SYSTEMS WITH RANDOM PERTURBATIONS

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ABSTRACT

The behavior of trajectories of random process $x_{n+1} = f(x_n) + \epsilon_n$ is studied where f is a continuous selfmapping of a cartesian product of m compact real intervals I^m, $x_0 \in I^m$ and ϵ_n are independent identically distributed random variables with support in a δ-sphere with center 0. It is shown that similarly as in one dimensional case, any trajectory is eventually transitive with probability 1 in a finite union of compact connected sets. Generic properties of these processes are studied in the 2-dimensional case.

The study of dynamical systems generated by iterates of continuous maps of the interval was motivated by applications in modelling. In numerous situations, model containing random elements seems to be more reasonable than deterministic one, since it can be interpreted as an error in measurement. This leads to systems with random perturbations. This possibility of introduction uncertainty into discrete dynamical systems has been utilized in the papers of Boyarski [2], Ruelle [6] or in the monograph of Lasota and Mackey [5]. Another possibility leading to similar problems originates in the study "dynamical" properties of discrete time random processes, cf., e.g., [3] and [7].

In this paper we use the first approach to study higher dimensional systems with random perturbations. In the following $I^m = [0, 1]^m$. For any continuous selfmapping f of I^m we consider a continuous extension to R^m (this will be also denoted by f) with

$$f(R^m \setminus I^m) \subset f(\mathrm{Fr}\, I^m),$$

where Fr denotes the boundary of a set. For $\delta > 0$ and $r \in I^m$ $S(r, \delta)$ denotes a

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δ-sphere with center r and diameter δ, i.e.,

$$S(r, \delta) = \{z \in I^m : \rho_m(z, r) < \delta\}.$$

The metric ρ_m is the usual Euclidean metric in R^m (if there is no ambiguity, it will be denoted only by ρ).

1.1. Definition. A sequence $\{x_n\}$ of random variables is eventually transitive in a set M with probability 1 if there is m_0 such that $P(x_n \in M$ for all $n > m_0) = 1$ and for any open set $J \subset M$, $P(x_n \in J$ for infinitely many $n) = 1$

For a continuous $f : I^m \to I^m$ and $\delta > 0$ we shall consider the random process $x_{n+1} = f(x_n) + \epsilon_n$ where ϵ_n are identically distributed independent random variables with support in $S(0, \delta)$ and such that for any open set $M \subset S(0, \delta)$ $P(\epsilon_n \in M) > 0$. This process will be called (f, δ)-process.

1.2. Remark. This definition is an analogue of the deterministic concept of transitivity of a function $f : X \to X$, X a compact metric space. A function f is said to be transitive if for every pair of nonempty open sets U and V in X there is a positive integer k such that $f^k(U) \cap V \neq \emptyset$ (see, e.g., [1]). It follows by Lemma 6 from [4] that if f is transitive then the corresponding (f, δ)-process is transitive in X with probability 1.

2. Behavior of trajectories of (f, δ) processes

We shall prove the following

2.1. Theorem. *Any (f, δ) process is eventually transitive with probability 1 in a finite union of compact connected sets of diameter greater or equal to δ.*

For any set $M \subset I^m$ denote

$$B(M) = \bigcap_{m=1}^{\infty} \overline{\bigcup_{n=m}^{\infty} (f^*)^n(M)}, \tag{1}$$

where f^* is a set valued map from I^m to the system $\mathcal{P}(R^m)$ of subsets of R^m defined by

$$f^*(u) = S(f(u), \delta).$$

The proof will follow from a sequence of lemmas.

2.2. Lemma. *For any set $J \subset I^m$ $B(J)$ is a finite union of compact connected sets. Any of these sets contains an open δ-sphere. Moreover, $f^*(B(J)) = B(J)$.*

Proof. By (1) $B(J) = \bigcap_{k=1}^{\infty} B_k(J)$ where $\{B_k(J)\}_{k=1}^{\infty}$ is a nonincreasing sequence of sets and every $B_k(J)$ consists of a finite number of compact connected components which contain an open δ-sphere. This implies the first assertion of the lemma. To show that $f^*(B(J)) = B(J)$ similarly as in the proof of Lemma 6 in [4] we obtain that for any nonincreasing sequence of compact sets $\{A_k\}_{k=1}^{\infty}$, $A_k \subset I^m$ and any compact set $D \subset I^m$ we have

$$f\Big(\bigcap_{k=1}^{\infty} A_k\Big) + D = \bigcap_{k=1}^{\infty}(f(A_k) + D).$$

Now put $A_k = B_k(J)$. We have

$$f^*(B(J)) = f\Big(\bigcap_{k=1}^{\infty} B_k(J)\Big) + S(0, \delta) = \bigcap_{k=1}^{\infty} B_{k+1}(J) = B(J),$$

where we have used that

$$f^*(B_k(J)) = f^*\Big(\bigcup_{i=k}^{\infty} \overline{(f^*)^i(J)}\Big) = \overline{f^*\Big(\bigcup_{i=k}^{\infty}(f^*)^i(J)\Big)} = B_{k+1}(J). \quad \square$$

2.3. Lemma. *Let $\mathcal{Z} = \{B(M), M \subset I^m\}$. $\mathcal{Z}$ is partially ordered with respect to inclusion and contains at least one minimal element. An element $M \in \mathcal{Z}$ is minimal if and only if for any set $H \subset M$ we have $B(H) = M$.*

Proof. Follows by the Zorn Lemma. Let $\{B_t\}_{t \in T}$ be a linearly ordered set, such that if $t_1 < t_2$, then $B_{t_2} \subset B_{t_1}$. Let $B_t = B(M_t)$, where $M_t \subset I^m$ is compact. Put $M = \bigcap_{t \in T} B(M_t)$. Then the set $B(M) \in \mathcal{Z}$ is a lower boundary of the set $\{B_t\}$ because

$$B(M) = B\Big(\bigcap_{t \in T} B(M_t)\Big) = B(B(M_t)) = \bigcap_{t \in T} B(M_t). \quad \square$$

Further we denote by $\mathcal{M}$ the system of all minimal sets of $\mathcal{Z}$.

2.4. Lemma. *Let $B \in \mathcal{M}$, $K \subset B$ a set with nonempty interior* (*such sets exist*, *see Lemma* 2.2). *Then there are a nonnegative integer r and $\lambda \in (0, 1]$ such that for any $x_0 \in B$ we have*

$$P(x_i \in K \textit{ for some } i \le r) \ge \lambda.$$

Proof. The set K contains a sphere $S(v, \sigma)$, $\sigma > 0$, $v \in I^m$. Denote $L = S(v, \frac{1}{3}\sigma)$. Since by Lemma 2.3 for any set $H \subset B$ we have $B(H) = B$ there exists nonnegative integer $k(M)$ minimal such that $(f^*)^{k(M)}(M) \cap L \neq \emptyset$. Let

$$r = \max\{k(H),\, H \subset B \text{ contains a } \delta\text{-sphere}\}.$$

By the continuity of f there is $\eta = \eta(r) > 0$ with the property that if $u_{i_0}, u_{i_1} = f(u_{i_0}) + \xi_{i_0}, \ldots u_{i_s} = f(u_{i_{s-1}}) + \xi_{i_{s-1}}, i \in \{1, 2\}$ are two δ-chains of length $s \leq r$ such that $u_{1_0}, u_{2_0} \in I^m$, $\rho(u_{1_0}, u_{2_0}) < \eta$, $\rho(\xi_{1_j}, \xi_{2_j}) < \eta$, then $\rho(u_{1_s}, u_{2_s}) < \frac{1}{3}\sigma$. Now let $x_0 \in B$. By the definition of r the sphere $S(z_0, \delta)$ contains the first member of a δ-chain $y_0, f(y_0) + \delta_0, f(y_1) + \delta_1 \ldots$ of length $\leq r$ ending in r. Now consider $\lambda = \beta^r$, where $\beta = \min\{\mu(J), J$ contains an open sphere of diameter η and μ is the distribution of $\epsilon_n\}$. Then

$$P(x_i \in K \text{ for some } i \leq r) \geq \beta^r. \quad \square$$

2.5. Lemma. *Let $\mathcal{M} = \{B_1 \ldots B_k\}$, B_i compact connected, $B_i \cap B_j \neq \emptyset$ for $i \neq j$. Then there is nonnegative integer r and $\lambda \in (0, 1]$ such that for any $x_0 \in I^m$*

$$P(x_i \in B_1 \cup \cdots \cup B_k \textit{ for some } i \leq r) \geq \lambda.$$

Proof. Since $f^*(M) = M$, M is a neighbourhood of $V = f(M)$. For any nonempty set L there is a nonnegative integer $k(L)$ such that $(f^*)^{k(L)}(L) \cap V \neq \emptyset$. Put

$$r = \sup\{k(S),\, S \text{ a } \delta\text{-sphere}\} < \infty$$

and take $x_0 \in I^m$. By the definition of r there is in $S(f(x_0), \delta)$ a point y which is the first member of a δ chain of length $\leq r$ ending in V. Let $\eta(r)$ be the same as in Lemma 2.4. Take $\lambda = \sigma^r$, where

$$\sigma = \min\{\mu(S(z, \eta)),\, z \in S(0, \delta)\}.$$

Then

$$P(x_i \in B_1 \cup \cdots \cup B_k \text{ for } i \leq r) \geq \sigma^r > 0. \quad \square$$

Proof of Theorem 2.1. Let $\{x_n\}$ be a trajectory of any (f, δ)-process. By Lemma 2.5 there is nonnegative integer r and $\lambda \in (0, 1]$ with $P(x_i \in \bigcup \mathcal{M} \text{ for } i \leq r) \geq \lambda > 0$. Applying Lemma 6 from [4] we obtain that $P($ there is i with $x_i \in \bigcup \mathcal{M}) = 1$ However, $\mathcal{M}$ is invariant with respect to f^* hence $P($ for some $n_0\, x_n \in \bigcup \mathcal{M}$ for all $n \geq n_0) = 1$ Now let $B \in \bigcup \mathcal{M}$ and $K \subset B$ be any open set. By Lemma 2.4 there are nonnegative integer r and $\lambda \in (0, 1]$ such that

$$P(x_n \in K, n \leq r) \geq \lambda.$$

Applying Lemma 6 from[4] we obtain that

$$P(\text{ for some } n\ x_n \in K) = 1.$$

Hence any trajectory is eventually transitive with probability 1 in $\bigcup \mathcal{M}$ $\square$

3. Generic behavior of (f, δ)-processes

Let $C(I^2, I^2)$ be the space of all continuous functions from I^2 into itself with the metric of uniform convergence. By $\mathcal{S}(f, \epsilon)$ we shall denote an open sphere in $C(I^2, I^2)$ with center f and diameter ϵ. For any $\epsilon > 0$ and $\lambda \in (0, 1)$ consider the set $F(\epsilon, \lambda)$ of all functions $f \in C(I^2, I^2)$ such that there exists δ_0 depending on ϵ that for any $\delta \in (\lambda\delta_0, \delta_0)$ any (f, δ)-process has components of transitivity of diameter less than ϵ.

3.1. Lemma. *$F(\epsilon, \lambda)$ contains an open dense set in $C(I^2, I^2)$.*

Proof. Let $f \in C(I^2, I^2)$ be arbitrary and consider its neighbourhood $\mathcal{S}(f, \xi)$, $\xi > 0$. We shall construct a function $g \in \mathcal{S}(f, \xi) \cap F(\epsilon, \lambda)$ such that there exists a neighbourhood $\mathcal{S}(g, \eta) \subset \mathcal{S}(f, \xi) \cap F(\epsilon, \lambda)$. By the uniform continuity of f there is $\gamma > 0$, $\gamma < \frac{1}{4}\xi$ such that for any $x, y \in I^2$, $\rho(x, y) < \gamma$ we have $\rho(f(x), f(y)) < \frac{1}{4}\xi$. Put $\gamma_1 = \min\{\gamma, \frac{1}{2}\epsilon\}$. Consider the division of I^2 on squares and rectangles as depicted on Fig. 1 so that the edge of any square and the shorter edge of a rectangle is less than γ_1. Denote by S_i, $i = 1 \dots R$ colored squares and c_i corresponding centres. We shall define mapping $g : I^2 \to I^2$ in the following way: If x belongs to a colored square, say S_i, put $g(x) = c_j$,where c_j is a center of the colored square with the property

$$\operatorname{dist}(f(S_i), c_j) = \min_{1 \le k \le R} \operatorname{dist}(f(S_i), c_k).$$

Outside of colored squares we define g arbitrarily so that$g \in \mathcal{S}(f, \frac{1}{2}\xi)$. It is possible since for $x \in \bigcup_{i=1}^{R} S_i$ we have

$$\rho(f(x), g(x)) \le \tfrac{1}{4}\xi + \tfrac{1}{4}\xi = \tfrac{1}{2}\xi.$$

Now let $h \in C(I^2, I^2) \cap \mathcal{S}(g, \frac{1}{2}\xi)$. We show that any (h, δ)-process has components of transitivity of diameter less than ϵ for $\delta \in (\lambda\gamma_1, \gamma_1)$. Really, if x_0 belongs to $\bigcup_{i=1}^{R} S_i$, then $h(x_0) \in \bigcup_{i=1}^{R} S_i$ and also $x_1 \in \bigcup_{i=1}^{R} S_i$. Similarly, $x_n \in \bigcup_{i=1}^{R} S_i$ for any n. Hence

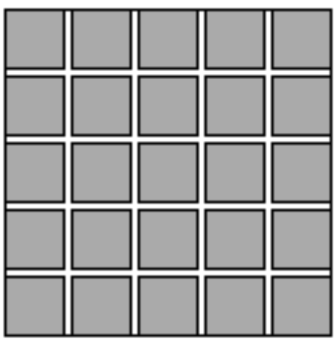

Fig. 1. The case when $r = 25$

components of transitivity have diameters less than $\gamma_1 < \epsilon$. If x_0 belongs to $I \setminus \bigcup_{i=1}^{R} S_i$, then $P(x_1 \in \bigcup_{i=1}^{R} S_i) \geq \omega > 0$. By Lemma 6 in [4] $P(x_n \in \bigcup_{i=1}^{R} S_i$ for all $n \geq n_0) = 1$. □

Consider the set F of all functions $f \in C(I^2, I^2)$ such that for any $\epsilon > 0$ there is δ_0 that for any $\delta \in (0, \delta_0)$ any (f, δ)-process has components of transitivity of diameter less than ϵ. By the previous lemma F contains a G_δ set dense in $C(I^2, I^2)$. We obtain the following

3.2. Corollary. *Generically, functions in $C(I^2, I^2)$ have the following property: For any $\epsilon > 0$ there is $\delta > 0$ such that any (f, δ)-process has components of transitivity of diameter less than ϵ.*

3.3. Remark. The set F does not contain an open dense set, since its complement is dense in $C(I^2, I^2)$, as the following example shows.

Example. Let $\mathcal{S}(f, \eta)$ be arbitrary open sphere in $C(I^2, I^2)$. Let $p \in I^2$ be fixed point of f. Take $\delta > 0$ from the uniform continuity of f such that $\rho(f(x), f(y)) < \eta$ whenever $\rho(x, y) < \delta$ and consider a square S with diameter $\frac{1}{2}\delta$ in I^2 such that $p \in S$. Define $g : I^2 \to I^2$ so that $g(x) = x$ for all $x \in S$. Evidently, any (g, δ)-process has components of transitivity of diameter $\delta + \frac{1}{2}\delta$.

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