ENVELOPES OF FAMILIES OF CURVES RELATED TO SOME ITERATIVE FUNCTIONAL EQUATIONS

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ABSTRACT

We examine the envelopes of one-parameter family of the graphs of "affine type" of functions $(\phi_\alpha)_{\alpha\in\mathbb{R}}$, where $\phi_\alpha(x) = g(p(x)-\alpha)+g(\alpha)$ (as well as three other types), where g and p are given functions. These families of functions appear in connection with some Cauchy functional equations of iterative type.

Introduction. In the present paper we examine the envelopes of the one-parameter families of curves being the graphs of functions ϕ_α which have one of the following forms

$$\phi_\alpha(x) := g(p(x) - \alpha) + g(\alpha), \qquad x, \alpha \in \mathbb{R},$$

$$\phi_\alpha(x) := g(p(x) - \alpha)g(\alpha), \qquad x, \alpha \in \mathbb{R},$$

$$\phi_\alpha(x) := g\Big(\frac{p(x)}{\alpha}\Big) + g(\alpha), \qquad x, \alpha > 0,$$

$$\phi_\alpha(x) := g\Big(\frac{p(x)}{\alpha}\Big)g(\alpha), \qquad x, \alpha > 0,$$

where g and p are given functions. In the case when $p = \mathrm{id}$ these functions appeared in

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a natural way in connection with means and some functional equations of iterative type (cf. [3]), and the respective problem of envelopes was considered in [4]. For obvious reason, coming from the suitable Cauchy functional equation, we say that the above families of curves are of *affine*, *exponential*, *logarithmic*, and *power type*, respectively.

In particular we show that, depending on the type of the family, the strict convexity of g, $\log \circ g$, $g \circ \exp$, and $\log \circ g \circ \exp$ (i.e., the geometrical convexity of g), guarantees a simple form of the envelope.

1. Envelopes for families of affine type

By $\mathbb{R}$ we denote the set of reals. For an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, and $p : \mathbb{R} \to \mathbb{R}$ define the one-parameter family of functions $\phi_\alpha : \mathbb{R} \to \mathbb{R}$ by

$$\phi_\alpha(x) := g\left(p(x) - \alpha\right) + g(\alpha), \qquad x, \alpha \in \mathbb{R},$$

by $\mathcal{G}(g, p)$ denote the family of curves being the graphs of ϕ_α, $\alpha > 0$. Let g and p be differentiable. Denote by $E_{g,p}$ the set of all points $(x, y) \in \mathbb{R}^2$ such that for some $\alpha \in \mathbb{R}$ we have

$$y = g\left(p(x) - \alpha\right) + g(\alpha), \qquad -g'\left(p(x) - \alpha\right) + g'(\alpha) = 0.$$

According to a well known fact (cf. for instance [1]) the envelope of the family $\mathcal{G}(g, p)$ either coincides with $E_{g,p}$ or is a proper subset of $E_{g,p}$. Since the derivative of the function p does not appear in the above special system of equations, we shall not assume any regularity conditions for p. In the sequel the set $E_{g,p}$ is called a generalized envelope or, shortly, envelope of the family $\mathcal{G}(g, p)$.

We often identify a function and its graph. Therefore we frequently write the generalized envelope $E_{g,p}$ in the form $y = E_{g,p}(x)$, $x \in \mathbb{R}$.

Remark 1.1. If $g(x) = cx + g(0)$, $x \in \mathbb{R}$, where c and $g(0)$ are arbitrary real constants, then $\mathcal{G}(g, p) = \{g \circ p + g(0)\}$ is a singleton, and $E_{g,p}$, the envelope of $\mathcal{G}(g, p)$, obviously coincides with the graph of the function $g \circ p + g(0)$.

It turns out that, under some general conditions, the converse implication holds true.

Proposition 1.1. *Let $g, p : \mathbb{R} \to \mathbb{R}$ be given functions, and suppose that $0 \in p(\mathbb{R})$. If $\mathcal{G}(g, p)$ is a singleton then*

$$g(\alpha + y) + g(0) = g(\alpha) + g(y), \qquad \alpha \in \mathbb{R}, y \in p(\mathbb{R}).$$

If, moreover, g is continuous at least at one point, and there are $u, v \in p(\mathbb{R})$, $u \neq 0 \neq v$, such that u/v is irrational, then

$$g(x) = cx + g(0), \qquad x \in \mathbb{R},$$

for some $c \in \mathbb{R}$.

Proof. The family $\mathcal{G}(g, p)$ is a singleton if and only if

$$g(p(x) - \alpha) + g(\alpha) = g(p(x) - \beta) + g(\beta),$$

for all $x, \alpha, \beta \in \mathbb{R}$. Setting $y := p(x)$ gives $g(y - \alpha) + g(\alpha) = g(y - \beta) + g(\beta)$, $\alpha, \beta \in \mathbb{R}$, $y \in p(\mathbb{R})$. Taking $\beta := y,\ y \in p(\mathbb{R})$ we have

$$g(y - \alpha) + g(\alpha) = g(0) + g(y), \qquad \alpha \in \mathbb{R}, y \in p(\mathbb{R}).$$

Hence, for $\psi : \mathbb{R} \to \mathbb{R}$ defined by the formula $\psi(\alpha) := g(\alpha) - g(0), \alpha \in \mathbb{R}$, one gets

$$\psi(y - \alpha) + \psi(\alpha) = \psi(y), \qquad \alpha \in \mathbb{R}, y \in p(\mathbb{R}). \tag{1}$$

Taking here $\alpha = y$ gives $\psi(0) = 0$. By assumption $0 \in p(\mathbb{R})$. Hence, setting $y = 0$ in this equation we obtain $\psi(-\alpha) = -\psi(\alpha), \alpha \in \mathbb{R}$. Replacing α by $-\alpha$ in (1) gives

$$\psi(\alpha + y) = \psi(\alpha) + \psi(y), \qquad \alpha \in \mathbb{R}, y \in p(\mathbb{R}), \tag{2}$$

which means that $g(\alpha + y) + g(0) = g(\alpha) + g(y), \alpha \in \mathbb{R}, y \in p(\mathbb{R})$. Taking in (2) $y = u$ and $y = v$ we see that $\psi : \mathbb{R} \to \mathbb{R}$ satisfies the simultaneous system of two functional equations

$$\psi(\alpha + u) = \psi(\alpha) + \psi(u), \qquad \psi(\alpha + v) = \psi(\alpha) + \psi(v), \qquad \alpha \in \mathbb{R},$$

Since ψ is continuous at a point, and u/v is irrational, it follows that there is a $c \in \mathbb{R}$ such that $\psi(\alpha) = c\alpha$ for all $\alpha \in \mathbb{R}$ (cf. [2]). This completes the proof. □

Remark 1.2. Suppose that $p = a$, where $a \in \mathbb{R}, a \neq 0$. Then the family $\mathcal{G}(g, p)$ is a singleton if and only if $g(a - \alpha) + g(\alpha) = g(a - \beta) + g(\beta), \alpha, \beta \in \mathbb{R}$. It is easy to check that the general solution $g : \mathbb{R} \to \mathbb{R}$ of this functional equation has the form

$$g(t) = \begin{cases} \gamma(t) & t > \frac{1}{2}a, \\ c & t = \frac{1}{2}a, \\ 2c - \gamma(a - t) & t < \frac{1}{2}a, \end{cases}$$

where $\gamma : (\frac{1}{2}a, \infty) \to \mathbb{R}$ and $c \in \mathbb{R}$ are arbitrary.

The main result of this section reads as follows:

Theorem 1.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then the graph of the function*

$$\mathbb{R} \ni x \to 2g\left[\frac{p(x)}{2}\right],$$

is contained in the envelope of the family $\mathcal{G}(g, p)$. If the function g' is one-to-one, then the envelope $E_{g,p}$ has the representation

$$y = E_{g,p}(x) = 2g\left[\frac{p(x)}{2}\right], \qquad x \in \mathbb{R}.$$

Proof. In view of the classical method (cf. for instance [1]), to find the envelope of the family of curves $\mathcal{G}(g, p)$ it is enough to eliminate the parameter α from the system of equations

$$y = g\,(p(x) - \alpha) + g(\alpha), \quad -g'\,(p(x) - \alpha) + g'(\alpha) = 0, \quad x, y, \alpha \in \mathbb{R}. \quad (3)$$

The second equation can be written in the equivalent form $g'\,(p(x) - \alpha) = g'(\alpha)$, $x, \alpha \in \mathbb{R}$. If the function g' is one-to-one, it follows that $p(x) - \alpha = \alpha$, and consequently, $\alpha = p(x)/2$, $x \in \mathbb{R}$. Setting $\alpha = p(x)/2$ into the first of the equations we get the function

$$y = 2g\left[\frac{p(x)}{2}\right], \qquad x \in \mathbb{R}, \quad (4)$$

the graph of which is the envelope of the considered family of curves.

If the function g' is not one-to-one, then, as the point $\alpha = p(x)/2$ satisfies the second equation of the system (3), every point of the graph of the function (4) is a point of the envelope. This completes the proof. □

Corollary 1.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable and strictly convex (or strictly concave) function. Then the graph of the function*

$$y = 2g\left[\frac{p(x)}{2}\right], \qquad x \in \mathbb{R}.$$

is the envelope $E_{g,p}$ of the family $\mathcal{G}(g, p)$.

2. Envelopes of families of exponential type

Let $g : \mathbb{R} \to (0, \infty)$ and $p : \mathbb{R} \to \mathbb{R}$ be given. Similarly as in the previous section we consider the one-parameter family $\mathcal{G}(g, p)$ of functions $\phi_\alpha : \mathbb{R} \to (0, \infty)$, defined by $\phi_\alpha(x) := g\,(p(x) - \alpha)\, g(\alpha)$, $x, \alpha \in \mathbb{R}$.

Remark 2.1. If $g(x) = g(0)\, e^{cx}$, $x \in \mathbb{R}$, where $c \in \mathbb{R}$ and $g(0) > 0$ are arbitrary constants, then $\mathcal{G}(g, p) = \{g(0)\, g \circ p\}$ is a singleton, and $E_{g,p}$, the envelope of $\mathcal{G}(g, p)$, coincides with the graph of the function $g(0)\, g \circ p$.

Proposition 2.1. *Let $g : \mathbb{R} \to (0, \infty)$, $p : \mathbb{R} \to \mathbb{R}$, be given function and suppose that $0 \in p(\mathbb{R})$. If $\mathcal{G}(g, p)$ is a singleton then*

$$g(0)\, g(\alpha + y) = g(\alpha)\, g(y), \qquad \alpha \in \mathbb{R}, y \in p(\mathbb{R}).$$

If, moreover, g is continuous at least at one point, and there are $u, v \in p(\mathbb{R})$, $u \neq 0 \neq v$, such that u/v is irrational, then

$$g(x) = g(0)\, e^{cx}, \qquad x \in \mathbb{R},$$

for some $c \in \mathbb{R}$.

Theorem 2.1. *Let* $g : \mathbb{R} \to (0, \infty)$, *and* $p : \mathbb{R} \to \mathbb{R}$ *be fixed functions. If* g *is differentiable then the graph of the function*

$$\mathbb{R} \ni x \longrightarrow \left[g\left(\frac{p(x)}{2}\right)\right]^2,$$

is contained in the envelope of the family $\mathcal{G}(g, p)$. *If the function* g'/g *is one-to-one*, *then the envelope* $E_{g,p}$ *has the representation*

$$y = E_{g,p}(x) = \left[g\left(\frac{p(x)}{2}\right)\right]^2, \qquad x > 0.$$

Corollary 2.1. *Let* $g : \mathbb{R} \to (0, \infty)$ *be a differentiable function. If* $\log \circ g$ *is a strictly convex* (*or strictly concave*) *function*, *then the graph of the function*

$$y = \left[g\left(\frac{p(x)}{2}\right)\right]^2, \qquad x \in \mathbb{R}.$$

is the envelope $E_{g,p}$ *of the family* $\mathcal{G}(g, p)$.

3. Envelopes of families of logarithmic type

For a function $g : (0, \infty) \to \mathbb{R}$, and $p : (0, \infty) \to (0, \infty)$ define the one-parameter family $\mathcal{G}(g, p)$ of functions $\phi_\alpha : (0, \infty) \to \mathbb{R}$ by

$$\phi_\alpha(x) := g\left(\frac{p(x)}{\alpha}\right) + g(\alpha), \qquad x, \alpha > 0.$$

Remark 3.1. If $g(x) = c \log x + g(1)$, $x > 0$, where c and $g(1)$ are arbitrary real constants, then $\mathcal{G}(g, p) = \{g \circ p + \log g(1)\}$ is a singleton, and $E_{g,p}$, the envelope of $\mathcal{G}(g, p)$, obviously coincides with the graph of $g \circ p + \log g(1)$.

Proposition 3.1. *Let* $g : (0, \infty) \to \mathbb{R}$, $p : (0, \infty) \to (0, \infty)$ *be given function*, *and suppose that* $1 \in p(\mathbb{R})$. *If* $\mathcal{G}(g, p)$ *is a singleton then*

$$g(1) + g(\alpha y) = g(\alpha) + g(y), \qquad \alpha \in \mathbb{R}, y \in p\left((0, \infty)\right).$$

If, *moreover*, g *is continuous at least at one point*, *and there are* $u, v \in p\left((0, \infty)\right)$, $u \neq 1 \neq v$, *such that* $\log u / \log v$ *is irrational*, *then*

$$g(x) = c \log x + g(1), \qquad x > 0,$$

for some $c \in \mathbb{R}$.

Theorem 3.1. *Let* $g : (0, \infty) \to \mathbb{R}$ *be a differentiable function. Then the graph of the function*

$$(0, \infty) \ni x \longrightarrow 2g\left[\sqrt{p(x)}\right],$$

is contained in the envelope of the family $\mathcal{G}(g, p)$. *If the function* $(0, \infty) \ni x \to g'(x)x$ *is one-to-one, then the envelope* $E_{g,p}$ *has the representation*

$$y = E_{g,p}(x) = 2g\left[\sqrt{p(x)}\right], \qquad x > 0.$$

Corollary. 3.1. *Let* $g : (0, \infty) \to \mathbb{R}$ *be a differentiable function. If the function* $g \circ \exp$ *is convex* (*or concave*) *in* $(0, \infty)$, *then the envelope* $E_{g,p}$ *of the family* $\mathcal{G}(g, p)$ *has the representation*

$$y = E_{g,p}(x) = 2g\left[\sqrt{p(x)}\right], \qquad x > 0.$$

4. Envelopes of families of power type

For a function $g : (0, \infty) \to \mathbb{R}$, and $p : (0, \infty) \to (0, \infty)$ define the one-parameter family $\mathcal{G}(g, p)$ of functions $\phi_\alpha : (0, \infty) \to (0, \infty)$ by

$$\phi_\alpha(x) := g\left(\frac{p(x)}{\alpha}\right) g(\alpha), \qquad x, \alpha > 0.$$

Remark 4.1. If $g(x) = g(1)\, x^c$, $x > 0$, where $c \in \mathbb{R}$ and $g(1) > 0$ are arbitrary constants then $\mathcal{G}(g, p) = \{g(1)\, g \circ p\}$ is a singleton, and $E_{g,p}$, the envelope of $\mathcal{G}(g, p)$, coincides with the graph of $g(1)\, g \circ p$.

Proposition 4.1. *Let* $g : (0, \infty) \to (0, \infty)$ *be an arbitrary function. If* $\mathcal{G}(g, p)$ *is a singleton then*

$$g(1)g(\alpha y) = g(\alpha)g(y), \qquad \alpha > 0, y \in p\left((0, \infty)\right).$$

If, moreover, g *is continuous at least at one point, and there are* $u, v \in p\left((0, \infty)\right)$, $u \neq 1 \neq v$, *such that* $\log u / \log v$ *is irrational, then*

$$g(x) = g(1)\, x^c, \qquad x > 0,$$

for some $c \in \mathbb{R}$.

Theorem 4.1. *Let* $g : (0, \infty) \to (0, \infty)$ *be a differentiable function. Then the graph of the function*

$$(0, \infty) \ni x \longrightarrow \left[g\left(\sqrt{p(x)}\right)\right]^2$$

is contained in the envelope $E_{g,p}$ of the family $\mathcal{G}(g, p)$. If the function

$$(0, \infty) \ni x \longrightarrow \frac{g'(x)}{g(x)}\, x$$

is one-to-one, then the envelope curve has the representation

$$y = E_g(x) = \left[g\left(\sqrt{p(x)}\right)\right]^2, \qquad x > 0.$$

Corollary 4.1. *Let $g : (0, \infty) \to (0, \infty)$ be a differentiable function. If the function* $\log \circ g \circ \exp$ *is convex* (*concave*), *i.e.*, *if g is convex* (*concave*) *with respect to the geometric mean*, *then the envelope curve $E_{g,p}$ of the family $\mathcal{G}(g, p)$ has the representation*

$$y = E_g(x) = \left[g\left(\sqrt{p(x)}\right)\right]^2, \qquad x > 0.$$

5. Final remarks

Remark 1. Taking $p = \mathrm{id}$ in all theorems we get the basic results of paper [4] (cf. also [3] where some motivations are given).

Remark 2. The main idea of the proof of Theorem 1.1 is based on the fact that for a one-parameter family of curves of the Cauchy affine type $\phi_\alpha(x) := g\,(p(x) - \alpha) + g(\alpha)$, $x, \alpha \in \mathbb{R}$, the second equation of the basic envelope system (3) written in the form $g'\,(p(x) - \alpha) = g'(\alpha)$, $x, \alpha \in \mathbb{R}$, together with the injectivity of the derivative g' permit, without difficulties, to eliminate the parameter. We are going to show that using this idea does not allow to prove more general results than Theorem 1.1 (as well as Theorems 2.1, 3.1, and 4.1). In fact, suppose that $g, k, m, r : \mathbb{R} \to \mathbb{R}$ are given functions, and consider the one-parameter family of functions

$$\phi_\beta(x) = g\,(r(x) + k(\beta)) + g\,(m(\beta))\,, \qquad x,\ \beta \in \mathbb{R}. \tag{5}$$

(Note that the function on the right-hand side is similar to the relevant one in the affine type family of curves, but of course more general). Assume that g, k, m are differentiable. Now one of the equations for the envelope of the family of curves $(\phi_\beta)_{\beta\in\mathbb{R}}$ is of the form

$$g'\,(r(x) + k(\beta))\,k'(\beta) + g'\,(m(\beta))\,m'(\beta) = 0,$$

and, if $k'(\beta) \neq 0$,

$$g'\,(r(x) + k(\beta)) = -\frac{m'(\beta)}{k'(\beta)}\, g'\,(m(\beta))\,. \tag{6}$$

Thus, to apply the same method for the determination of β as a function of x (to eliminate β from the system of envelope equations) it is necessary to assume that $-m'(\beta)/k'(\beta) = 1$, $\beta \in \mathbb{R}$, i.e., that $k(\beta) = -m(\beta) + c$, $\beta \in \mathbb{R}$, where $c \in \mathbb{R}$ is a constant. Setting this into equation (6) we obtain $g'(r(x) - m(\beta) + c) = g'(m(\beta))$, $\beta \in \mathbb{R}$. Setting $p(x) := r(x) + c$, $x \in \mathbb{R}$ into (5) and (6) gives

$$y = \phi_\beta(x) = g(p(x) - m(\beta)) + g(m(\beta)), \qquad x, \beta \in \mathbb{R},$$

and $g'(p(x) - m(\beta)) = g'(m(\beta))$, $x, \beta \in \mathbb{R}$. Now, if g' is one-to-one, the last equation implies $m(\beta) = \frac{1}{2}p(x)$, $x, \beta \in \mathbb{R}$, and, consequently, $y = 2g(\frac{1}{2}p(x))$, $x \in \mathbb{R}$, is the envelope of the family of curves $(\phi_\beta)_{\beta \in \mathbb{R}}$. Thus, setting $\alpha := m(\beta)$, we get the family of curves $(\phi_\alpha)_{\alpha \in m(\mathbb{R})}$:

$$y = g(p(x) - \alpha) + g(\alpha), \qquad x \in \mathbb{R}, \alpha \in m(\mathbb{R})$$

being a subfamily of curves of affine Cauchy type considered in section 1.

Remark 3. Looking for a generalization of the problem one could go beyond the affine type for the equation of the family of curves (considered in section 1). Take for instance the family of functions ϕ_α of the form

$$y = \phi_\alpha(x) = A(x)g(a(x)m(\alpha) + n(\alpha)) + B(x)g(b(x)k(\alpha) + l(\alpha)), \tag{7}$$

for all $x, \alpha \in \mathbb{R}$, where $A, B, a, b, m, n, k, l : \mathbb{R} \to \mathbb{R}$ are given functions. Suppose that m, n, k, and l are differentiable. Assuming that the suitable functions do not vanish, we can write the counterpart of (3) in the form

$$\begin{aligned} &g'(a(x)m(\alpha) + n(\alpha)) \\ &\quad = -\frac{B(x)\,[\,b(x)k'(\alpha) + l'(\alpha)\,]}{A(x)\,[\,a(x)m'(\alpha) + n'(\alpha)\,]}\, g'(b(x)k(\alpha) + l(\alpha)). \end{aligned} \tag{8}$$

(Note that the family of functions given by (7) is essentially the most general one which permits to make advantage of the injectivity of the function g'.) To eliminate the parameter α with the aid of the assumption that g' is one-to-one, we should have

$$\frac{B(x)\,[\,b(x)k'(\alpha) + l'(\alpha)\,]}{A(x)\,[\,a(x)m'(x) + n'(\alpha)\,]} = -1, \qquad x, \alpha \in \mathbb{R},$$

or, equivalently,

$$\frac{b(x)k'(\alpha) + l'(\alpha)}{a(x)m'(\alpha) + n'(\alpha)} = -\frac{A(x)}{B(x)}, \qquad x, \alpha \in \mathbb{R}.$$

The function on the left-hand side does not depend on α.

Suppose that $n'(\alpha) \neq 0$ for $\alpha \in \mathbb{R}$. It follows that there exist $c_1, c_2, c_3 \in \mathbb{R}$ such that $m'(\alpha) = c_1 n'(\alpha)$, $k'(\alpha) = c_2 n'(\alpha)$, $l'(\alpha) = c_3 n'(\alpha)$, $\alpha \in \mathbb{R}$. Consequently, $m(\alpha) = c_1 n(\alpha) + d_1$, $k(\alpha) = c_2 n(\alpha) + d_2$, $l(\alpha) = c_3 n(\alpha) + d_3$, $\alpha \in \mathbb{R}$, for some constants $d_1, d_2, d_3 \in \mathbb{R}$, and

$$\frac{c_2 b(x) + c_3}{c_1 a(x) + 1} = -\frac{A(x)}{B(x)}, \qquad x \in \mathbb{R}.$$

From the last relation we have

$$B(x) = -A(x)\frac{c_1 a(x) + 1}{c_2 b(x) + c_3}, \qquad x \in \mathbb{R}.$$

It follows that the functions ϕ_α given by (7) have to be of the form

$$\begin{aligned} y &= \phi_\alpha(x) \\ &= A(x)g(a(x)[c_1 n(\alpha) + d_1] + n(\alpha)) \\ &\quad - A(x)\frac{c_1 a(x) + 1}{c_2 b(x) + c_3} g(b(x)[c_2 n(\alpha) + d_2] + c_3 n(\alpha) + d_3), \end{aligned} \tag{9}$$

for all $x, \alpha \in \mathbb{R}$. The envelope equation (8) takes the form

$$g'\left(a(x)[c_1 n(\alpha) + d_1] + n(\alpha)\right) = g'\left(b(x)[c_2 n(\alpha) + d_2] + c_3 n(\alpha) + d_3\right),$$

for all $x, \alpha \in \mathbb{R}$. If g' is one-to-one we hence get

$$a(x)[c_1 n(\alpha) + d_1] + n(\alpha) = b(x)[c_2 n(\alpha) + d_2] + c_3 n(\alpha) + d_3, \qquad x, \alpha \in \mathbb{R}.$$

Assuming that $c_1 a(x) - c_2 b(x) + 1 - c_3 \neq 0$, $x \in \mathbb{R}$, we get

$$n(\alpha) = \frac{d_2 b(x) - d_1 a(x) + d_3}{c_1 a(x) - c_2 b(x) + 1 - c_3}.$$

Substituting this value of $n(\alpha)$ into equation (9) we obtain the envelope of the family of graphs of $(\phi_\alpha)_{\alpha \in \mathbb{R}}$.

The above discussion proves that, even if we give up the affine type of the family of curves, the method applied in this paper does not allow to treat essentially more general families of functions.

It is easy to observe that the counterparts of Remarks 2 and 3 for exponential, logarithmic, and power type families of functions also remain true.

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