SYMBOL SEQUENCES AND INVARIANT MEASURES FOR PIECEWISE LINEAR TRANSFORMATIONS OF THE INTERVAL

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ABSTRACT

For a piecewise linear transformation S of the unit interval [0, 1] into itself the absolutely continuous S-invariant measures are constructed explicitly by means of the so called minus and plus symbol sequences of the separation points $s_1, \ldots, s_{N-1}$.

1. Introduction

Definition 1.1. A transformation S from the closed unit interval $I = [0, 1]$ into itself is called *piecewise linear on I with the separation points* $s_1, s_2, \ldots, s_{N-1}$, if there is a natural number $N \geq 2$ and if there are real numbers $s_1, s_2, \ldots, s_{N-1}$ with $0 = s_0 < s_1 < \cdots < s_{N-1} < s_N = 1$ and if, for $i = 1, \ldots, N$, there are real numbers k_i and d_i with $|k_i| > 1$ such that

$$s_{i-1} < x < s_i \quad \text{implies} \quad S(x) = k_i x + d_i \, .$$

The interval (s_{i-1}, s_i) is called a *full interval* if $k_i s_{i-1} + d_i, k_i s_i + d_i \in \{0, 1\}$.

In the sequel let S denote a piecewise linear transformation on I with the separation points $s_1, s_2, \ldots, s_{N-1}$. N need not be minimal and S need not be continuous at the separation points $s_1, \ldots, s_{N-1}$. The values $S(s_j)$ can be arbitrary chosen in I for $j = 0, \ldots, N$. Let $S^0(x) = x$ and $S^1(x) = S(x)$ for all $x \in I$. For $k \geq 2$ let S^k denote the k-fold composition of S with itself.

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For every $x \in I$ we introduce infinite sequences $\sigma_-^x = (\sigma_-^x(0), \sigma_-^x(1), \dots)$ and $\sigma_+^x = (\sigma_+^x(0), \sigma_+^x(1), \dots)$ of integers $\sigma_-^x(k), \sigma_+^x(k) \in \{1, \dots, N\}$ as follows:

Definition 1.2. For $x \in (0, 1]$ the kth term of the sequence $\sigma_-^x = (\sigma_-^x(0), \sigma_-^x(1), \dots)$ is defined by

$$\sigma_-^x(k) = j \quad \text{iff} \quad s_{j-1} < S^k(y) < s_j \text{ for } y \in I,\ y < x \text{ and } x - y \text{ arbitrary small.} \tag{1.2}$$

For $x \in [0, 1)$ the kth term of the sequence $\sigma_+^x = (\sigma_+^x(0), \sigma_+^x(1), \dots)$ is given by

$$\sigma_+^x(k) = j \quad \text{iff} \quad s_{j-1} < S^k(y) < s_j \text{ for } y \in I, x < y \text{ and } y - x \text{ arbitrary small.} \tag{1.3}$$

The sequences σ_-^x and σ_+^x are called minus resp. plus symbol sequences of the point x and do not depend on the values $S(s_j)$ for $j = 0, \dots, N$.

Without loss of generality we assume $\lim_{x\to 0} S(x), \lim_{x\to 1} S(x) \in \{0, 1\}$. This can be achieved by enlarging and transforming the interval I by a suitable affine transformation. Thereby the symbol sequences $\sigma_-^{s_j}$ and $\sigma_+^{s_j}$ $(1 \le j \le N - 1)$ remain unchanged.

A. Lasota and J. Yorke [3] gave a positive answer to the conjecture of S. Ulam [7, p. 74] that piecewise linear transformations S possess absolutely continuous S-invariant measures.

In this note we will construct the absolutely continuous S-invariant measures by means of the symbol sequences $\sigma_-^{s_j}$ and $\sigma_+^{s_j}$ for $j = 1, \dots, N - 1$. The construction is symmetric at the separation points and is independent of the values $S(s_j)$ for $j = 0, \dots, N$.

For previous results on absolutely continuous S-invariant measures see, e.g., [6], [1], [2].

2. S-Expansions

By means of the symbol sequences σ_-^x and σ_+^x we define orbit sequences $o_-^x = (o_-^x(0), o_-^x(1), \dots)$ and $o_+^x = (o_+^x(0), o_+^x(1), \dots)$ of real numbers $o_-^x(k), o_+^x(k) \in I$ in the following way:

Definition 2.1. For $x \in (0, 1]$ we define

$$o_-^x(0) = x \quad \text{and} \quad o_-^x(i + 1) = k_{\sigma_-^x(i)} o_-^x(i) + d_{\sigma_-^x(i)} \quad \text{for } i \ge 0. \tag{2.1}$$

For $x \in [0, 1)$ we set

$$o_+^x(0) = x \quad \text{and} \quad o_+^x(i + 1) = k_{\sigma_+^x(i)} o_+^x(i) + d_{\sigma_+^x(i)} \quad \text{for } i \ge 0. \tag{2.2}$$

Theorem 2.2. *The orbit sequences o_-^x and o_+^x are determined by the symbol sequences σ_-^x and σ_+^x by the following relations, which hold for every $i \geq 0$:*

$$o_-^x(i) = -\sum_{n=i}^{\infty} \frac{d_{\sigma_-^x(n)}}{\prod_{m=i}^{n} k_{\sigma_-^x(m)}} \quad \text{for all } x \in (0,1] \tag{2.3}$$

and

$$o_+^x(i) = -\sum_{n=i}^{\infty} \frac{d_{\sigma_+^x(n)}}{\prod_{m=i}^{n} k_{\sigma_+^x(m)}} \quad \text{for all } x \in [0,1). \tag{2.4}$$

Proof. Assume $i \geq 0$. For $x \in (0,1]$ we obtain by application of (2.1)

$$\begin{aligned} o_-^x(i) &= \frac{1}{k_{\sigma_-^x(i)}} o_-^x(i+1) - \frac{d_{\sigma_-^x(i)}}{k_{\sigma_-^x(i)}} \\ &= \frac{1}{k_{\sigma_-^x(i)} k_{\sigma_-^x(i+1)}} o_-^x(i+2) - \frac{d_{\sigma_-^x(i+1)}}{k_{\sigma_-^x(i)} k_{\sigma_-^x(i+1)}} - \frac{d_{\sigma_-^x(i)}}{k_{\sigma_-^x(i)}} \\ &= \frac{1}{k_{\sigma_-^x(i)} \dots k_{\sigma_-^x(n)}} o_-^x(n+1) - \frac{d_{\sigma_-^x(n)}}{k_{\sigma_-^x(i)} \dots k_{\sigma_-^x(n)}} - \dots - \frac{d_{\sigma_-^x(i)}}{k_{\sigma_-^x(i)}} \end{aligned}$$

for all $n \geq i$. The relation (2.3) now follows by $n \to \infty$. For $x \in [0,1)$ we start with

$$o_+^x(i) = \frac{1}{k_{\sigma_+^x(i)}} o_+^x(i+1) - \frac{d_{\sigma_+^x(i)}}{k_{\sigma_+^x(i)}} .$$

The relation (2.4) is shown by an analogous argument.

3. Fundamental matrix

Lemma 3.1. *For $i = 1, \dots, N$ let k_i and d_i denote real numbers such that $k_i \neq 1$. The following assertions are equivalent*:

(1) *The intersections of the diagonal $y = x$ with the lines $y = k_i x + d_i$ coincide for $i = 1, \dots, N$, i.e., there is a real number r such that $d_i = r(1 - k_i)$ for $i = 1, \dots, N$.*

(2) *The vectors*
$$\begin{pmatrix} d_1 \\ \vdots \\ d_N \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} k_1 - 1 \\ \vdots \\ k_N - 1 \end{pmatrix}$$
are linearly dependent.

(3) *$d_i(k_j - 1) = d_j(k_i - 1)$ for $1 \leq i, j \leq N$.*

Let 1_A denote the indicator function of the set A. For $i = 1, \ldots, N$, $y \in (0, 1)$ and $\alpha \in \{-, +\}$ we define real numbers $K_\alpha^y(i)$ by

$$K_\alpha^y(i) = \sum_{n \geq 0} \kappa_\alpha^y(o, n) 1_{\{i\}}(\sigma_\alpha^y(n)), \tag{3.1}$$

where $\kappa_\alpha^y(m, n)$ is given by

$$\kappa_\alpha^y(m, n) = \begin{cases} 1 & \text{for } n < m, \\ \left(\prod_{j=m}^{n} k_{\sigma_\alpha^y(j)} \right)^{-1} & \text{for } n \geq m. \end{cases}$$

It follows from (2.1)–(2.4) that

$$\sum_{i=1}^{N} d_i K_\alpha^y(i) = \sum_{n \geq 0} \kappa_\alpha^y(o, n) d_{\sigma_\alpha^y(n)} = -o_\alpha^y(0) = -y. \tag{3.2}$$

Furthermore we obtain the relation

$$\sum_{i=1}^{N} (k_i - 1) K_\alpha^y(i) = 1. \tag{3.3}$$

Definition 3.2. The $N \times (N-1)$-matrix $M = (M_{i,j})$, defined by

$$M_{i,j} = K_+^{s_j}(i) - K_-^{s_j}(i) \quad \text{for } 1 \leq i \leq N \text{ and } 1 \leq j \leq N-1, \tag{3.4}$$

is called fundamental matrix of the piecewise linear transformation S with the separation points $s_1, \ldots, s_{N-1}$.

The fundamental matrix M is completely determined by the symbol sequences $\sigma_-^{s_j}$ and $\sigma_+^{s_j}$ for $j = 1, \ldots, N-1$. Closely related to the fundamental matrix is the kneading matrix, which was introduced for piecewise monotone and continuous transformations by J. Milnor and W. Thurston in [5].

By the relations (3.2) and (3.3) we obtain for $j = 1, \ldots, N-1$

$$\sum_{i=1}^{N} d_i M_{i,j} = 0 \tag{3.5}$$

and

$$\sum_{i=1}^{N} (k_i - 1) M_{i,j} = 0. \tag{3.6}$$

Note that $k_1 = k_2 = \cdots = k_N$ implies $\sum_{i=1}^{N} M_{i,j} = 0$ for $j = 1, \ldots, N-1$.

As a consequence of (3.5) and (3.6) we obtain

$$\sum_{i=1}^{N} (d_t(k_i - 1) - d_i(k_t - 1)) M_{i,j} = 0 \quad \text{for } t = 1, \ldots, N. \tag{3.7}$$

Relation (3.7) together with Lemma 3.1 imply that after dropping the tth row of the fundamental matrix M the remaining $N-1$ rows are linearly dependent since not all intersections of the diagonal $y = x$ with the lines $y = k_i x + d_i$ $(1 \le i \le N)$ coincide.

Thus, for a fixed $t = 1, \dots, N$, there is at least one real, nonzero vector

$$\gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{N-1} \end{pmatrix}$$

such that

$$\sum_{j=1}^{N-1} \gamma_j M_{i,j} = 0 \quad \text{for } \; i = 1, \dots, N, \; i \neq t$$

and, by (3.6),

$$(k_t - 1) \sum_{j=1}^{N-1} \gamma_j M_{t,j} = - \sum_{\substack{i=1 \\ i \neq t}}^{N} (k_i - 1) \sum_{j=1}^{N-1} \gamma_j M_{i,j} = 0 \, .$$

So the nonzero vector γ satisfies $M\gamma = 0$.

Theorem 3.3. *Let $M = (M_{i,j})$ denote the fundamental matrix of S, defined by*

$$M_{i,j} = \sum_{n \ge 0} \kappa_+^{s_j}(o, n) 1_{\{i\}}(\sigma_+^{s_j}(n)) - \sum_{n \ge 0} \kappa_-^{s_j}(o, n) 1_{\{i\}}(\sigma_-^{s_j}(n)) \tag{3.8}$$

for $1 \le i \le N$ and $1 \le j \le N-1$. Then the system

$$\sum_{j=1}^{N-1} \gamma_j M_{i,j} = 0 \quad (1 \le i \le N)$$

has a real, nontrivial solution

$$\gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{N-1} \end{pmatrix} .$$

In the case $N = 2$ we obtain from (3.2) and (3.3) for $y \in (0, 1)$ and $\alpha \in \{-, +\}$

$$K_\alpha^y(1) = \sum_{n \ge 0} \kappa_\alpha^y(o, n) 1_{\{1\}}(\sigma_\alpha^y(n)) = \frac{(k_2 - 1)y + d_2}{d_2(k_1 - 1) - d_1(k_2 - 1)},$$
$$K_\alpha^y(2) = \sum_{n \ge 0} \kappa_\alpha^y(o, n) 1_{\{2\}}(\sigma_\alpha^y(n)) = \frac{(k_1 - 1)y + d_1}{d_1(k_2 - 1) - d_2(k_1 - 1)}$$

and

$$M = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Hence $M\gamma = 0$ for any real number γ.

In the case $N = 3$, since the vectors

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} k_1 - 1 \\ k_2 - 1 \\ k_3 - 1 \end{pmatrix}$$

are linearly independent, we obtain from (3.5) and (3.6) for $j = 1, 2$

$$\begin{pmatrix} M_{1,j} \\ M_{2,j} \\ M_{3,j} \end{pmatrix} = m_j \begin{pmatrix} d_2(k_3 - 1) - d_3(k_2 - 1) \\ d_3(k_1 - 1) - d_1(k_3 - 1) \\ d_1(k_2 - 1) - d_2(k_1 - 1) \end{pmatrix}$$

with suitable chosen real numbers m_j. Hence

$$M \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = 0 \quad \text{iff} \quad m_1\gamma_1 + m_2\gamma_2 = 0.$$

Lemma 3.4. *Let* $G = (G_{i,j})$ *denote a* $N \times (N-1)$*-matrix, defined by*

$$G_{i,j} = \sum_{n\geq 0} \kappa_+^{s_j}(o, n-1) 1_{\{i\}}(\sigma_+^{s_j}(n)) - \sum_{n\geq 0} \kappa_-^{s_j}(o, n-1) 1_{\{i\}}(\sigma_-^{s_j}(n)) \tag{3.9}$$

for $1 \leq i \leq N$ *and* $1 \leq j \leq N-1$, *and let*

$$\gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{N-1} \end{pmatrix}$$

denote a real $(N-1)$*-vector.*

The following assertions are equivalent:

(1) $M\gamma = 0$.

(2) $G\gamma = 0$.

(3) $$\sum_{j=1}^{N-1} \gamma_j \left[\sum_{n\geq 1} \kappa_+^{s_j}(0, n-1) 1_{\{1,\ldots,t\}}\big(\sigma_+^{s_j}(n)\big) - \sum_{n\geq 1} \kappa_-^{s_j}(0, n-1) 1_{\{1,\ldots,t\}}\big(\sigma_-^{s_j}(n)\big) \right] = \gamma_t \qquad \text{for } t = 1, \ldots, N-1.$$

(4) $$\sum_{j=1}^{N-1} \gamma_j \left[\sum_{n\geq 1} \kappa_+^{s_j}(0, n-1) 1_{\{t+1,\ldots,N\}}\big(\sigma_+^{s_j}(n)\big) - \sum_{n\geq 1} \kappa_-^{s_j}(0, n-1) 1_{\{t+1,\ldots,N\}}\big(\sigma_-^{s_j}(n)\big) \right] = -\gamma_t \qquad \text{for } t = 1, \ldots, N-1.$$

Proof. The equivalence of (1) and (2) follows from the relation $G_{i,j} = k_i M_{i,j}$ for $i = 1, \ldots, N$ and $j = 1, \ldots, N-1$.

Let $A \subset \{1, \ldots, N\}$. Then, for $\alpha \in \{-, +\}$,

$$\sum_{i \in A} \sum_{n \geq 0} \kappa_\alpha^{s_j}(0, n-1) 1_{\{i\}}(\sigma_\alpha^{s_j}(n)) = \sum_{n \geq 0} \kappa_\alpha^{s_j}(0, n-1) 1_A(\sigma_\alpha^{s_j}(n)) \tag{3.10}$$

and, by (3.10),

$$\begin{aligned} &\sum_{i \in A} \sum_{j=1}^{N-1} \gamma_j \Big[\sum_{n \geq 0} \kappa_+^{s_j}(o, n-1) 1_{\{i\}}\big(\sigma_+^{s_j}(n)\big) - \sum_{n \geq 0} \kappa_-^{s_j}(o, n-1) 1_{\{i\}}\big(\sigma_-^{s_j}(n)\big) \Big] \\ &\quad = \sum_{j=1}^{N-1} \gamma_j \Big[1_A\big(\sigma_+^{s_j}(0)\big) - 1_A\big(\sigma_-^{s_j}(0)\big) \Big] \\ &\qquad + \sum_{j=1}^{N-1} \gamma_j \Big[\sum_{n \geq 1} \kappa_+^{s_j}(0, n-1) 1_A\big(\sigma_+^{s_j}(n)\big) - \sum_{n \geq 1} \kappa_-^{s_j}(0, n-1) 1_A\big(\sigma_-^{s_j}(n)\big) \Big]. \end{aligned} \tag{3.11}$$

From assertion (2) we deduce assertion (3) and (4) and the equivalence of (3) and (4) by setting $A = \{1, \ldots, t\}$, $A = \{t+1, \ldots, N\}$ for $t = 1, \ldots, N-1$ and $A = \{1, \ldots, N\}$ in the relation (3.11). In the latter case we obtain

$$\sum_{j=1}^{N-1} \gamma_j \Big[\sum_{n \geq 0} \kappa_+^{s_j}(0, n) - \sum_{n \geq 0} \kappa_-^{s_j}(0, n) \Big] = 0. \tag{3.12}$$

Assertion (2) follows from (3) resp. (4) by setting successively $A = \{1\}$, $A = \{1, 2\}$, $\ldots$, $A = \{1, \ldots, N-1\}$ resp. $A = \{N\}$, $A = \{N-1, N\}$, $\ldots$, $A = \{2, \ldots, N-1, N\}$ in the relation (3.11).

Note that for $j = 1, \ldots, N-1$

$$\sum_{i=1}^{N} \frac{d_i}{k_i} G_{i,j} = 0 \qquad \text{and} \qquad \sum_{i=1}^{N} \frac{(k_i - 1)}{k_i} G_{i,j} = 0.$$

4. Absolutely continuous S-invariant measures

Definition 4.1. Let $x \in I$ and set

$$H^S(x) = \big\{ m \;\big|\; 1 \leq m \leq N, \exists\, x_m \in I, s_{m-1} < x_m < s_m \text{ such that } S(x_m) = x \big\}.$$

By P^S we denote the Ruelle–Perron–Frobenius operator associated with S, defined on the space L^1 of all real-valued, measurable and Lebesgue-integrable functions on I by

$$P^S h(x) = \frac{d}{dx} \int_{S^{-1}([0,x])} h(y)\, dy \qquad \text{for } h \in L^1 \text{ and } x \in I.$$

A L^1-function h on I is called S-invariant if h is a fixpoint of the Ruelle–Perron–Frobenius operator P^S or, equivalently, if the Ruelle–Perron–Frobenius equation

$$\sum_{m \in H^S(x)} \frac{1}{|k_m|} h(x_m) = h(x) \tag{4.1}$$

holds for almost all $x \in I$ with respect to the Lebesgue measure λ.

A function $h \in L^1$ is S-invariant if and only if the absolutely continuous measure $d\nu = h\, d\lambda$ is S-invariant, i.e., $\nu(B) = \nu(S^{-1}(B))$ for every measurable set $B \subset I$.

Theorem 4.2. *Let S denote a piecewise linear transformation on the unit interval I with the separation points $0 < s_1 < \cdots < s_{N-1} < 1$. Let M denote the fundamental matrix of S and let γ be a nontrivial solution of $M\gamma = 0$. For $y \in (0,1)$ and $\alpha \in \{-,+\}$ let L_α^y denote a real-valued function on I given by*

$$L_\alpha^y(x) = \sum_{n \geq 0} \kappa_\alpha^y(0,n) 1_{[0, o_\alpha^y(n+1))}(x) \qquad \textit{for } x \in I. \tag{4.2}$$

Then the real-valued, measurable and Lebesgue-integrable function l_γ, defined by

$$l_\gamma(x) = \sum_{j=1}^{N-1} \gamma_j \big(L_+^{s_j}(x) - L_-^{s_j}(x)\big) \qquad \textit{for } x \in I, \tag{4.3}$$

is S-invariant.

Proof. First we need some preliminaries. Assume $1 \leq j \leq N-1$.

For $x \in I$ let

$$e_-^{s_j}(x) = \begin{cases} -\dfrac{1}{k_j} 1_{(o_-^{s_j}(1), 1]}(x) & \text{for } k_j > 0, \\ \dfrac{1}{k_j} 1_{[0, o_-^{s_j}(1))}(x) & \text{for } k_j < 0 \end{cases}$$

and

$$e_+^{s_j}(x) = \begin{cases} \dfrac{1}{k_{j+1}} 1_{[0, o_+^{s_j}(1))}(x) & \text{for } k_{j+1} > 0, \\ -\dfrac{1}{k_{j+1}} 1_{(o_+^{s_j}(1), 1]}(x) & \text{for } k_{j+1} < 0. \end{cases}$$

Then

$$\sum_{i=1}^{N} \frac{1}{|k_i|} - \sum_{j=1}^{N-1} (e_+^{s_j}(x) - e_-^{s_j}(x)) = \sum_{m \in H^S(x)} \frac{1}{|k_m|} \quad \text{for } \lambda\text{-almost all } x \in I.$$

Let

$$\eta_-^{s_j} = \begin{cases} \dfrac{1 - o_-^{s_j}(1)}{k_j} & \text{for } k_j > 0, \\ -\dfrac{o_-^{s_j}(1)}{k_j} & \text{for } k_j < 0 \end{cases}$$

and

$$\eta_+^{s_j} = \begin{cases} -\dfrac{o_+^{s_j}(1)}{k_{j+1}} & \text{for } k_{j+1} > 0, \\ \dfrac{1 - o_+^{s_j}(1)}{k_{j+1}} & \text{for } k_{j+1} < 0. \end{cases}$$

Then

$$\sum_{i=1}^{N} \frac{1}{|k_i|} + \sum_{j=1}^{N-1} (\eta_+^{s_j} - \eta_-^{s_j}) = 1.$$

For $j = 1, \dots, N-1$ and $\alpha \in \{-, +\}$ we obtain

$$e_\alpha^{s_j}(x) + \eta_\alpha^{s_j} = \frac{1_{[0, o_\alpha^{s_j}(1))}(x) - o_\alpha^{s_j}(1)}{k_{\sigma_\alpha^{s_j}(0)}} \qquad \text{for } \lambda\text{-almost all } x \in I. \tag{4.4}$$

Let

$$c_i = \begin{cases} \displaystyle\sum_{j=1}^{i-1} \frac{1}{|k_j|} & \text{for } k_i > 0 \\ \displaystyle\sum_{j=1}^{i} \frac{1}{|k_j|} & \text{for } k_i < 0. \end{cases}$$

It is easy to see that

$$\sum_{m \in H^S(x)} \frac{1}{|k_m|} 1_{[0, o_\alpha^y(n))}(x_m) \tag{4.5}$$

$$= c_{\sigma_\alpha^y(n)} + \frac{1}{k_{\sigma_\alpha^y(n)}} 1_{[0, o_\alpha^y(n+1))}(x) - \sum_{t=1}^{N-1} \big(e_+^{s_t}(x) - e_-^{s_t}(x)\big) 1_{\{t+1, \dots, N\}}\big(\sigma_\alpha^y(n)\big)$$

for $y \in (0,1)$, $\alpha \in \{-, +\}$, $n \geq 0$ and for λ-almost all $x \in I$.

The assertion

$$o_\alpha^y(i) = -\sum_{n \geq i} \kappa_\alpha^y(i, n) d_{\sigma_\alpha^y(n)}$$

of Theorem 2.2 together with the evident formula

$$-\frac{d_{\sigma_\alpha^y(n)}}{k_{\sigma_\alpha^y(n)}} = c_{\sigma_\alpha^y(n)} + \sum_{t=1}^{N-1} (\eta_+^{s_t} - \eta_-^{s_t}) 1_{\{t+1, \dots, N\}}(\sigma_\alpha^y(n))$$

yields

$$\sum_{n\geq i}\kappa_\alpha^y(i,n-1)c_{\sigma_\alpha^y(n)} \tag{4.6}$$
$$= o_\alpha^y(i) - \sum_{n\geq i}\kappa_\alpha^y(i,n-1)\sum_{t=1}^{N-1}(\eta_+^{s_t}-\eta_-^{s_t})1_{\{t+1,\dots,N\}}\big(\sigma_\alpha^y(n)\big)$$

for $y \in (0,1)$, $\alpha \in \{-,+\}$ and $i \geq 0$.

By (4.5), (4.6) and (4.4) we deduce

$$\sum_{m\in H^S(x)}\frac{1}{|k_m|}L_\alpha^{s_j}(x_m) \tag{4.7}$$
$$= \sum_{n\geq 0}\kappa_\alpha^{s_j}(0,n)\sum_{m\in H^S(x)}\frac{1}{|k_m|}1_{[0,o_\alpha^{s_j}(n+1))}(x_m)$$
$$= \sum_{n\geq 0}\kappa_\alpha^{s_j}(0,n)\Bigg[c_{\sigma_\alpha^{s_j}(n+1)} + \frac{1}{k_{\sigma_\alpha^{s_j}(n+1)}}1_{[0,o_\alpha^{s_j}(n+2))}(x)$$
$$- \sum_{t=1}^{N-1}\big(e_+^{s_t}(x)-e_-^{s_t}(x)\big)1_{\{t+1,\dots,N\}}\big(\sigma_\alpha^{s_j}(n+1)\big)\Bigg]$$
$$= \kappa_\alpha^{s_j}(0,0)\sum_{n\geq 1}\kappa_\alpha^{s_j}(1,n-1)c_{\sigma_\alpha^{s_j}(n)} + \sum_{n\geq 1}\kappa_\alpha^{s_j}(0,n)1_{[0,o_\alpha^{s_j}(n+1))}(x)$$
$$- \sum_{n\geq 1}\kappa_\alpha^{s_j}(0,n-1)\sum_{t=1}^{N-1}\big(e_+^{s_t}(x)-e_-^{s_t}(x)\big)1_{\{t+1,\dots,N\}}\big(\sigma_\alpha^{s_j}(n)\big)$$
$$= \kappa_\alpha^{s_j}(0,0)o_\alpha^{s_j}(1) + \sum_{n\geq 1}\kappa_\alpha^{s_j}(0,n)1_{[0,o_\alpha^{s_j}(n+1))}(x)$$
$$- \sum_{n\geq 1}\kappa_\alpha^{s_j}(0,n-1)\sum_{t=1}^{N-1}\big(e_+^{s_t}(x)+\eta_+^{s_t}-e_-^{s_t}(x)-\eta_-^{s_t}\big)1_{\{t+1,\dots,N\}}\big(\sigma_\alpha^{s_j}(n)\big)$$
$$= L_\alpha^{s_j}(x) - \big(e_\alpha^{s_j}(x)+\eta_\alpha^{s_j}\big)$$
$$- \sum_{t=1}^{N-1}\big(e_+^{s_t}(x)+\eta_+^{s_t}-e_-^{s_t}(x)-\eta_-^{s_t}\big)\sum_{n\geq 1}\kappa_\alpha^{s_j}(0,n-1)1_{\{t+1,\dots,N\}}\big(\sigma_\alpha^{s_j}(n)\big)$$

for $\alpha \in \{-,+\}$ and for λ-almost all $x \in I$.

With the abbreviation

$$E^{s_t} = e_+^{s_t}(x) + \eta_+^{s_t} - e_-^{s_t}(x) - \eta_-^{s_t} \quad (1 \le t \le N-1)$$

we obtain relation (4.1) by application of (4.7) and by assertion (4) of Lemma 3.4:

$$\begin{aligned}
&\sum_{m \in H^S(x)} \frac{1}{|k_m|} l_\gamma(x_m) \qquad (4.8)\\
&= \sum_{j=1}^{N-1} \gamma_j \sum_{m \in H^S(x)} \frac{1}{|k_m|} \big(L_+^{s_j}(x_m) - L_-^{s_j}(x_m)\big)\\
&= \sum_{j=1}^{N-1} \gamma_j \big(L_+^{s_j}(x) - L_-^{s_j}(x)\big) - \sum_{j=1}^{N-1} \gamma_j E^{s_j}\\
&\quad - \sum_{t=1}^{N-1} E^{s_t} \sum_{j=1}^{N-1} \gamma_j \Big[\sum_{n \ge 1} \kappa_+^{s_j}(0, n-1) 1_{\{t+1,\dots,N\}}\big(\sigma_+^{s_j}(n)\big)\\
&\qquad - \sum_{n \ge 1} \kappa_-^{s_j}(0, n-1) 1_{\{t+1,\dots,N\}}\big(\sigma_-^{s_j}(n)\big)\Big]\\
&= l_\gamma(x)
\end{aligned}$$

for λ-almost all $x \in I$.

For $y \in (0,1)$ and $\alpha \in \{-,+\}$ let R_α^y denote a real-valued function on I given by

$$R_\alpha^y(x) = \sum_{n \ge 0} \kappa_\alpha^y(0,n) 1_{(o_\alpha^y(n+1),1]}(x) \quad \text{for } x \in I. \qquad (4.9)$$

Then

$$L_\alpha^y(x) + R_\alpha^y(x) = \sum_{n \ge 0} \kappa_\alpha^y(0,n) \quad \text{for λ-almost all } x \in I. \qquad (4.10)$$

By (3.12) and (4.10) we conclude

$$\sum_{j=1}^{N-1} \gamma_j \big(R_-^{s_j}(x) - R_+^{s_j}(x)\big) = \sum_{j=1}^{N-1} \gamma_j \big(L_+^{s_j}(x) - L_-^{s_j}(x)\big) \quad \text{for λ-almost all } x \in I$$

and

$$r_\gamma(x) = \sum_{j=1}^{N-1} \gamma_j \big(R_-^{s_j}(x) - R_+^{s_j}(x)\big) \quad \text{for } x \in I \qquad (4.11)$$

is another representation of the S-invariant function l_γ. For further representations of S-invariant functions see [2]. Note that l_γ has bounded variation since every function L_α^y can be written as the difference of two monotone functions.

5. Additional separation point

Let S again denote a piecewise linear transformation on I with the separation points $s_1, \dots, s_{N-1}$ such that $0 = s_0 < s_1 < \cdots < s_{N-1} < s_N = 1$ and let s_c be an arbitrary point in I such that $s_{l-1} < s_c < s_l$ with $1 \le l \le N$. Let S^* denote a piecewise linear transformation with the separation points $s_1, \dots, s_{l-1}, s_c, s_l, \dots, s_{N-1}$ such that

$$S^*(x) = S(x) \quad \text{for all} \quad x \in I \setminus \{0, s_1, \dots, s_{l-1}, s_c, s_l, \dots, s_{N-1}, 1\},$$

i.e., S^* differs from S by the additional separation point s_c.

We now investigate the relations between the fundamental matrices M and M^* of S resp. S^* and between the vectors γ and γ^* with $M\gamma = 0$ resp. $M^*\gamma^* = 0$. We will show that the two systems (I, S) and (I, S^*) have the same invariant functions.

We denote the interval (s_{l-1}, s_c) with the symbol a and the interval (s_c, s_l) with the symbol b . Thus the symbol sequences σ_-^x and σ_+^x with respect to the transformation S^* are sequences over the alphabet $\{1, \dots, l-1, a, b, l+1, \dots, N\}$ with $\sigma_-^{s_c}(0) = a$, $\sigma_+^{s_c}(0) = b$ and determine the fundamental matrix M^* of S^* according to (3.8). Note that $k_a = k_l = k_b$.

The fundamental matrix

$$M^* = (M^*_{i,j}), \quad \begin{array}{l} i = 1, \dots, l-1, a, b, l+1, \dots, N, \\ j = 1, \dots, l-1, c, l, \dots, N-1 \end{array}$$

of the transformation S^* satisfies

$$M^*_{i,j} = M_{i,j} \quad \begin{array}{l} \text{for } i = 1, \dots, l-1, l+1, \dots, N \text{ and} \\ \text{for } j = 1, \dots, l-1, l, \dots, N-1 \end{array} \tag{5.1}$$

and

$$M^*_{a,j} + M^*_{b,j} = M_{l,j} \quad \text{for } j = 1, \dots, l-1, l, \dots, N-1, \tag{5.2}$$

where the fundamental matrix $M = (M_{i,j})_{i=1,\dots,N, j=1,\dots,N-1}$ of the transformation S is defined by the symbol sequences σ_-^x and σ_+^x with respect to S according to (3.8).

Note that (5.2) implies

$$\sum_{j=1}^{N-1} \gamma_j M_{l,j} = 0 \quad \text{if and only if} \quad \sum_{j=1}^{N-1} \gamma_j M^*_{a,j} = -\sum_{j=1}^{N-1} \gamma_j M^*_{b,j} \, .$$

The following lemma follows by straightforward calculation:

Lemma 5.1. *Let S denote a piecewise linear transformation on I with the separation points $s_1, \dots, s_{N-1}$ such that $0 = s_0 < s_1 < \cdots < s_{N-1} < s_N = 1$ and let S^* denote a piecewise linear transformation which differs from S by the additional separation* *point* $s_c \in (s_{l-1}, s_l)$ *with* $1 \leq l \leq N$. *Then every* $(N-1)$*-vector*

$$\gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{N-1} \end{pmatrix}$$

with $M\gamma = 0$ *has a corresponding* N*-vector*

$$\gamma^* = \begin{pmatrix} \gamma_1^* \\ \vdots \\ \gamma_{l-1}^* \\ \gamma_c^* \\ \gamma_l^* \\ \vdots \\ \gamma_{N-1}^* \end{pmatrix}$$

with $M^*\gamma^* = 0$ *and vice versa .*

If $S^n(s_c) \notin \{ s_1, \ldots, s_{N-1}\}$ *for all* $n \geq 1$ *the correspondence between* γ *and* γ^* *is given by the relations*

$$\gamma_c^* = -\frac{1}{M^*_{a,c}} \sum_{j=1}^{N-1} \gamma_j M^*_{a,j} = -\frac{1}{M^*_{b,c}} \sum_{j=1}^{N-1} \gamma_j M^*_{b,j},$$

$$\gamma_j^* = \gamma_j \quad \text{for } j = 1, \ldots, N-1.$$

If $S^n(s_c) \in \{ s_1, \ldots, s_{N-1}\}$ *for some* $n \geq 1$ *and if* n_0 *is minimal with this property and* $S^{n_0}(s_c) = s_{j_0}$, *then the correspondence between* γ *and* γ^* *is given by the relations*

$$\gamma_c^* = k_l \sum_{j=1}^{N-1} \gamma_j M^*_{a,j} = -k_l \sum_{j=1}^{N-1} \gamma_j M^*_{b,j},$$

$$\gamma_{j_0}^* = \gamma_{j_0} - |\kappa_+^{s_c}(0, n_0 - 1)|\gamma_c^*,$$

$$\gamma_j^* = \gamma_j \quad \text{for } j = 1, \ldots, N-1,\ j \neq j_0.$$

For two corresponding vectors γ *and* γ^* *the* S*- resp.* S^**-invariant functions* l_γ *and* l_{γ^*} *coincide* λ*-almost everywhere on* I.

6. $\gamma \mapsto l_\gamma$ is an isomorphism

Assume a real $(N-1)$-vector γ with $M\gamma = 0$ and let l_γ denote the S-invariant function defined by (4.3). For $i = 1, \ldots, N-1$ we now determine the limits

$$l_\gamma(s_i-) = \lim_{\substack{x \to s_i \\ x < s_i}} l_\gamma(x)$$

and

$$l_\gamma(s_i+) = \lim_{\substack{x \to s_i \\ x > s_i}} l_\gamma(x)$$

and show that $l_\gamma = 0$ if $l_\gamma(s_i-) = 0$ for $i = 1, \ldots, N-1$ or if $l_\gamma(s_i+) = 0$ for $i = 1, \ldots, N-1$.

Lemma 6.1. *Let S denote a piecewise linear transformation on the closed unit interval with the separation points* $0 < s_1 < \ldots < s_{N-1} < 1$ *and let* l_γ *denote a S-invariant function defined by* (4.3). *Then, for* $i = 1, \ldots, N-1$,

$$l_\gamma(s_i-) = +\sum_{j=1}^{N-1} \gamma_j \Bigg[\sum_{\substack{n \geq 0 \\ \sigma_+^{s_j}(n)=i \\ o_+^{s_j}(n)=s_i}} \kappa_+^{s_j}(0, n-1) \; - \sum_{\substack{n \geq 0 \\ \sigma_-^{s_j}(n)=i \\ o_-^{s_j}(n)=s_i}} \kappa_-^{s_j}(0, n-1) \Bigg]$$

and

$$l_\gamma(s_i+) = -\sum_{j=1}^{N-1} \gamma_j \Bigg[\sum_{\substack{n \geq 0 \\ \sigma_+^{s_j}(n)=i+1 \\ o_+^{s_j}(n)=s_i}} \kappa_+^{s_j}(0, n-1) \; - \sum_{\substack{n \geq 0 \\ \sigma_-^{s_j}(n)=i+1 \\ o_-^{s_j}(n)=s_i}} \kappa_-^{s_j}(0, n-1) \Bigg].$$

Proof. By the relation (4) of Lemma 3.4 we obtain

$$l_\gamma(s_i-) = \sum_{j=1}^{N-1} \gamma_j \sum_{n \geq 0} \big[\kappa_+^{s_j}(0, n) 1_{[0, o_+^{s_j}(n+1))}(s_i-) - \kappa_-^{s_j}(0, n) 1_{[0, o_-^{s_j}(n+1))}(s_i-) \big]$$

$$= \sum_{j=1}^{N-1} \gamma_j \sum_{n \geq 1} \big[\kappa_+^{s_j}(0, n-1) 1_{\{i+1,\ldots,N\}}\big(\sigma_+^{s_j}(n)\big) - \kappa_-^{s_j}(0, n-1) 1_{\{i+1,\ldots,N\}}\big(\sigma_-^{s_j}(n)\big) \big]$$

$$+ \sum_{j=1}^{N-1} \gamma_j \Bigg[\sum_{\substack{n \geq 1 \\ \sigma_+^{s_j}(n)=i \\ o_+^{s_j}(n)=s_i}} \kappa_+^{s_j}(0, n-1) - \sum_{\substack{n \geq 1 \\ \sigma_-^{s_j}(n)=i \\ o_-^{s_j}(n)=s_i}} \kappa_-^{s_j}(0, n-1) \Bigg]$$

$$= + \sum_{j=1}^{N-1} \gamma_j \Bigg[\sum_{\substack{n \geq 0 \\ \sigma_+^{s_j}(n)=i \\ o_+^{s_j}(n)=s_i}} \kappa_+^{s_j}(0, n-1) - \sum_{\substack{n \geq 0 \\ \sigma_-^{s_j}(n)=i \\ o_-^{s_j}(n)=s_i}} \kappa_-^{s_j}(0, n-1) \Bigg]$$

and

$$l_\gamma(s_i+) = \sum_{j=1}^{N-1}\gamma_j \sum_{n\geq 0}\left[\kappa_+^{s_j}(0,n)1_{[0,o_+^{s_j}(n+1))}(s_i+) - \kappa_-^{s_j}(0,n)1_{[0,o_-^{s_j}(n+1))}(s_i+)\right]$$

$$= \sum_{j=1}^{N-1}\gamma_j \sum_{n\geq 1}\left[\kappa_+^{s_j}(0,n-1)1_{\{i+1,\dots,N\}}(\sigma_+^{s_j}(n)) - \kappa_-^{s_j}(0,n-1)1_{\{i+1,\dots,N\}}(\sigma_-^{s_j}(n))\right]$$

$$-\sum_{j=1}^{N-1}\gamma_j\Bigg[\sum_{\substack{n\geq 1\\ \sigma_+^{s_j}(n)=i+1\\ o_+^{s_j}(n)=s_i}}\kappa_+^{s_j}(0,n-1) - \sum_{\substack{n\geq 1\\ \sigma_-^{s_j}(n)=i+1\\ o_-^{s_j}(n)=s_i}}\kappa_-^{s_j}(0,n-1)\Bigg]$$

$$= -\sum_{j=1}^{N-1}\gamma_j\Bigg[\sum_{\substack{n\geq 0\\ \sigma_+^{s_j}(n)=i+1\\ o_+^{s_j}(n)=s_i}}\kappa_+^{s_j}(0,n-1) - \sum_{\substack{n\geq 0\\ \sigma_-^{s_j}(n)=i+1\\ o_-^{s_j}(n)=s_i}}\kappa_-^{s_j}(0,n-1)\Bigg].$$

Lemma 6.2. *Define for* $1 \leq j, i \leq N-1$,

$$a_{j,i} = \begin{cases} \kappa_+^{s_j}(0, n_{j,i}-1) & \text{if there exists a number } n \geq 1 \text{ with } \sigma_+^{s_j}(n) = i \text{ and} \\ & o_+^{s_j}(n) = s_i \text{ and if } n_{j,i} \text{ is minimal with this property,} \\ 0 & \text{else.} \end{cases}$$

For $1 \leq j, i \leq N-1, j \neq i$ *we set*

$$b_{j,i} = \begin{cases} \kappa_-^{s_j}(0, m_{j,i}-1) & \text{if there exists a number } m \geq 1 \text{ with } \sigma_-^{s_j}(m) = i \text{ and} \\ & o_-^{s_j}(m) = s_i \text{ and if } m_{j,i} \text{ is minimal with this property,} \\ 0 & \text{else} \end{cases}$$

and for $1 \leq i \leq N-1$ *we set* $b_{i,i} = 1$.

Furthermore, for $1 \leq j, i \leq N-1$, *let*

$$c_{j,i} = \begin{cases} \kappa_-^{s_j}(0, u_{j,i}-1) & \text{if there exists a number } u \geq 1 \text{ with } \sigma_-^{s_j}(u) = i+1 \text{ and} \\ & o_-^{s_j}(u) = s_i \text{ and if } u_{j,i} \text{ is minimal with this property,} \\ 0 & \text{else.} \end{cases}$$

For $1 \leq j, i \leq N-1, j \neq i$ *let*

$$d_{j,i} = \begin{cases} \kappa_+^{s_j}(0, v_{j,i}-1) & \text{if there exists a number } v \geq 1 \text{ with } \sigma_+^{s_j}(v) = i+1 \text{ and} \\ & o_+^{s_j}(v) = s_i \text{ and if } v_{j,i} \text{ is minimal with this property} \\ 0 & \text{else} \end{cases}$$

and for $1 \leq i \leq N-1$ *we set* $d_{i,i} = 1$.

Then the following assertions hold:
(1) $-1 < a_{j,i} \le 0 \le b_{j,i} \le 1$ *for all* $1 \le j, i \le N-1$.
(2) *Let*

$$A = \{j \mid 1 \le j \le N-1 \text{ there is an } i \text{ with } 1 \le i \le N-1 \text{ and } a_{j,i} < 0\}.$$

For every $j \in A$ *there is a uniquely defined* $a(j) \in \{1, \ldots, N-1\}$ *such that* $a_{j,i} = a_{j,a(j)} b_{a(j),i}$ *for every* $i = 1, \ldots, N-1$. *Let* $\mathcal{Z}(a)$ *denote the set of all cycles of the mapping* $a : A \to \{1, \ldots, N-1\}$, *i.e.*, $(j_{n,1}, j_{n,2}, \ldots, j_{n,z_n}) \in \mathcal{Z}(a)$ *iff* $z_n \ge 1$, $j_{n,k} \in A$ *for* $1 \le k \le z_n$, $j_{n,k} \ne j_{n,l}$ *for* $1 \le k < l \le z_n$, $a(j_{n,k}) = j_{n,k+1}$ *for* $k = 1, \ldots, z_n - 1$ *and* $a(j_{n,z_n}) = j_{n,1}$. *Then the determinants of the matrices* $(b_{j,i} - a_{j,i})_{1 \le j,i \le N-1}$ *and* $(b_{j,i})_{1 \le j,i \le N-1}$ *satisfy the relation*

$$\det((b_{j,i} - a_{j,i})_{1 \le j,i \le N-1}) = \det((b_{j,i})_{1 \le j,i \le N-1}) \times \prod_{(j_{n,1}, j_{n,2}, \ldots, j_{n,z_n}) \in \mathcal{Z}(a)} (1 - a_{j_{n,1}, j_{n,2}} a_{j_{n,2}, j_{n,3}} \ldots a_{j_{n,z_n}, j_{n,1}}).$$

(3) $\det((b_{j,i})_{1 \le j,i \le N-1}) > 0$.
(4) *Analogous results hold for* $c_{j,i}$ *and* $d_{j,i}$.

Theorem 6.3. *Let* S *denote a piecewise linear transformation on the closed unit interval* I *with the separation points* $0 < s_1 < \ldots < s_{N-1} < 1$ *and let* l_γ *denote a* S*-invariant function defined by* (4.3). *The following assertions are equivalent*:
(1) $l_\gamma(s_i-) = 0$ *for* $i = 1, \ldots, N-1$.
(2) $l_\gamma(s_i+) = 0$ *for* $i = 1, \ldots, N-1$.
(3) $\sum_{1 \le j \le N-1} \gamma_j (b_{j,i} - a_{j,i}) = 0$ *for* $i = 1, \ldots, N-1$.
(4) $\sum_{1 \le j \le N-1} \gamma_j (d_{j,i} - c_{j,i}) = 0$ *for* $i = 1, \ldots, N-1$.
(5) $\gamma = 0$.
(6) $l_\gamma = 0$ λ*-almost everywhere on* I.

Proof. (1) ⇒ (3): For $i = 1, \ldots, N-1$ we set

$$\beta_i = \sum_{\substack{n \ge 0 \\ \sigma_-^{s_i}(n) = i \\ o_-^{s_i}(n) = s_i}} \kappa_-^{s_i}(0, n-1)$$

$$= \begin{cases} \dfrac{k_{\sigma_-^{s_i}(0)} \ldots k_{\sigma_-^{s_i}(p_{i,i}-1)}}{k_{\sigma_-^{s_i}(0)} \ldots k_{\sigma_-^{s_i}(p_{i,i}-1)} - 1} & \text{if there exists a number } p \ge 1 \text{ with } \sigma_-^{s_i}(p) = i \text{ and } o_-^{s_i}(p) = s_i \text{ and if } p_{i,i} \text{ is minimal with this property,} \\ 1 & \text{else.} \end{cases}$$

Then for all $1 \le j, i \le N-1$ we have

$$\sum_{n\ge 0, \sigma_+^{s_j}(n)=i, o_+^{s_j}(n)=s_i} \kappa_+^{s_j}(0, n-1) - \sum_{n\ge 0, \sigma_-^{s_j}(n)=i, o_-^{s_j}(n)=s_i} \kappa_-^{s_j}(0, n-1) = -(b_{j,i} - a_{j,i})\beta_i$$

and, by Lemma 6.1, $l_\gamma(s_i-) = 0$ for $i = 1, \dots, N-1$ iff $\sum_{j=1}^{N-1} \gamma_j(b_{j,i} - a_{j,i}) = 0$ for $i = 1, \dots, N-1$.

(3) $\Rightarrow$ (5): From Lemma 6.2 we deduce the regularity of the matrix

$$(b_{j,i} - a_{j,i})_{1\le j,i\le N-1}$$

which implies (5).

The implications (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are shown in a similar way.

Lemma 6.4. *Let A be a finite union of closed disjoint subintervals of I of positive length such that A is S-invariant*, *i.e.*, $S(A) \subset A$ *λ-almost everywhere. The following assertions hold*:

(1) *The set A contains at least one discontinuity point of S or of S' in its interior.*

(2) *There is a nontrivial* $(N-1)$*-vector*

$$\gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{N-1} \end{pmatrix}$$

with $M\gamma = 0$ *such that* $\gamma_j = 0$ *if s_j is not contained in the interior of A* $(1 \le j \le N-1)$ *and such that the S-invariant function l_γ, defined in* (4.3), *vanishes λ-almost everywhere on $I \setminus A$.*

Proof. We assume that $\lambda(A) < 1$ and that the endpoints of all subintervals of A are contained in the set $\{0, s_1, \dots, s_{N-1}, 1\}$ (see Section 5).

Let A_0 denote a subinterval of A of maximal length. If A_0 does not contain a discontinuity point of S or of S' in its interior then the closure of $S(A_0)$ contains an interval of length greater then the length of A_0 which is a contradiction to the S-invariance of A.

We define a piecewise linear transformation S_A on I in the following way: Let $S_A(x) = S(x)$ for $x \in A$ and let every interval of $I \setminus A$ be full with respect to S_A. Then A is S_A-invariant and $S_A^{-1}(I \setminus A) \subset I \setminus A$ λ-almost everywhere. Let M_A denote the fundamental matrix of S_A. If the vector γ_A satifies $M_A\gamma_A = 0$ then the S_A-invariant function l_{γ_A} defined by (4.3) vanishes on $I \setminus A$ λ-almost everywhere since

$$\int_{S_A^{-n}(I\setminus A)} l_{\gamma_A}\, d\lambda = \int_{(I\setminus A)} l_{\gamma_A}\, d\lambda \quad \text{for all } n \ge 1$$

and since $\lim_{n\to\infty} \lambda(S_A^{-n}(I \setminus A)) = 0$.

Let $s_t \in (0, 1)$ be a separation point of S_A and an endpoint of a subinterval of A. If s_t is a left-sided endpoint, then

$$\begin{aligned} L_\alpha^{s_j}(s_t-) &= \sum_{n\geq 0} \kappa_\alpha^{s_j}(0, n) 1_{[0, \sigma_\alpha^{s_j}(n+1))}(s_t-) \\ &= \sum_{n\geq 1} \kappa_\alpha^{s_j}(0, n-1) 1_{\{t+1,\ldots,N\}}(\sigma_\alpha^{s_j}(n)) \end{aligned}$$

for $\alpha \in \{+, -\}$ and for all separation points s_j of S_A and assertion (4) of Lemma 3.4 implies that the component of γ_A corresponding to the separation point s_t is zero since $l_{\gamma_A} = 0$ on $I \setminus A$.

If s_t is a right-sided endpoint, then

$$\begin{aligned} R_\alpha^{s_j}(s_t+) &= \sum_{n\geq 0} \kappa_\alpha^{s_j}(0, n) 1_{(\sigma_\alpha^{s_j}(n+1), 1]}(s_t+) \\ &= \sum_{n\geq 1} \kappa_\alpha^{s_j}(0, n-1) 1_{\{1,\ldots,t\}}\big(\sigma_\alpha^{s_j}(n)\big) \end{aligned}$$

for $\alpha \in \{+, -\}$ and for all separation points s_j of S_A and assertion (3) of Lemma 3.4 implies that the component of γ_A corresponding to the separation point s_t is zero since the S_A-invariant function r_{γ_A} defined in (4.11) vanishes on $I \setminus A$.

We find a vector

$$\gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{N-1} \end{pmatrix}$$

with the desired properties in the following way: Assume $1 \leq j \leq N - 1$. If $s_j \in A$, let γ_j be equal to the corresponding component of γ_A. If $s_j \notin A$, set $\gamma_j = 0$.

If some endpoints of subintervals of A are not separation points of S, we add them to the separation points, carry out the above construction and drop the vanishing components of γ, which correspond to the added separation points. The remaining components form a vector with the desired properties.

Theorem 6.5. *Let S denote a piecewise linear transformation on I with separation points $0 < s_1 < \cdots < s_{N-1} < 1$ and let M denote the fundamental matrix of S. Let U denote the subspace consisting of all real $(N-1)$-vectors γ with $M\gamma = 0$ and let V denote the subspace of L^1 consisting of all $S-$invariant functions on I. Then the linear mapping $\gamma \mapsto l_\gamma$ from U into V, established by Theorem 4.2, is an isomorphism between U and V.*

Proof. In Theorem 1 of A. Lasota and J. Yorke [3] it is shown that every S-invariant function f^* equals almost everywhwere some function of bounded variation. Using this result T. Li and J. Yorke [4] have shown that every S-invariant function f^* equals almost everywhwere some linear combination $\sum_{1 \leq j \leq n} a_j f_j$, where a_j denotes a suitable chosen real constant and where f_j denotes a S-invariant function with a support F_j consisting of a finite union of closed intervals and containing at least one discontinuity point of S or of S' in its interior $(1 \leq j \leq n)$. Furthermore $S(F_j) \subset F_j$ λ-almost everywhere and $F_i \cap F_j$ contains at most a finite number of points for $i \neq j$. The assertion of the Theorem now follows by Lemma 6.4.

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