ON CONTINUOUS ITERATION GROUPS OF SOME HOMEOMORPHISMS OF THE PLANE

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ABSTRACT

We consider the problem of determining all continuous iteration groups $\{f^t : t \in \mathbb{R}\}$ of a free mapping f. Some of free mappings which are not Sperner homeomorphisms are considered. We prove that the plane can be covered by a family of simply connected domains such that in each of them $\{f^t : t \in \mathbb{R}\}$ is conjugate with a group of translations.

We are interested in finding the form of all continuous iteration groups $\{f^t : t \in \mathbb{R}\}$ of a *free mapping* f (i.e., a homeomorphism of the plane onto itself which preserves the orientation and has no fixed points). In this paper we restrict our attention to some particular free mappings. The idea presented here can be applied to continuous iteration groups of any free mapping. The general case will be discussed in a forthcoming publication.

1. For every free mapping f it can be introduced the *codivergency relation* defined by

$$p \sim q \quad \text{iff} \quad \begin{array}{l} p \text{ and } q \text{ are endpoints of some arc } L \\ \text{for which } f^n[L] \to \infty \text{ as } n \to \pm\infty \end{array}$$

(see [1]). This relation is an equivalence relation. The equivalence classes of the relation are called *fundamental domains* of f.

In [9] it has been proved that every continuous iteration group of a *Sperner homeomorphism* f defined on a simply connected region $D \subset \mathbb{R}^2$ (i.e., a homeomorphism mapping D onto itself such that for every Jordan curve $J \subset D$ the set $J \cup \operatorname{ins} J$ meets

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at most finite number of its images $f^n[J \cup \operatorname{ins} J]$, $n \in \mathbb{Z}$) is given by the formula

$$f^t(x) = \varphi^{-1}(\varphi(x) + (t, 0)), \qquad x \in D, t \in \mathbb{R},$$

where φ is a homeomorphic solution of the Abel equation

$$\varphi(f(x)) = \varphi(x) + (1, 0), \qquad x \in D.$$

The construction of all homeomorphic solutions of this equation in the case when f is a Sperner homeomorphism of the plane has been given in [8].

In the present paper we consider continuous iteration groups $\{f^t : t \in \mathbb{R}\}$ of a free mapping f which is not a Sperner homeomorphism (i.e., a free mapping which has more than one fundamental domain). S. Andrea [1] has proved that a free mapping cannot have exactly two fundamental domains, so the smallest number of them for free mappings considered here is 3.

Every continuous iteration group $\{f^t : t \in \mathbb{R}\}$ of a free mapping f has its Kaplan diagram made up by polygonoids inscribed in the unit disc and chords of the disc which are parallel to the sides of the polygonoids (see Kaplan [5], [6] and Beck [2, Ch. 11]). Moreover, each of these polygonoids has exactly one open side (see [2], p. 357), and if two of the polygonoids have a common side, then the side must be closed (see [2], p. 360).

On account of the Whitney–Bebutov theorem (see [3], p. 50) every continuous iteration group $\{f^t : t \in \mathbb{R}\}$ of a free mapping is locally parallelizable. The local parallelizability of such a group has also been obtained by M. Hmissi [4] as a consequence of his result which says that the plane can be covered by a family of maximal invariant domains such that $\{f^t : t \in \mathbb{R}\}$ restricted to each of them is parallelizable.

2. First we consider the case when the Kaplan diagram of $\{f^t : t \in \mathbb{R}\}$ includes only one polygonoid and the polygonoid is a triangle. Thus each of the sides of the triangle cut off a segment which is exhausted by chords parallel to this side. Each of the segments corresponds to a fundamental domain of f (see [6]), whence f has three fundamental domains. Moreover, only one of them is open, since the triangle from the Kaplan diagram must have exactly one open side.

Theorem 1. *Let $\{f^t : t \in \mathbb{R}\}$ be a continuous iteration group of a free mapping f which has the Kaplan diagram including only one triangle and no other polygonoids. Denote by G_0 the fundamental domain of f which is open, and by G_1, G_2 the others. Then there exist homeomorphisms φ_1, φ_2 mapping $G_0 \cup G_1$, $G_0 \cup G_2$, respectively, onto $\mathbb{R}^2$ such that*

$$(1) \qquad f^t(x) = \begin{cases} \varphi_1^{-1}(\varphi_1(x) + (t, 0)), & x \in G_0 \cup G_1, \\ \varphi_2^{-1}(\varphi_2(x) + (t, 0)), & x \in G_0 \cup G_2, \end{cases}$$

and the function $\psi := \varphi_2 \circ (\varphi_1|_{G_0})^{-1} : \varphi_1[G_0] \longrightarrow \varphi_2[G_0]$ *is given by*

$$\psi(y_1, y_2) = (y_1 + \alpha(y_2), \beta(y_2)), \qquad (y_1, y_2) \in \varphi_1[G_0], \tag{2}$$

where α *is a continuous function of* $(0, +\infty)$ *into* $\mathbb{R}$ *and* β *is a homeomorphism of* $(0, +\infty)$ *onto itself.*

Proof. We first prove that $G_0 \cup G_1$ is a simply connected set. Denote by K_1 and K_2 the orbits of $\{f^t : t \in \mathbb{R}\}$ corresponding to the sides of the triangle of the Kaplan diagram which cut off the segments corresponding to G_1 and G_2, resp. Fix any $p \in K_1$ and $q \in G_0$. Then $p \in K_1 \cup K_2$. Since the set $K_1 \cup K_2$ is closed and locally connected, there exists an arc L with endpoints p and q such that $L \setminus \{p\} \subset \mathbb{R}^2 \setminus (K_1 \cup K_2)$ (see [7], p. 365). Hence $L \setminus \{p\} \subset G_0$, since $q \in G_0$. By the fact that every equivalence class of the codivergency relation is arcwise connected it follows that so is $G_0 \cup G_1$. Consequently $G_0 \cup G_1$ is a connected set.

Let J be a Jordan curve such that $J \subset G_0 \cup G_1$. Fix any $p \in \operatorname{ins} J$. Then there exists a point q which belongs to the intersection of J and the orbit of p, since every orbit of a continuous iteration group of a free mapping is unbounded. Therefore $q = f^t(p)$ for some $t \in \mathbb{R}$, whence $p = f^{-t}(q)$. Hence $p \in G_0 \cup G_1$, since each of the fundamental domains of f is invariant under f^t for every $t \in \mathbb{R}$ (see [1]). Thus $\operatorname{ins} J \subset G_0 \cup G_1$, and consequently $G_0 \cup G_1$ is simply connected.

Let us recall that a free mapping cannot have exactly two fundamental domains. Since $G_0 \cup G_1$ is a simply connected region which is a sum of two fundamental domains, the function $f|_{G_0 \cup G_1}$ is a Sperner homeomorphism. Hence there exists a homeomorphism φ_1 mapping $G_0 \cup G_1$ onto $\mathbb{R}^2$ such that

$$f^t(x) = \varphi_1^{-1}(\varphi_1(x) + (t, 0)), \qquad x \in G_0 \cup G_1, t \in \mathbb{R}$$

(see [9]).

In the same way we can prove that $G_0 \cup G_2$ is simply connected and $f|_{G_0 \cup G_2}$ is a Sperner homeomorphism. Therefore there exists a homeomorphism φ_2 mapping $G_0 \cup G_2$ onto $\mathbb{R}^2$ such that

$$f^t(x) = \varphi_2^{-1}(\varphi_2(x) + (t, 0)), \qquad x \in G_0 \cup G_2, t \in \mathbb{R}.$$

Thus f^t is given by (1).

Let $x_0 \in G_0$. Then by (1)

$$(\varphi_1^{-1} \circ T^t \circ \varphi_1)(x_0) = (\varphi_2^{-1} \circ T^t \circ \varphi_2)(x_0), \quad t \in \mathbb{R},$$

where $T^t(x) = x + (t, 0)$ for $x \in \mathbb{R}^2$, $t \in \mathbb{R}$. Hence

$$(T^t \circ \varphi_1)(x_0) = (\varphi_1 \circ \varphi_2^{-1} \circ T^t \circ \varphi_2)(x_0), \quad t \in \mathbb{R},$$

whence

$$T^t(y_0) = (\varphi_1 \circ \varphi_2^{-1} \circ T^t \circ \varphi_2 \circ \varphi_1^{-1})(y_0), \quad t \in \mathbb{R},$$

where $y_0 = \varphi_1(x_0)$. Thus

$$T^t(y_0) = (\psi^{-1} \circ T^t \circ \psi)(y_0), \quad t \in \mathbb{R},$$

where $\psi = \varphi_2 \circ (\varphi_1|_{G_0})^{-1}$.

Let $\psi(y_1, y_2) = (\psi_1(y_1, y_2), \psi_2(y_1, y_2))$ for $(y_1, y_2) \in \varphi_1[G_0]$. Then

$$\psi_1(y_1 + t, y_2) = \psi_1(y_1, y_2) + t$$

and

$$\psi_2(y_1 + t, y_2) = \psi_2(y_1, y_2)$$

for $(y_1, y_2) \in \varphi_1[G_0]$ and $t \in \mathbb{R}$. Hence

$$\psi_1(y_1, y_2) = \psi_1(0, y_2) + y_1$$

and

$$\psi_2(y_1, y_2) = \psi_2(0, y_2)$$

for $(y_1, y_2) \in \varphi_1[G_0]$. Thus

$$\psi(y_1, y_2) = (y_1 + \alpha(y_2), \beta(y_2)),$$

where $\alpha := \psi_1(0, \cdot)$ is a continuous function of $(0, +\infty)$ into $\mathbb{R}$ and $\beta := \psi_2(0, \cdot)$ is a homeomorphism of $(0, +\infty)$ onto itself. This completes the proof.

3. In this section we deal with continuous iteration groups $\{f^t : t \in \mathbb{R}\}$ such that their Kaplan diagram has exactly two polygonoids and the polygonoids are triangles. Let us start with the case when the triangles have no common side. We assume that their open sides are border chords of the strip between the triangles.

Theorem 2. *Let $\{f^t : t \in \mathbb{R}\}$ be a continuous iteration group of a free mapping f such that its Kaplan diagram has exactly two triangles which have no common side and no other polygonoids. Denote by G_{11}, G_{12} and G_{21}, G_{22} the fundamental domains of f which corresponds to the segments cut off by the sides of the first triangle and by the sides of the second one, respectively. Assume that the fundamental domain corresponding to the part of the unit disc which lies between the triangles is open and denote it by G_0. Then there exist homeomorphisms $\varphi_{11}, \varphi_{12}, \varphi_{21}, \varphi_{22}$ mapping $G_0 \cup G_{11}, G_0 \cup G_{12}, G_0 \cup G_{21}, G_0 \cup G_{22}$, respectively, onto $\mathbb{R}^2$ such that*

$$f^t(x) = \begin{cases} \varphi_{11}^{-1}(\varphi_{11}(x) + (t, 0)), & x \in G_0 \cup G_{11}, \\ \varphi_{12}^{-1}(\varphi_{12}(x) + (t, 0)), & x \in G_0 \cup G_{12}, \\ \varphi_{21}^{-1}(\varphi_{21}(x) + (t, 0)), & x \in G_0 \cup G_{21}, \\ \varphi_{22}^{-1}(\varphi_{22}(x) + (t, 0)), & x \in G_0 \cup G_{22}, \end{cases} \tag{3}$$

and the functions $\psi_0 := \varphi_{12}\circ(\varphi_{11}|_{G_0})^{-1}$, $\psi_1 := \varphi_{21}\circ(\varphi_{11}|_{G_0})^{-1}$, $\psi_2 := \varphi_{22}\circ(\varphi_{11}|_{G_0})^{-1}$ *are given by*

$$\psi_i(y_1, y_2) = (y_1 + \alpha_i(y_2), \beta_i(y_2)), \qquad (y_1, y_2) \in \varphi_{11}[G_0],\ i = 0, 1, 2, \tag{4}$$

where α_i *are continuous functions of* $(0, +\infty)$ *into* $\mathbb{R}$ *and* β_i *are homeomorphisms of* $(0, +\infty)$ *onto itself.*

Proof. In the same way as in the proof of Theorem 1 we can show that $G_0 \cup G_{11}$, $G_0 \cup G_{12}$, $G_0 \cup G_{21}$, $G_0 \cup G_{22}$ are simply connected sets and f restricted to each of the sets is a Sperner homeomorphism. Using the result of [9] to each of the functions $f|_{G_0\cup G_{11}}$, $f|_{G_0\cup G_{12}}$, $f|_{G_0\cup G_{21}}$, $f|_{G_0\cup G_{22}}$ we get the existence of homeomorphisms φ_{11}, φ_{12}, φ_{21}, φ_{22} satisfying (3). Relation (4) can be obtained by the same method as (2) in the proof of Theorem 1. This completes the proof.

Now let us consider the case when the two triangles occurring in the Kaplan diagram of $\{f^t : t \in \mathbb{R}\}$ have a common side. The same reasoning as before gives the following theorem.

Theorem 3. *Let* $\{f^t : t \in \mathbb{R}\}$ *be a continuous iteration group of a free mapping* f *such that its Kaplan diagram has exactly two triangles having a common side and no other polygonoids. Denote by* G_{10}, G_{11} *and* G_{20}, G_{21} *the fundamental domains of* f *which corresponds to the segments cut off by the sides of the first triangle and by the sides of the second one, respectively, in such a way that* G_{10} *and* G_{20} *are open sets. Let* K *denotes the orbit of* $\{f^t : t \in \mathbb{R}\}$ *corresponding to the common side of the triangles. Then there exist homeomorphisms* φ_0, φ_1, φ_2 *mapping* $G_{10} \cup K \cup G_{20}$, $G_{10} \cup G_{11}$, $G_{20} \cup G_{21}$, *respectively, onto* $\mathbb{R}^2$ *such that*

$$f^t(x) = \begin{cases} \varphi_0^{-1}(\varphi_0(x) + (t, 0)), & x \in G_{10} \cup K \cup G_{20}, \\ \varphi_1^{-1}(\varphi_1(x) + (t, 0)), & x \in G_{10} \cup G_{11}, \\ \varphi_2^{-1}(\varphi_2(x) + (t, 0)), & x \in G_{20} \cup G_{21}, \end{cases}$$

and the functions $\psi_1 := \varphi_1 \circ (\varphi_0|_{G_{10}})^{-1}$, $\psi_2 := \varphi_2 \circ (\varphi_0|_{G_{20}})^{-1}$ *are given by*

$$\psi_i(y_1, y_2) = (y_1 + \alpha_i(y_2), \beta_i(y_2)), \qquad (y_1, y_2) \in \varphi_0[G_{i0}],\ i = 1, 2,$$

where α_i *are continuous functions of* $(0, +\infty)$ *into* $\mathbb{R}$ *and* β_i *are homeomorphisms of* $(0, +\infty)$ *onto itself.*

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