \in \mu(\Sigma)$ such that $t_n \ne 1$, $t_n > 0$, and $\lim_{n\to\infty} t_n = 1$. We may assume, without any loss of generality, that $t_n > 1$ for all $n \in \mathbb{N}$. Since f^{t_n} is subadditive for every $n \in \mathbb{N}$, in view of Corollary 4, every element of the iteration semi-group $(f^t)_{t>1}$ is subadditive. Thus for every $t \in (0,1)$, the function f^t is superadditive, and consequently, the condidion (iii) of Theorem 3 is fulfilled.

Put $s := \mu(A)$, and $r := \mu(B)$. Taking $C = A$, and next $C = B$ in (2) shows that the functions $f^s = \phi \circ (s\phi^{-1})$ and $f^r = \phi \circ (r\phi^{-1})$ are subadditive. Since $0 < s < 1 < r$, the condition (iv) of Theorem 3 is also fulfilled. By Theorem 3 the function $\phi = \gamma^{-1}$ is a power function, and consequently, the inequality (1) becomes an equality. This completes the proof. $\square$

Remark 4. Assuming additionally that inequality (1) holds true for two linearly independent functions $x = \chi_C$ and $y = \chi_D$, $C, D \in \Sigma$, $\mu(C), \mu(D) \in (0,\infty)$, one can show that $p \ge 1$.

It can be easily verified that, in a similar way as Theorem 5, one can prove the following more general

Theorem 6. *Let* (Ω, Σ, μ) *be a measure space such that* 1 *is an accumulation point of the range of measure* $\mu(\Sigma)$, *and suppose that there exist two sets* $A, B \in \Sigma$ *such that*

$$0 < \mu(A) < 1 < \mu(B) < \infty.$$

If $\gamma : (0, \infty) \to (0, \infty)$ *is a continuous bijection such that*

$$\gamma^{-1}\left(\int_{\Omega} \gamma \circ (x + y)\, d\mu\right) \leq \gamma^{-1}\left(\int_{\Omega} \gamma \circ x\, d\mu\right) + \gamma^{-1}\left(\int_{\Omega} \gamma \circ y\, d\mu\right) \tag{3}$$

for all $x,\ y \in \mathbf{Lin}_{+}(\chi_C)$, $C \in \Sigma$, $0 < \mu(C) < \infty$, *then there exists* a $p \neq 0$ *such that* $\gamma(t) = \gamma(1)t^p$, $t > 0$. *Moreover the both sides of inequality* (3) *are equal.*

Remark 5. Some converses of the Minkowski's inequality, for essentially stronger inequality than (3), have been proved in [5] (cf. also [6]).

4. Iteration groups with convex or concave functions

Take $M(x, y) := (x + y)/2$ for $x, y > 0$. A function $f : (0, \infty) \to (0, \infty)$ is M-convex iff

$$f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2}, \quad x,\ y > 0,$$

i.e., f is Jensen convex. Of course, f is M-concave and M-affine if, respectively, f is Jensen concave, Jensen affine. Applying Theorem 1, similarly as in the proof of Theorem 3, one can prove the following

Theorem 7. *Suppose that* $(f^t)_{t>0}$ *is a continuous multiplicative iteration group such that*

(*i*) *for every* $t > 0$, f^t *is a homeomorphism of* $(0, \infty)$;
(*ii*) *for some* $t > 0$, f^t *has no fixed points*;
(*iii*) *for every* $t > 0$, f^t *is convex or concave*;
(*iv*) *there exist* s *and* r, $0 < s < 1 < r$, *such that* f^s *and* f^r *are both convex* (*or both concave*).

Then there exists $p \in \mathbb{R}\backslash\{0\}$ *such that*

$$f^t(x) = t^p x, \quad x,\ t > 0.$$

Moreover, the generator ϕ *of the iteration group is the power function, i.e.,* $\phi(x) = \phi(1)x^p$, $x > 0$, *for some* $p \in \mathbb{R}$, $p \neq 0$, *and* $\phi(1) > 0$ *is an arbitrary constant.*

Remark 6. Because every increasing and convex homeomorphism of the interval $(0, \infty)$ is superadditive, Theorem 7 is an immediate consequence of Theorem 3.

Remark 7. In view of Theorem 2, the assumption (iii) in the above theorem is satisfied if there exists a sequence $t_k > 0$, $t_k \neq 1$, $\lim_{k\to\infty} t_k = 1$ such that for every $k \in \mathbb{N}$ the function f^{t_k} is increasing and convex or concave.

Theorem 8. *Let a differentiable bijection $\phi : (0,\infty) \to (0,\infty)$ be a generator of the iteration group $(f^t)_{t>0}$, i.e., $f^t = \phi \circ (t\phi^{-1})$ for all $t > 0$.*

1^0 *If $\phi' > 0$ and ϕ' is geometrically convex, then f^t is convex for every $t > 1$, and concave for every $t \in (0,1)$.*

2^0 *If $\phi' > 0$ and ϕ' is geometrically concave, then f^t is concave for every $t > 1$, and convex for every $t \in (0,1)$.*

3^0 *If $\phi' < 0$ and $-\phi'$ is geometrically convex, then f^t is concave for every $t > 1$, and convex for every $t \in (0,1)$.*

4^0 *If $\phi' < 0$ and $-\phi'$ is geometrically concave, then f^t is convex for every $t > 1$, and concave for every $t \in (0,1)$.*

Proof. 1^0 The geometrical convexity of ϕ' implies the existence of the right derivative $\phi^{(2)} := (\phi')'_+$ exists everywhere in $(0,\infty)$ and the function

$$x \to \frac{\phi^{(2)}(x)}{\phi'(x)}x, \quad x > 0,$$

is increasing. Thus

$$\frac{\phi^{(2)}(tx)}{\phi'(tx)}tx \geq \frac{\phi^{(2)}(x)}{\phi'(x)}x, \quad x > 0,\ t > 1,$$

or, equivalently,

$$\phi^{(2)}(tx)t\phi'(x) - \phi'(tx)\phi^{(2)}(x) \geq 0, \quad x > 0,\ t > 1\,.$$

Multiplying the both sides by the positive function $t/[\phi'(x)]^3$ gives

$$\phi^{(2)}(tx)t^2[\phi'(x)]^{-2} - \phi'(tx)t\phi^{(2)}(x)[\phi'(x)]^{-3} \geq 0, \quad x > 0,\ t > 1.$$

Replacing x by $\phi^{-1}(x)$ yields the inequality

$$\frac{\phi^{(2)}(t\phi^{-1}(x))t^2}{[\,\phi'(\phi^{-1}(x))\,]^{-2}} - \frac{\phi'(t\phi^{-1}(x))t\phi^{(2)}(\phi^{-1}(x))}{[\,\phi'(\phi^{-1}(x))\,]^{-3}} \geq 0, \quad x > 0,\ t > 1\,.$$

which means that

$$\frac{d^2 f^t}{dx^2}(x) = \frac{d^2}{dx^2}\phi \circ (t\phi^{-1})(x) \geq 0, \quad x > 0,\ t > 1\,.$$

It follows that f^t is convex and consequently the proof of 1^0 is completed.
We omit analogous proofs of statements 2^0–4^0. □

5. Iteration groups with geometrically convex and geometrically concave elements

Take $M(x, y) := \sqrt{xy}$ for $x, y > 0$. A function $f : (0, \infty) \to (0, \infty)$ is M-convex iff

$$f(\sqrt{xy}) \leq \sqrt{f(x)\, f(y)}, \quad x,\ y > 0,$$

i.e., f is Jensen geometrically convex. Of course, f is M-concave, and M-affine if, respectively, f is Jensen geometrically concave, Jensen geometrically affine. Applying Theorem 1, similarly as in the proof of Theorem 3, one can prove the following

Theorem 9. *Suppose that* $(f^t)_{t>0}$ *is a continuous multiplicative iteration group such that*

(i) *for every* $t > 0$, f^t *is a homeomorphism of* $(0, \infty)$;
(ii) *for some* $t > 0$, f^t *has no fixed points*;
(iii) *for every* $t > 0$, f^t *is geometrically convex or geometrically concave*;
(iv) *there exist* s *and* r, $0 < s < 1 < r$, *such that* f^s *and* f^r *are both geometrically convex* (*or both geometrically concave*).

Then either

1^0 *there exists a* $q \in \mathbb{R}\backslash\{0\}$ *such that*

$$f^t(x) = t^q x, \quad x,\ t > 0,$$

and the generator ϕ *of the iteration group is the power function* $\phi(t) = \phi(1)t^q$, $t > 0$, *for some* $q \in \mathbb{R}$, $q \neq 0$, *and* $\phi(1) > 0$ *is an arbitrary constant*; *or*

2^0 *there exist* $p \in \mathbb{R}$, $p \neq 0$, *and* $c > 0$, *such that*

$$f^t(x) = c^{t^p - 1} x^{t^p}, \quad x, t > 0;$$

moreover, the generator ϕ *of the iteration group is of the form*

$$\phi(t) = \phi(1) c^{t^p - 1}, \quad t > 0,$$

where $\phi(1) > 0$ *is an arbitrary constant.*

Proof. Let $\phi : I \to (0, \infty)$ be a generator of the iteration group. In view of Theorem 1 we have

$$f^t(\sqrt{xy}) = \phi\left(t\,\phi^{-1}\left(\sqrt{xy}\right)\right) = \sqrt{\phi\left(t\,\phi^{-1}(x)\right)\,\phi\left(\phi^{-1}(y)\right)}, \quad x,\ y,\ t > 0.$$

Setting $x = e^u$, $y = e^v$, and taking the logarithm of both the sides, we can write this equation in the equivalent form

$$\log \phi\left(t\,\phi^{-1}\left(e^{\frac{1}{2}(u+v)}\right)\right) = \frac{\log \phi(t\phi^{-1}(e^u)) + \log \phi(t\phi^{-1}(e^v))}{2}, \quad u,\ v \in \mathbb{R},$$

which means that for each $t > 0$ the function $\log \circ \phi \circ (t\phi^{-1}) \circ \exp$ is a Jensen one. Since it is continuous, there exist (cf. Aczél [1] or Kuczma [4]) the real numbers $a(t)$ and $b(t)$ such that

$$\log \phi(t\phi^{-1}(e^u)) = a(t)u + b(t), \quad u \in \mathbb{R},\ t > 0\,.$$

Moreover the continuity of ϕ and ϕ^{-1} implies that the functions $a, b : (0, \infty) \to \mathbb{R}$ are continuous. Putting $B(t) := \exp(b(t))$, we have $B(t) > 0$ for all $t > 0$, and

$$\phi(t\phi^{-1}(x)) = B(t)x^{a(t)}, \quad x,\ t > 0\,. \tag{4}$$

Hence

$$\phi(st\phi^{-1}(x)) = B(st)x^{a(st)}, \quad x,\ s,\ t > 0\,.$$

On the other hand, composing the function $\phi \circ (s\phi^{-1})$ and $\phi \circ (t\phi^{-1})$, we get

$$\phi(st\phi^{-1}(x)) = B(s)\,[\,B(t)\,]^{a(s)}\,x^{a(s)a(t)}, \quad x,\ s,\ t > 0\,.$$

From the last two relations we obtain

$$B(st)x^{a(st)} = B(s)\,[\,B(t)\,]^{a(s)}\,x^{a(s)a(t)}, \quad x,\ s,\ t > 0\,.$$

It follows that the function a is multiplicative, i.e.,

$$a(st) = a(s)\,a(t), \quad s,\ t > 0\,,$$

and B satisfies the functional equation

$$B(st) = B(s)\,[\,B(t)\,]^{a(s)}\,, \quad s,\ t > 0\,. \tag{5}$$

The continuity of a implies the existence of $p \in \mathbb{R}$ (cf. [1], [4]) such that

$$a(t) = t^p, \quad t > 0\,.$$

If $p = 0$ then $a \equiv 1$, and by equation (5), the function B is multiplicative. Thus there exists a $q \in \mathbb{R}$ such that

$$B(t) = t^q, \quad t > 0\,.$$

Setting $x := \phi(1)$ in (4) gives $\phi(t) = \phi(1)B(t)$, and consequently

$$\phi(t) = \phi(1)t^q, \quad t > 0\,,$$

where $q \neq 0$.

If $p \neq 0$, then (5) has the form

$$B(st) = B(s)\,[\,B(t)\,]^{s^p}\,, \quad s,\ t > 0\,.$$

Taking $s = 1$ we infer that $B(1) = 1$. The symmetry of the left side of this relation gives

$$B(s)\,[\,B(t)\,]^{s^p} = B(t)\,[\,B(s)\,]^{t^p}\,, \quad s,\ t > 0\,,$$

i.e.,

$$[B(t)]^{(t^p-1)^{-1}} = [B(s)]^{(s^p-1)^{-1}}, \quad s,\ t > 0.$$

Thus there is a constant $K > 0$ such that

$$[B(t)]^{(t^p-1)^{-1}} = K, \quad t > 0,$$

and, consequently,

$$B(t) = K^{t^p-1}, \quad t > 0.$$

By (4) we obtain

$$\phi(t\phi^{-1}(x)) = K^{t^p-1}x^{t^p}, \quad x,\ t > 0.$$

For $x := \phi(1)$ we hence get

$$\phi(t) = K^{t^p-1}(\phi(1))^{t^p} = \phi(1)(\phi(1)K)^{t^p-1}, \quad t > 0.$$

Putting $c := \phi(1)K$ gives

$$\phi(t) = \phi(1)c^{t^p-1}, \quad t > 0,$$

and, as ϕ is one-to-one, we infer that $c \neq 1$. Of course, we also have

$$\phi(t\phi^{-1}(x)) = \left(\frac{c}{\phi(1)}\right)^{t^p-1} x^{t^p}, \quad x,\ t > 0.$$

This completes the proof. □

Final remarks

The basic notions of Section 1, the M-convexity, M-concavity, and M-affinity of a function can be extended in the following way. Let $M_1, M_2 : (0,\infty) \times (0,\infty) \to (0,\infty)$ be arbitrary functions. A function $f : (0,\infty) \to (0,\infty)$ is said to be (M_1, M_2)-*convex* if

$$f(M_1(x, y)) \leq M_2(f(x), f(y)), \quad x,\ y > 0.$$

If the inequality is reversed, f is (M_1, M_2)-*concave*. The function f is (M_1, M_2)-*affine* if

$$f(M_1(x, y)) = M_2(f(x), f(y)), \quad x,\ y > 0.$$

Now the question arises whether Theorem 1 can be extended to iteration groups with (M_1, M_2)-convex and (M_1, M_2)-concave functions where $M_1 \neq M_2$. To show that the answer is negative consider the following

Example 2. Take $M_1(x, y) := (xy)^{1/2}$, $M_2(x, y) := (x + y)/2$, for $x,\ y > 0$. It is easy to verify that $(f^t)_{t>0}$ where

$$f^t(x) := (x + 1)^t - 1, \quad x > 0,$$

is a multiplicative continuous iteration group (a generator of this group is $\phi(x) := e^x - 1, x > 0$). For every fixed $t > 0$, the function

$$f^t(e^x) := (e^x + 1)^t - 1, \quad x \in \mathbb{R},$$

(i.e., the function $f^t \circ \exp$) is convex on $\mathbb{R}$. In fact, we have

$$\frac{\partial^2 f^t}{\partial x^2}(x) = t^2 e^x + t > 0, \quad x \in \mathbb{R}.$$

Consequently, for every fixed $t > 0$, the function $f^t \circ \exp$ is Jensen convex:

$$2f^t(e^{(x+y)/2}) \leq f^t(e^x) + f^t(e^y), \quad x, y \in \mathbb{R},$$

or, equivalently,

$$f^t((xy)^{1/2}) \leq \frac{f^t(x) + f^t(y)}{2}, \quad x,\ y > 0.$$

Thus we have proved that

$$f^t(M_1(x, y)) \leq M_2(f^t(x), f^t(y)), \quad x,\ y,\ t > 0.$$

This inequality shows that all functions of the considered iteration group $(f^t)_{t>0}$ are (M_1, M_2)-convex. Since for every $t > 0$, $\partial^2 f^t/\partial x^2$ is positive in $(0, \infty)$, we have

$$f^t(M_1(x, y)) < M_2(f^t(x), f^t(y)), \quad x,\ y,\ t > 0,\ \ x \neq y.$$

It follows that for every $t > 0$, the function f^t is not (M_1, M_2)-affine.

Remark 8. Writing the last two inequalities in their explicite forms we obtain the following inequalities

$$2((xy)^{1/2} + 1)^t \leq (x + 1)^t + (y + 1)^t, \quad x,\ y,\ t > 0,$$
$$2((xy)^{1/2} + 1)^t < (x + 1)^t + (y + 1)^t, \quad x,\ y,\ t > 0,\ \ x \neq y.$$

Remark 9. All the considered iteration groups were assumed to be continuous. Instead of the continuity it is enough to assume that they are measurable (cf. M.C. Zdun [7]).

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