SERIES OF ITERATES SUMMING UP TO THE IDENTITY FUNCTION

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The following problem seems to be quite natural and elementary: Given a function $F : D \to D$ (where D is a set of reals or, more generally, a subset of a linear space) and numbers $a_1, \dots, a_k$ find all functions $f : D \to D$ satisfying the equation

$$\sum_{i=1}^{k} a_i f^i(x) = F(x). \tag{1}$$

(Here f^i denotes the ith iterate of f.) Observe that the question posed in such a generality contains among others the classical problem of fractional iterates which has a vast literature (see [4, Chapter XV], [5, Chapter 11], and [7, Chapter 2]). Some other special cases of equation (1) were studied in various settings by many authors (see, for instance, the references of [3]). In particular, the paper [3] deals with the equation

$$\sum_{i=1}^{k} a_i f^i(x) = x. \tag{2}$$

To formulate the main result of it let us introduce the following notation useful also in further considerations. Given a (finite or not) sequence a of elements of $[-\infty, \infty]$ define the support supp a of a as the set of all i's for which a_i does not vanish. By the greatest common divisor of a non-void set $I \neq \{0\}$ of integers we mean the maximal number $p \in \mathbb{N}$ such that $I \subset p\mathbb{Z}$.

Consider the characteristic equation

$$\sum_{i=1}^{k} a_i \lambda^i = 1 \tag{3}$$

of (2). Observe that if $a_1, \dots, a_k$ are non-negative and at least one of them is positive then there is a unique positive root, say c, of (3).

Theorem A ([3]). *Let $a_1, \dots, a_k$ be non-negative not all zero reals such that the greatest common divisor of* supp $(a_1, \dots, a_k)$ *equals* 1.

If either $D \subset (-\infty, 0)$, or $D \subset (0, \infty)$ and $f : D \to D$ satisfies equation (2) *then*

$$f(x) = cx$$

for every $x \in D$.

Recently, Jacek Tabor and Józef Tabor [6] using completely different methods have generalized Theorem A getting the same conclusion for the equation

$$\sum_{i=0}^{k} a_i f^i(x) = 0$$

where the numbers $a_0, \dots, a_k$ satisfy conditions weaker than those above (in particular, they need not have the same sign).

As a main tool in the proof of Theorem A we used [2, Theorem 1.1]. In the present paper we extend Theorem A to the infinite order equation

$$\sum_{i=1}^{\infty} a_i f^i(x) = x \tag{4}$$

applying the following "two-dimensional" case of [2, Theorem 1.1], originally proved by K. Baron and the author.

Lemma B ([1]). *If $\alpha : \mathbb{N}_0^2 \to \mathbb{R}$ is a non-negative solution of the equation*

$$\alpha(m+1, n) + \alpha(m, n+1) = \alpha(m, n)$$

then

$$\alpha(m, n)^2 \le \alpha(m-p, n-q)\,\alpha(m+p, n+q)$$

for every $(m, n) \in \mathbb{N}_0^2$ and $(p, q) \in \mathbb{Z}^2$ such that $|p| \le m$ and $|q| \le n$.

Although formally weaker Lemma B suitably used will allow us to obtain the Theorem below which is essentially more general than Theorem A.

Given a sequence $a : \mathbb{N} \to \mathbb{R}$ of non-negative reals consider the characteristic equation

$$\sum_{i=1}^{\infty} a_i \lambda^i = 1 \tag{5}$$

and observe that it has at most one positive root. (Now, contrary to the case of (3), it need not have a positive root at all even if all a_i's are positive.)

The main result reads as follows.

Theorem. *Let* $a : \mathbb{N} \to \mathbb{R}$ *be a non-zero sequence of non-negative reals such that the greatest common divisor of* supp a *equals* 1.

If either $D \subset (-\infty, 0)$, *or* $D \subset (0, \infty)$ *and* $f : D \to D$ *satisfies equation* (4) *then there is exactly one positive root* c *of equation* (5) *and*

$$f(x) = cx$$

for every $x \in D$.

Before proving this result we shall make some preliminary remarks.

In the set $[0, \infty]^{\mathbb{N}}$ we introduce the convolution operation $\star$ by putting

$$(a \star b)_j := \sum_{i=1}^{j-1} a_i b_{j-i}, \qquad j \in \mathbb{N},$$

where, by definition, $0 \cdot \infty = \infty \cdot 0 := 0$ and the sum over the empty set of indices equals zero. It is an easy observation that the operation $\star$ is associative and commutative. Moreover,

$$\operatorname{supp} a \star b = \operatorname{supp} a + \operatorname{supp} b$$

for all $a, b \in [0, \infty]^{\mathbb{N}}$.

Given a sequence $a \in [0, \infty]^{\mathbb{N}}$ and a positive integer n denote by $a^{(n)}$ the nth convolution power of a:

$$a^{(n)} := \underbrace{a \star \ldots \star a}_{n \text{ times}}.$$

Then

$$a_i^{(n)} = \sum_{\substack{i_1, \ldots, i_n \in \mathbb{N} \\ i_1 + \ldots + i_n = i}} a_{i_1} \cdot \ldots \cdot a_{i_n}, \qquad i \in \mathbb{N}, \tag{6}$$

and

$$\operatorname{supp} a^{(n)} = \underbrace{\operatorname{supp} a + \ldots + \operatorname{supp} a}_{n \text{ times}}. \tag{7}$$

In particular, we have $a^{(1)} = a$. Additionally we put

$$a_i^{(\infty)} := \sum_{m=1}^{\infty} 2^{-m} a_i^{(m)}, \qquad i \in \mathbb{N}. \tag{8}$$

Lemma. *Let* $a : \mathbb{N} \to [0, \infty]$ *and* $n \in \mathbb{N} \cup \{\infty\}$. *Then*:

(a) *a positive real number* λ *is a root of equation* (5) *if and only if*

$$\sum_{i=1}^{\infty} a_i^{(n)} \lambda^i = 1;$$

(b) *if* $D \subset (0,\infty)$ *then every solution* $f : D \to D$ *of equation* (4) *is a solution of the equation*

$$\sum_{i=1}^{\infty} a_i^{(n)} f^i(x) = x.$$

Proof. Fix a number $\lambda \in (0,\infty)$ and put $\mu = \sum_{i=1}^{\infty} a_i \lambda^i$. Clearly $\mu \in [0,\infty]$. If n is finite then, by (6),

$$\sum_{i=1}^{\infty} a_i^{(n)} \lambda^i = \sum_{i=1}^{\infty} \sum_{\substack{i_1,\dots,i_n \in \mathbb{N} \\ i_1+\dots+i_n=i}} a_{i_1} \cdot \ldots \cdot a_{i_n} \lambda^{i_1+\dots+i_n} = \sum_{i_1=1}^{\infty} a_{i_1}\lambda^{i_1} \cdot \ldots \cdot \sum_{i_n=1}^{\infty} a_{i_n}\lambda^{i_n} = \mu^n.$$

Hence, using also (8), we have

$$\sum_{i=1}^{\infty} a_i^{(\infty)} \lambda^i = \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} 2^{-m} a_i^{(m)} \lambda^i = \sum_{m=1}^{\infty} 2^{-m} \sum_{i=1}^{\infty} a_i^{(m)} \lambda^i = \sum_{m=1}^{\infty} 2^{-m} \mu^m.$$

Thus assertion (a) follows.

To prove part (b) we proceed inductively. Assume that $D \subset (0,\infty)$ and $f : D \to D$ satisfies (4) and the condition

$$\sum_{i=1}^{\infty} a_i^{(m)} f^i(x) = x, \qquad x \in D, \tag{9}$$

for an $m \in \mathbb{N}$. Then for any $x \in D$ we have

$$\begin{aligned}\sum_{j=1}^{\infty} a_j^{(m+1)} f^j(x) &= \sum_{j=1}^{\infty} (a^{(m)} \star a)_j f^j(x) = \sum_{j=1}^{\infty} \sum_{i=1}^{j-1} a_i^{(m)} a_{j-i} f^j(x) \\ &= \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} a_i^{(m)} a_{j-i} f^j(x) = \sum_{i=1}^{\infty} a_i^{(m)} \sum_{j=1}^{\infty} a_j f^j(f^i(x)) = \sum_{i=1}^{\infty} a_i^{(m)} f^i(x) = x.\end{aligned}$$

Having (9) proved for all $m \in \mathbb{N}$ we get

$$\sum_{i=1}^{\infty} a_i^{(\infty)} f^i(x) = \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} 2^{-m} a_i^{(m)} f^i(x) = \sum_{m=1}^{\infty} 2^{-m} \sum_{i=1}^{\infty} a_i^{(m)} f^i(x) = \sum_{m=1}^{\infty} 2^{-m} x = x$$

for every $x \in D$ which completes the proof. □

Proposition. *Let* $a : \mathbb{N} \to [0,\infty]$ *be a non-zero sequence and let* $r \in \mathbb{N}$. *If* $c = a^{(\infty)}$ *and* $b = c^{(r)}$ *then*:

(i) *a positive real number* λ *is a root of equation* (5) *if and only if*

$$\sum_{i=1}^{\infty} b_i \lambda^i = 1; \tag{10}$$

(ii) *if* $D \subset (0, \infty)$ *then every solution of equation* (4) *is a solution of the equation*

$$\sum_{i=1}^{\infty} b_i f^i(x) = x; \tag{11}$$

(iii) supp b *is an additive subsemigroup of* $\{i \in \mathbb{N} : \; i \geq r\}$.
If, in addition, the greatest common divisor of supp a *equals* 1 *then*
(iv) $\{i \in \mathbb{N} : \; i \geq p\} \subset$ supp b *for an integer* $p \geq r$.

Proof. Conditions (i) and (ii) follows immediately from the Lemma. Moreover, equalities (8) and (7) yield

$$\text{supp}\, c = \text{supp}\, a^{(\infty)} = \bigcup_{n=1}^{\infty} \text{supp}\, a^{(n)} = \bigcup_{n=1}^{\infty} \Big(\underbrace{\text{supp}\, a + \ldots + \text{supp}\, a}_{n \text{ times}} \Big)$$

and

$$\text{supp}\, b = \underbrace{\text{supp}\, c + \ldots + \text{supp}\, c}_{r \text{ times}}. \tag{12}$$

Consequently, supp c is an additive subsemigroup of $\mathbb{N}$ so, in particular, also (iii) holds true.

Now assume that the greatest common divisor of supp a is 1. Since supp c − supp c is an additive group of integers,

$$\text{supp}\, c - \text{supp}\, c = d\mathbb{Z}$$

for a $d \in \mathbb{N}$. If $n \in$ supp c then

$$\text{supp}\, c - nd \subset d\mathbb{Z},$$

whence

$$\text{supp}\, a \subset \text{supp}\, c \subset d\mathbb{Z} + nd = d\mathbb{Z}.$$

Consequently $d = 1$, that is

$$\text{supp}\, c - \text{supp}\, c = \mathbb{Z}.$$

In particular,

$$1 \in \; \text{supp}\, c - \text{supp}\, c$$

which means that there is a positive integer q with the property

$$q, q+1 \in \text{supp}\, c. \tag{13}$$

If $q = 1$ then supp $c = \mathbb{N}$ and (cf. (12)) supp $b = \{i \in \mathbb{N} : \; i \geq r\}$. So assume that $q \geq 2$. Since supp c is a semigroup it follows from (13) by induction that

$$iq, iq+1, \ldots, iq+i \in \text{supp}\, c, \qquad i \in \mathbb{N}. \tag{14}$$

Put $s := (q-1)q$ and fix an integer $i \geq s$. Then $i \geq q$, so we can choose numbers $j \in \mathbb{N}$ and $t \in \{0, \ldots, q-1\}$ in such a way that $i = jq + t$. Since $i \geq (q-1)q$ we have $j \geq q-1$, whence $t \in \{0, \ldots, j\}$ and, according to (14), $i \in \operatorname{supp} c$. Therefore we have shown that $\{i \in \mathbb{N} : \; i \geq s\} \subset \operatorname{supp} c$. Putting $p := rs$ and taking into account (12) we see that $\{i \in \mathbb{N} : \; i \geq p\} \subset \operatorname{supp} b$. $\square$

Proof of the Theorem. Assume that $D \subset (0, \infty)$ and let $f : D \to D$ be a solution of (4). Then f satisfies equation (11) with the sequence $b = a^{(\infty)}$. Since $D \subset (0, \infty)$ it follows from (11) that b takes the finite values only. Moreover, by the inclusion supp $a \subset \operatorname{supp} b$ the greatest common divisor of b equals 1. Consequently, by virtue of the Proposition (with $r = 1$) we may additionally assume that

$$\{i \in \mathbb{N} : \; i \geq p\} \subset \operatorname{supp} a \tag{15}$$

for a positive integer p.

We shall prove that

$$\text{the sequence } \left(\frac{f^{(n+1)i}(x)}{f^{ni}(x)} : \; n \in \mathbb{N}_0 \right) \text{ is increasing}, \quad x \in D, \; i \geq p. \tag{16}$$

To this aim fix a point $x \in D$ and an integer $i \geq p$ and define a sequence $\alpha : \mathbb{N}_0^2 \to [0, \infty]$ by

$$\alpha(m, 0) = a_i^m f^{mi}(x)$$

and

$$\alpha(m, n) = \sum_{i_1, \ldots, i_n \in \mathbb{N} \setminus \{i\}} a_i^m a_{i_1} \cdot \ldots \cdot a_{i_n} f^{mi + i_1 + \ldots + i_n}(x)$$

whenever $n \in \mathbb{N}$. In the latter case

$$\alpha(m, n) \leq a_i^m \sum_{i_1, \ldots, i_n \in \mathbb{N}} a_{i_1} \cdot \ldots \cdot a_{i_n} f^{i_1 + \ldots + i_n}\left(f^{mi}(x)\right) = a_i^m f^{mi}(x).$$

Thus, in fact, α is a sequence of finite numbers. If $m \in \mathbb{N}_0$ and $n = 0$ then

$$\begin{aligned}
&\alpha(m+1, n) + \alpha(m, n+1) \\
&\quad = a_i^{m+1} f^{(m+1)i}(x) + \sum_{j \in \mathbb{N} \setminus \{i\}} a_i^m a_j f^{mi+j}(x) \\
&\quad = a_i^m \sum_{j=1}^{\infty} a_j f^j(f^{mi}(x)) = a_i^m f^{mi}(x) = \alpha(m, n)
\end{aligned}$$

and in the case $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$

$$\begin{aligned}
\alpha(m, n+1) &= \sum_{i_1,\dots,i_{n+1}\in\mathbb{N}\setminus\{i\}} a_i^m a_{i_1}\cdot\ldots\cdot a_{i_{n+1}} f^{mi+i_1+\ldots+i_{n+1}}(x) \\
&= \sum_{i_1,\dots,i_n\in\mathbb{N}\setminus\{i\}} a_i^m a_{i_1}\cdot\ldots\cdot a_{i_n} \sum_{j\in\mathbb{N}\setminus\{i\}} a_j f^j\big(f^{mi+i_1+\ldots+i_n}(x)\big) \\
&= \sum_{i_1,\dots,i_n\in\mathbb{N}\setminus\{i\}} a_i^m a_{i_1}\cdot\ldots\cdot a_{i_n}\Big(f^{mi+i_1+\ldots+i_n}(x) - a_i f^i\big(f^{mi+i_1+\ldots+i_n}(x)\big)\Big) \\
&= \alpha(m,n) - \sum_{i_1,\dots,i_n\in\mathbb{N}\setminus\{i\}} a_i^{m+1} a_{i_1}\cdot\ldots\cdot a_{i_n} f^{(m+1)i+i_1+\ldots+i_n}(x) \\
&= \alpha(m,n) - \alpha(m+1,n).
\end{aligned}$$

Therefore it follows from Lemma B that

$$\alpha(1,0)^2 \le \alpha(0,0)\,\alpha(2,0),$$

that is

$$(a_i f^i(x))^2 \le x a_i^2 f^{2i}(x),$$

whence, by (15),

$$(f^i(x))^2 \le x f^{2i}(x).$$

Thus we have shown that

$$\frac{f^i(x)}{x} \le \frac{f^{2i}(x)}{f^i(x)}$$

for all $x \in D$ and $i \ge p$ which implies property (16).

By virtue of (16) the limit

$$\lambda(x,i) = \lim_{n\to\infty}\left[\frac{f^{(n+1)i}(x)}{f^{ni}(x)}\right]^{1/i}$$

exists for every $x \in D$ and $i \ge p$. If $x \in D,\ i \ge p$ and $m \in \mathbb{N}$ then

$$\frac{f^{ni+mi}(x)}{f^{ni}(x)} = \frac{f^{(n+1)i}(x)}{f^{ni}(x)}\cdot\frac{f^{(n+2)i}(x)}{f^{(n+1)i}(x)}\cdot\ldots\cdot\frac{f^{(n+m)i}(x)}{f^{(n+m-1)i}(x)}.$$

Thus

$$\lim_{n\to\infty}\frac{f^{ni+mi}(x)}{f^{ni}(x)} = \lambda(x,i)^{mi}, \qquad x \in D,\ i \ge p,\ m \in \mathbb{N}. \tag{17}$$

We shall verify that the function $\lambda : D \times \{i \in \mathbb{N} : \ i \ge p\} \to (0,\infty]$ does not depend on the second variable. Fix a point $x \in D$ and numbers $i, j \ge p$. Then, by (17),

$$\lim_{n\to\infty}\frac{f^{(n+1)ij}(x)}{f^{nij}(x)} = \lim_{n\to\infty}\frac{f^{in\cdot j+ij}(x)}{f^{in\cdot j}(x)} = \lambda(x,j)^{ij}.$$

Hence, due to the symmetry of the left-hand side, $\lambda(x,i)=\lambda(x,j)$. So in the sequel we shall write $\lambda(x)$ instead of $\lambda(x,i)$.

Now we shall prove that

$$\lambda(f(x))=\lambda(x), \qquad x\in D. \tag{18}$$

Fix a point $x\in D$. On account of (17) we have

$$\lim_{n\to\infty}\frac{f^{np+(p+1)p}(x)}{f^{np}(x)}=\lambda(x)^{(p+1)p}$$

and

$$\lim_{n\to\infty}\frac{f^{np+n+1+p(p+1)}(x)}{f^{np+n+1}(x)}=\lim_{n\to\infty}\frac{f^{n(p+1)+p(p+1)}(f(x))}{f^{n(p+1)}(f(x))}=\lambda(f(x))^{p(p+1)}.$$

Thus, since $(np:\ n\in\mathbb{N})$ and $(np+n+1:\ n\in\mathbb{N})$ have a common subsequence, namely $((np-1)p+np:\ n\in\mathbb{N})$, we obtain

$$\lambda(f(x))^{p(p+1)}=\lambda(x)^{(p+1)p},$$

that is $\lambda(f(x))=\lambda(x)$.

Taking $r:=p$ in the Proposition we obtain a sequence $b:\mathbb{N}\to[0,\infty)$ such that condition (i) holds true, f satisfies equation (11),

$$\{i\in\mathbb{N}:\ i\ge q\}\subset \operatorname{supp} b \tag{19}$$

for an integer $q\ge p$ and, since b is the pth convolution power of a sequence,

$$\operatorname{supp} b\subset\{i\in\mathbb{N}:\ i\ge p\}. \tag{20}$$

Fix a point $x\in D$ and a number $j\in\mathbb{N}_0$. In view of (11) and (20) we have

$$\sum_{i=p}^{\infty} b_i\frac{f^{i+j}(x)}{f^{j}(x)}=1,$$

that is

$$\sum_{k=0}^{\infty}\sum_{m=p}^{2p-1} b_{kp+m}\frac{f^{j+m}(x)}{f^{j}(x)}\cdot\frac{f^{j+p}(f^m(x))}{f^{j}(f^m(x))}\cdot\ldots\cdot\frac{f^{j+kp}(f^m(x))}{f^{j+(k-1)p}(f^m(x))}=1. \tag{21}$$

Replace j by $(2p)!j$ in this equality and take into account property (16). The sequences

$$\left(\frac{f^{(2p)!j+m}(x)}{f^{(2p)!j}(x)}:\ j\in\mathbb{N}_0\right)\quad\text{and}\quad\left(\frac{f^{(2p)!j+lp}(f^m(x))}{f^{(2p)!j+(l-1)p}(f^m(x))}:\ j\in\mathbb{N}_0\right)$$

are increasing (by (17) to $\lambda(x)^m$ and $\lambda(f^m(x))^p$, respectively) for every $m\in\{p,\ldots,2p-1\}$ and $l\in\mathbb{N}$. Thus, tending with j to infinity, we obtain

$$\sum_{k=0}^{\infty}\sum_{m=p}^{2p-1} b_{kp+m}\lambda(x)^m(\lambda(f(x))^p)^k=1.$$

Hence and from (18) and (20) we infer that $\lambda(x)$ is a (positive) root of (10). Consequently, due to Proposition (i), equation (5) has exactly one positive root, say c, and

$$\lambda(x) = c, \quad x \in D.$$

Furthermore, due to (10) and (20), we can rewrite (21) as

$$\sum_{k=0}^{\infty}\sum_{m=p}^{2p-1} b_{kp+m}\left(c^{kp+m} - \frac{f^{j+m}(x)}{f^j(x)}\frac{f^{j+p}(f^m(x))}{f^j(f^m(x))}\cdot\ldots\cdot\frac{f^{j+kp}(f^m(x))}{f^{j+(k-1)p}(f^m(x))}\right) = 0.$$

Now, replacing here j by $(2p)!j$ and taking into account property (16) once again we see that

$$b_{kp+m}\left(c^{kp+m} - \frac{f^{(2p)!j+m}(x)}{f^{(2p)!j}(x)}\prod_{l=1}^{k}\frac{f^{(2p)!j+lp}(f^m(x))}{f^{(2p)!j+(l-1)p}(f^m(x))}\right) = 0$$

for every $j \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $m \in \{p, \ldots, 2p-1\}$. In particular, taking any $k \in \mathbb{N}$ such that $(k+1)p \geq q$ and making use of (19), we obtain

$$c^{kp+m} = \frac{f^{(2p)!j+m}(x)}{f^{(2p)!j}(x)}\prod_{l=1}^{k}\frac{f^{(2p)!j+lp}(f^m(x))}{f^{(2p)!j+(l-1)p}(f^m(x))} \tag{22}$$

for every $j \in \mathbb{N}_0$ and $m \in \{p, \ldots, 2p-1\}$. However, if j increases to infinity then the product occuring in (22) increases to c^{kp} whereas

$$\frac{f^{(2p)!j+m}(x)}{f^{(2p)!j}(x)} \tag{23}$$

increases to c^m. Consequently, (23) does not depend on $j \in \mathbb{N}_0$. In particular,

$$f^m(x) = c^m x, \qquad x \in D,\ m \in \{p, \ldots, 2p-1\}.$$

If $p = 1$ this means that

$$f(x) = cx, \qquad x \in D, \tag{24}$$

otherwise we obtain

$$f^p(x) = c^p x \qquad \text{and} \qquad f^{p+1}(x) = c^{p+1}x, \qquad x \in D.$$

Thus, if $x \in D$ then

$$c^p f(x) = f^p(f(x)) = f^{p+1}(x) = c^{p+1}x.$$

Consequently, if $p \geq 2$ then condition (24) holds true, too. This completes the proof in the case $D \subset (0, \infty)$. If $D \subset (-\infty, 0)$ it is enough to apply the just proved part of the theorem to the function $-D \ni x \mapsto -f(-x)$. $\square$

We finish the paper with the following result where no assumption concerning the support of a is needed.

Corollary. *Let $a : \mathbb{N} \to \mathbb{R}$ be a non-zero sequence of non-negative reals.*

If either $D \subset (-\infty, 0)$, or $D \subset (0, \infty)$ and $f : D \to D$ satisfies equation (4) *then there is exactly one positive root c of equation* (5) *and*

$$f^p(x) = c^p x$$

for every $x \in D$, where p is the greatest common divisor of supp a.

Proof. Putting $b_i = a_{pi}$, $i \in \mathbb{N}$, and $g = f^p$ we see that supp $a = p$ supp b, whence the greatest common divisor of supp b equals 1, and g satisfies the equation

$$\sum_{i=1}^{\infty} b_i g^i(x) = x.$$

Moreover,

$$\sum_{i=1}^{\infty} a_i \lambda^i = \sum_{i=1}^{\infty} a_{pi} \lambda^{pi} = \sum_{i=1}^{\infty} b_i (\lambda^p)^i$$

for every $\lambda \in (0, \infty)$. Therefore the statement follows immediately from the Theorem applied to the sequence b and the function g. □

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