SOME RESULTS ON ESSENTIALLY ALGEBRAIC COMPOSITION OPERATORS

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ABSTRACT

An operator C_a defined by the rule $(C_a f)(x) = f(a(x))$ on some function space is called a composition operator. We consider the question which composition operators are algebraic or essentially algebraic, i.e., for which composition operators there is a polynomial that makes the operator into the zero operator or into a compact operator. We also show how some properties of the operator and the function space can be obtained knowing these polynomials. Our main concern in this paper are operators on the function spaces l^p with weight, $C(X)$ and H^2.

1. Introduction

This paper consists of two parts. In the first part, constituting section 2, we give an overview of some results for composition operators on the weighted discrete Lebesgue spaces l^p and on the space of continuous functions $C(X)$ which have already been published. We only mention some interesting issues without proof and refer to the papers [3] and [4] for a more detailed representation of these facts. The main part of this paper contains results for composition operators on the Hardy space H^2 of analytic functions on the unit disk $\boldsymbol{D}$. These results are published here for the first time.

At first we wish to give some important definitions. Let F be a linear space of complex-valued functions on a set X. A mapping $a : X \to X$ with the property that $f \circ a$ belongs to F for all $f \in F$ defines a composition operator $C_a : F \to F$ by

$$(C_a f)(x) = f(a(x)), \quad \text{i.e.,} \quad C_a f = f \circ a.$$

A composition operator is said to be *algebraic* or *essentially algebraic* if there is a nonzero polynomial $p(z) = \sum_{i=0}^{n} p_i z^i$ such that $p(C_a) = \sum_{i=0}^{n} p_i C_a^i = 0$ or a nonzero polynomial $q(z)$ with $q(C_a) \in \mathcal{K}$, respectively. Here and in the following $\mathcal{K}$ stands for the ideal of compact operators. The polynomial of lowest degree and leading coefficient one is called *characteristic polynomial* $p_a(z)$ or *essentially characteristic polynomial* $q_a(z)$, respectively. We remark that the concept of algebraicity generalizes, e.g., that one of projectors and Carleman shifts. A projector is an operator P with $P^2 = P$, i.e., $p(P) = 0$ for $p(z) = z^2 - z$, while a composition operator is called a *Carleman shift* if $C_a^n = I$ for some $n \in \mathbb{N}$ (i.e., $p(C_a) = 0$ with $p(z) = z^n - 1$). We now define an *essential Carleman shift* to be a composition operator for which $C_a^n - I \in \mathcal{K}$ for some $n \in \mathbb{N}$ and call an operator with coinciding characteristic and essentially characteristic polynomial of the form $z^n - 1$ an *irreducible Carleman shift*. In other words, C_a is an irreducible Carleman shift if and only if $C_a^n - I = 0$ for some $n \in \mathbb{N}$ and no polynomial of lower degree makes C_a compact. In the paper [2] we gave some results on algebraic composition operators and part of them will be generalized here to the case of essentially algebraic composition operators. Further we remark that this concept contains the question of the compactness of composition operators. (Notice that an operator is compact if and only if its essentially characteristic polynomial equals the polynomial $q(z) = z$.)

The space l^p is considered as the space of functions $f : \mathbb{N} \to \mathbb{C}$ with the norm

$$\|f\|_p = \left(\sum_{x\in\mathbb{N}} \mu(x)|f(x)|^p\right)^{1/p} \qquad (1 \le p < \infty),$$

$$\|f\|_\infty = \sup_{x\in\mathbb{N}} \mu(x)|f(x)| \qquad (p = \infty),$$

where $\mu : \mathbb{N} \to (0, \infty)$ is a weight function. By $C(X)$ we mean the usual space of continuous functions on a compact Hausdorff space X and the Hardy space H^2 is defined to be the space of analytic functions f on the open unit disc $\boldsymbol{D}$ for which the norm defined by

$$\|f\|_{H^2}^2 = \sup_{r\to 1-0} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \, d\theta = \sup_{r\to 1-0} \int_{\boldsymbol{T}} |f(r\tau)|^2 \, d\lambda(\tau)$$

is finite. Here $\lambda(\tau)$ is normalized Lebesgue measure on the unit circle $\boldsymbol{T}$. For the norm of an analytic function $f(z) = \sum_{j=0}^{\infty} f_j z^j$ it is well known that

$$\|f\|_{H^2}^2 = \sum_{j=0}^{\infty} |f_j|^2.$$

The first question occuring is which self-maps $a : X \to X$ define a bounded composition operator. It is well known that a self-mapping a of X induces a composition operator on $C(X)$ if and only if a is continuous and on H^2 if and only if a is analytic. In the case of l^p the question is more difficult and depends on the weight (see, e.g.,

[13] for $p = 2$ and [21, Lemma 5.1.3] or [1, Theorem 1.1] for arbitrary p). However in all cases a composition operator is automatically bounded.

Operator theoretical properties of composition operators on H^2 such as spectra, compactness, norm, ... are well investigated (e.g., by C.C. Cowen, H. Kamowitz, E.A. Nordgren, J.H. Shapiro, C. Sundberg and P.D. Taylor; see [6], [7], [8], [12], [13], [16], [17], [18] and [19]). J.H. Shapiro [16] solved the problem of deciding whether a composition operator is compact or not and he gave a formula for the essential norm of any composition operator. The spectra of a composition operator were investigated by C.C. Cowen [6] and H. Kamowitz [12], but until now there are unsolved cases, which illustrates the difficulty of our subject.

2. The spaces l^p and $C(X)$

For $x \in \mathbb{N}$ or $x \in X$ we define

$$\begin{aligned} \mathrm{ent}(x) &= \min\{k \in \mathbb{N} \cup \{0\} : a^k(x) = a^{k+m}(x) \text{ for some } m \in \mathbb{N}\}, \\ \mathrm{cyc}(x) &= \min\{m \in \mathbb{N} : a^{\mathrm{ent}(x)+m}(x) = a^{\mathrm{ent}(x)}(x)\}, \\ \mathrm{ent}\,(a) &= \sup\nolimits_{x \in X} \mathrm{ent}(x), \\ \mathrm{Per}\,(a) &= \{\mathrm{cyc}(x) : x \in X\}. \end{aligned}$$

It turns out that merely knowing ent (a) and Per (a), which are defined using only the orbit structure of the self-mapping a, enables us to decide if the composition operator C_a is algebraic and to determine its characteristic polynomial.

2.1. Theorem. *For a composition operator $C_a \in \mathcal{L}(C(X))$ or $C_a \in \mathcal{L}(l^p)$ we have: C_a is algebraic if and only if* ent $(a) < \infty$ *and* $|\mathrm{Per}\,(a)| < \infty$, *where* $|\mathrm{Per}\,(a)|$ *stands for the number of elements of the set* Per (a). *In this case, the characteristic polynomial of C_a is*

$$p_a(z) = z^n \prod_{\lambda \in G} (z - \lambda), \tag{1}$$

where $n =$ ent (a) *and*

$$G = \bigcup_{k \in \mathrm{Per}(a)} \boldsymbol{G}_k, \quad \boldsymbol{G}_k := \{\lambda \in \mathbb{C} : \lambda^k = 1\}.$$

In a similar (but more complicated) way one may define a number went(a) and a set Wper (a); see [3]. Then an analogous theorem can be stated for essentially algebraic composition operators.

2.2. Theorem. *For a composition operator $C_a \in \mathcal{L}(C(X))$ or $C_a \in \mathcal{L}(l^p)$ we have: C_a is essentially algebraic if and only if* went$(a) < \infty$ *and* $|\mathrm{Per}\,(a)| < \infty$. *In this* *case, the essentially characteristic polynomial of* C_a *is*

$$q_a(z) = z^m \prod_{\lambda \in H} (z - \lambda), \tag{2}$$

with $m = \mathrm{went}(a)$ *and*

$$H = \bigcup_{k \in \mathrm{Wper}(a)} \boldsymbol{G}_k, \quad \boldsymbol{G}_k := \{\lambda \in \mathbb{C} : \lambda^k = 1\}.$$

One interesting difference between the spaces l^p and $C(X)$ is the following fact.

2.3. Theorem. *Every essentially algebraic composition operator* $C_a \in \mathcal{L}(C(X))$ *is algebraic, but an essentially algebraic composition operator on* $\mathcal{L}(l^p)$ *need not necessarily be algebraic.*

In the case of $C(X)$ the situation seems a little bit boring, but this does not mean that the characteristic and essentially characteristic polynomials have to coincide. This opens a gate for a surprising fact. We remark that obviously q_a is a divisor of p_a, and hence we may define the list

$$\boldsymbol{L}_Q(X) = \{p_a(z)/q_a(z) : C_a \text{ is algebraic}\}$$

of all quotients of characteristic and essentially characteristic polynomials which can occur for algebraic composition operators. It turns out that this list contains all information about the connectivity of X.

2.4. Theorem. *Suppose* X *is connected and* C_a *is algebraic on* $C(X)$. *Then either* $q_a(z) = p_a(z)$ *or* $q_a(z) = z^m$ *and* $p_a(z) = z^m(z-1)$ *for some* $m \in \mathbb{N}$. *In the latter case* a^m *is a contraction of* X *to a single point. Thus* $\boldsymbol{L}_Q(X) = \{1, z-1\}$.

2.5. Theorem. *Let* $\varkappa(X) \in \mathbb{N} \cup \{\infty\}$ *denote the number of connected components of* X. *Then*

$$z^k - 1 \in \boldsymbol{L}_Q(X) \iff \varkappa(X) \geq k.$$

In other words, $\varkappa(X)$ *equals the number of different polynomials of the form* $z^l - 1$ $(l \geq 1)$ *in* $\boldsymbol{L}_Q(X)$.

3. Preliminaries concerning H^2

Now we come back to the Hardy space H^2 and collect some known facts and definitions which are helpful for our aims.

3.1. Linear fractional mappings and inner functions. A mapping of the form

$$a(z) = \frac{cz + d}{ez + f}$$

is called a *linear fractional mapping*. In this paper we also admit constant mappings a, which arise if and only if $cf - ed = 0$. Of course the case $e = f = 0$ has to be avoided. We remark that linear fractional mappings are frequently called *Möbius transformations* and *linear fractional transformations* if they are not constant.

A self-mapping $a : \mathbf{D} \to \mathbf{D}$ is said to be a *generalized rotation* or an *elliptic automorphism* if it is of the form

$$a(z) = b^{-1}(\alpha b(z)), \quad \text{with} \quad b(z) = \frac{z - \lambda}{1 - z\bar{\lambda}},$$

$\alpha \in \mathbf{T}$ and $\lambda \in \mathbf{D}$. We remark that a is obtained from an ordinary rotation around the origin by the angle $\arg(\alpha)$ moving the origin conformly into λ, the fixed point of a. We say that the generalized rotation a has the *period* k if $\alpha^k = 1$ and $\alpha^m \neq 1$ for $0 < m < k$. A bounded analytic function on the unit disk $\mathbf{D}$ is called an *inner function* if it has radial limits of modulus 1 almost everywhere on the boundary. Note that generalized rotations are inner functions and linear fractional mappings.

3.2. Logarithmic integrability of the radial limit. [10, Theorem 2.2] For each function f analytic in the unit disc which is the quotient of two bounded analytic functions (i.e., for each f in the Nevanlinna class) the nontangential limit $f(e^{i\theta})$ exists almost everywhere, and $\log|f(e^{i\theta})|$ is integrable unless $f(z) \equiv 0$. If furthermore $f \in H^p$ for some $p > 0$, then $f(e^{i\theta}) \in L^p$ and

$$\|f\|_{H^2}^2 = \int_{\mathbf{T}} |f(\theta)|^2 \, d\lambda(\theta) = \|f\|_{L^2}^2. \tag{3}$$

3.3. Reproducing kernels. (See, e.g., [18, p.6]) Because H^2 is a Hilbert space and pointwise evaluation is continuous, we may with every $\omega \in \mathbf{D}$ associate a function $k_\omega \in H^2$ with the property

$$f(\omega) = \langle f, k_\omega \rangle = \int_{\mathbf{T}} f(\theta)\overline{k_\omega(\theta)} \, d\lambda(\theta). \tag{4}$$

These functions, the so-called *reproducing kernels*, play an important role and are well investigated. One has

$$k_\omega(z) = \frac{1}{1 - \bar{\omega} z} \in H^2 \tag{5}$$

and obviously,

$$\|k_\alpha\|^2 = \langle k_\alpha, k_\alpha \rangle = k_\alpha(\alpha) = \frac{1}{1 - |\alpha|^2}. \tag{6}$$

It follows that

$$\langle C_a^* k_\omega,\ f\rangle = \langle k_\omega, C_a f\rangle = \overline{\langle C_a f, k_\omega\rangle} = \overline{f(a(\omega))} = \overline{\langle f, k_{a(\omega)}\rangle} = \langle k_{a(\omega)},\ f\rangle$$

for all f and therefore,

$$C_a^* k_\omega = k_{a(\omega)}. \tag{7}$$

One interesting property of reproducing kernels is the following fact. In [13, p. 46] and [20, p. 783] it is shown that multiplication operators on H^2 are characterized by their adjoints having the reproducing kernels as eigenvectors, while an operator A on H^2 is a composition operator if and only if the set $\{k_\alpha : \alpha \in \boldsymbol{D}\}$ is invariant under A^*. In this latter case the mapping $a : \boldsymbol{D} \to \boldsymbol{D}$ is defined by

$$A^* K_\alpha = K_{a(\alpha)}$$

according to (7).

We call a point $w \in \boldsymbol{T}$ a *boundary fixed point* if $\lim_{r\to 1-0} a(rw) = w$ (i.e., if w is a fixed point in the sense of radial limits).

3.4. Theorem. *If an analytic function a, which is not an generalized rotation, maps $\boldsymbol{D}$ into $\boldsymbol{D}$, then there is a unique fixed point α in $\bar{\boldsymbol{D}}$ with $|a'(\alpha)| \le 1$. One has $a^n \to \alpha$ uniformly on compact subsets of $\boldsymbol{D}$. Furthermore, if $\alpha \in \boldsymbol{T}$ then $0 < a'(\alpha) \le 1$, and for any fixed point $\beta \in \boldsymbol{D}$ the inequality $|a'(\beta)| < 1$ holds.*

Proof. Parts of the theorem can be found in [8, p.5], [9] or [22]. A nice exposition of the whole topic is in [17, p.78]. □

The unique fixed point mentioned in the theorem is called the *Denjoy–Wolff point* of the mapping a. We remark that the Denjoy–Wolff point is defined only for mappings different from generalized rotations; also notice that some authors refer to α as the Denjoy–Wolff point only the case where it is located on the unit circle $\boldsymbol{T}$.

The Denjoy–Wolff point reflects some interesting properties of the self-mapping and the composition operator. The next lemma, which will be needed later, gives one example of this.

3.5. Lemma. *Let a be an analytic self-mapping of $\boldsymbol{D}$, which is not a generalized rotation, and let α denote its Denjoy–Wolff point.*

1. *If $|\alpha| = 1$ and $a'(\alpha) < 1$ then C_a is not essentially algebraic.*
2. *If a is analytic in $G \supset \bar{\boldsymbol{D}}$, not inner, $|\alpha| < 1$, and $C_a^k \notin \mathcal{K}\ \forall k \in \mathbb{N}$, then C_a is not essentially algebraic.*

Proof. In the first case a theorem by Cowen [6, Theorem 4.5.] gives that all λ with $a'(\alpha)^{1/2} < |\lambda| < a'(\alpha)^{-1/2}$ are eigenvalues of C_a. Therefore for every nonconstant polynomial p the spectrum $\sigma(p(C_a))$ is uncountable and hence $p(C_a)$ not compact.

Theorem 3.4 in [12] shows us that in the case considered in the second item there is a $c > 0$ with $\sigma(C_a) \supset \{\lambda : |\lambda| \le c\}$. As above we see that C_a cannot be essentially algebraic. □

3.6. Lemma. (Cowen [7, p. 151]) *A composition operator C_a generated by a linear fractional mapping a is compact if and only if a maps $\bar{\boldsymbol{D}}$ into $\boldsymbol{D}$.*

3.7. Lemma. (E.A. Nordgren [13, p. 57, Theorem 14]) *If for any composition operator C_a on H^2 there is a N such that C_a^N is compact, then a has a fixed point in $\boldsymbol{D}$.*

The following very nice lemma is due to J.H. Shapiro and C. Sundberg [18, Theorem 2.3, p. 13]. This lemma is very useful for our aims. That's why we will not omit the proof.

3.8. Lemma. *Suppose $a_1, a_2, \dots, a_n$ are* distinct *holomorphic self-mappings of $\boldsymbol{D}$, and $\alpha_1, \alpha_2, \dots, \alpha_n$ are complex scalars. Then*

$$\Big\| \sum_{j=1}^n \alpha_j C_{a_j} \Big\|_{\text{ess}}^2 \ge \sum_{j=1}^n |\alpha_j|^2 \lambda(E(a_j)),$$

where $E(a) = \{\theta \in \boldsymbol{T} : |\lim_{r\to 1-0} a(r\theta)| = 1\}$.

Proof. We denote by

$$f_\beta = \frac{k_\beta}{\|k_\beta\|}$$

the "normalized reproducing kernel" for the point $\beta \in \boldsymbol{D}$. Thus $\|f_\beta\| = 1$, and due to (6)

$$f_\beta(z) = \frac{\sqrt{1-|\beta|^2}}{1-\bar{\beta}z} \quad (z \in \boldsymbol{D}).$$

We have

$$\begin{aligned} \Big\| \sum_{j=1}^n \alpha_j C_{a_j} \Big\|_{\text{ess}} &= \inf_{K\in\mathcal{K}} \sup_{\|f\|=1} \Big\| \Big(\sum_{j=1}^n \alpha_j C_{a_j} + K \Big) f \Big\| \\ &\ge \inf_{K\in\mathcal{K}} \limsup_{r\to 1-0} \sup_{|\theta|=1} \Big(\Big\| \sum_{j=1}^n \alpha_j C_{a_j} f_{r\theta} \Big\| - \big\| K f_{r\theta} \big\| \Big). \end{aligned} \tag{8}$$

Since the family $\{f_\beta : \beta \in \boldsymbol{D}\}$ is bounded in H^2 and since $f_\beta \to 0$ uniformly on compact subsets of $\boldsymbol{D}$ as $|\beta| \to 1-0$, we have

$$\|K f_{r\theta}\| \to 0 \qquad \forall K \in \mathcal{K}(H^2),$$

and so (8) leads to

$$
\begin{aligned}
\Big\|\sum_{j=1}^{n}\alpha_j C_{a_j}\Big\|_{\text{ess}}^2 &\ge \limsup_{r\to 1-0}\sup_{|\theta|=1}\Big\|\sum_{j=1}^{n}\alpha_j C_{a_j} f_{r\theta}\Big\|^2\\
&\ge \limsup_{r\to 1-0}\int_{\boldsymbol{T}}\Big\|\sum_{j=1}^{n}\alpha_j C_{a_j} f_{r\theta}\Big\|^2\,d\lambda(\theta)\\
&= \limsup_{r\to 1-0}\int_{\boldsymbol{T}}\Big\langle\sum_{i=1}^{n}\alpha_i C_{a_i} f_{r\theta},\sum_{j=1}^{n}\alpha_j C_{a_j} f_{r\theta}\Big\rangle\,d\lambda(\theta)\\
&= \limsup_{r\to 1-0}\sum_{i,j=1}^{n}\alpha_i\overline{\alpha_j}\int_{\boldsymbol{T}}\langle C_{a_i} f_{r\theta}, C_{a_j} f_{r\theta}\rangle\,d\lambda(\theta).
\end{aligned}
\tag{9}
$$

Further we have

$$
\begin{aligned}
\langle C_{a_i} f_{r\theta}, C_{a_j} f_{r\theta}\rangle &= (1-r^2)\langle C_{a_i}k_{r\theta}, C_{a_j}k_{r\theta}\rangle\\
&= (1-r^2)\int_{\boldsymbol{T}}\frac{d\lambda(\tau)}{(1-r\bar{\theta}a_i(\tau))(1-r\theta\overline{a_j(\tau)})}\\
&= (1-r^2)\int_{\boldsymbol{T}}\overline{k_{ra_i(\tau)}(\theta)}k_{ra_j(\tau)}(\theta)\,d\lambda(\tau),
\end{aligned}
$$

and hence due to Fubini's Theorem and (4)

$$
\begin{aligned}
&\int_{\boldsymbol{T}}\langle C_{a_i} f_{r\theta}, C_{a_j} f_{r\theta}\rangle\,d\lambda(\theta)\\
&\quad= (1-r^2)\int_{\boldsymbol{T}}\int_{\boldsymbol{T}}\overline{k_{ra_i(\tau)}(\theta)}k_{ra_j(\tau)}(\theta)\,d\lambda(\tau)\,d\lambda(\theta)\\
&\quad= (1-r^2)\int_{\boldsymbol{T}}\langle k_{ra_j(\tau)}, k_{ra_i(\tau)}\rangle\,d\lambda(\tau)\\
&\quad= (1-r^2)\int_{\boldsymbol{T}}k_{ra_j(\tau)}(ra_i(\tau))\,d\lambda(\tau)\\
&\quad= \int_{\boldsymbol{T}}\frac{1-r^2}{1-r^2a_i(\tau)\overline{a_j(\tau)}}\,d\lambda(\tau).
\end{aligned}
$$

The integrand in the last line is bounded by 1, and as $r\to 1-0$ it converges to the characteristic function of the set

$$
E_{ij}\overset{\text{def}}{=}\Big\{\tau\in\boldsymbol{T}:\lim_{r\to 1-0}a_i(r\tau)\overline{a_j(r\tau)}=1\Big\}.
$$

Thus the Lebesgue Dominated Convergence Theorem yields

$$
\lim_{r\to 1-0}\int_{\boldsymbol{T}}\langle C_{a_i} f_{r\theta}, C_{a_j} f_{r\theta}\rangle\,d\lambda(\theta)=\lambda(E_{ij}),
$$

and (9) gives

$$\Big\|\sum_{j=1}^{n} \alpha_j C_{a_j}\Big\|_{\text{ess}}^2 \geq \sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \lambda(E_{ij}).$$

If $i = j$ then we have $E_{ij} = E(a_j)$ and if $i \neq j$, then $a_i \neq a_j$, and since by Lemma 3.2 the radial limits of two different bounded holomorphic functions cannot coincide on a subset of positive measure of $\boldsymbol{T}$, we get $\lambda(E_{ij}) = 0$. Connecting these facts with the last inequality completes the proof. □

The following criteria for compactness or non-compactness are well known. They are not very sharp but easy to check.

3.9. Lemma. *Let $a : \boldsymbol{D} \to \boldsymbol{D}$ be an analytic mapping.*

1) *If a maps the unit disc into a polygon inscribed in the unit circle, then C_a is compact on H^2.*
2) *If a is a conformal mapping which extends continuously to the closure $\overline{\boldsymbol{D}}$ of $\boldsymbol{D}$ and $\lambda(a(\boldsymbol{T}) \cap \boldsymbol{T}) > 0$ then C_a is not compact.*
3) *If a is analytic in some neighborhood of a point $w \in \boldsymbol{T}$ with $a(w) \in \boldsymbol{T}$, then C_a is not compact.*

Proof. Part 1) can, e.g., be found in [17, 2.3] as the Polygone Compactness Theorem. To prove part 2) we may use a corollary from [17, 2.5] which says: If a is a conformal mapping of $\boldsymbol{D}$ such that $a(\boldsymbol{D})$ contains a disc in $\boldsymbol{D}$ that is tangent to the unit circle, then C_a is not compact. Obviously these assumtions are fulfilled in our case. The part 3) comes from the Angular Derivative Criterion (see [17, 4.2]) because our assumptions guarantee the existence of the angular derivative at the point w. □

3.10. Lemma. *If p_a is a characteristic polynomial for some composition operator on H^2, then either $p_a(z) = z^n - 1$ for some $n \in \mathbb{N}$ or $p_a(z) = z^2 - z$, the last case occuring if and only if a is constant.*

Proof. This lemma is proved in [2] (see Theorem 3.1 and its proof). □

3.11. Lemma. *Let C_a be a composition operator on H^2. If there is a polynomial q with $q(0) \neq 0$ such that $q(C_a) \in \mathcal{K}$, then C_a is an irreducible Carleman shift and a is a generalized rotation.*

Proof. First we see that there are at least two coinciding iterates $a^m = a^{m+k}$ $(k > 0)$ of a, since otherwise Lemma 3.8 with $\lambda(E(\text{id})) = 1$ would imply that $\|q(C_a)\|_{\text{ess}}^2 \geq |q(0)|^2 > 0$. Let's assume first that a^m is not constant. Then $a^m(\boldsymbol{D})$ is open and a^k has to be the identical mapping. Let n denote the smallest positive integer for which

$a^n = \mathrm{id}$. Any proper divisor $p(z)$ of $z^n - 1$ does not have the root 0 and is of degree less than n. Lemma 3.8 gives that $\|p(C_a)\|^2_{\mathrm{ess}} \geq |p(0)|^2 > 0$ and therefore $p(C_a)$ is not compact. Theorem 3.4 shows that a is a generalized rotation, because otherwise $a^n \to \alpha$ on compact subsets of $\boldsymbol{D}$, which contradicts $a^n = \mathrm{id}$.

We are left with the case where a^m is constant, which means that a itself is constant. Then, by Lemma 3.9, C_a^j is compact for all $j > 0$. Therefore $q(C_a)$ is compact if and only if $q(0)I$ is compact (as a sum of compact operators). But $q(0)I$ is not compact because $q(0) \neq 0$. □

4. Main results

4.1. Theorem. *For each polynomial $q(z) \in \{z^k, z^k - 1 : k \in \mathbb{N}\}$ there is an analytic self-mapping a of $\boldsymbol{D}$ defining a composition operator $C_a : H^2 \to H^2$ with the essentially characteristic polynomial $q_a = q$.*

Proof. The rotation $a(z) = e^{2\pi i/k}z$ satisfies $a^k(z) = z$ and $C_a^k - I = 0 \in \mathcal{K}$ for any $k \in \mathbb{N}$. A divisor $p(z)$ of $q(z) = z^k - 1$ does not have the root 0, so Lemma 3.8 gives

$$\|p(C_a)\|^2_{\mathrm{ess}} \geq |p(0)|^2 \lambda(E(\mathrm{id})) = |p(0)|^2 > 0$$

and hence q is the essentially characteristic polynomial of C_a.

We are left with the much more difficult construction of a self-mapping a with the essentially characteristic polynomial z^k. The case $k = 1$ is trivial. One can choose $a(z) = \frac{1}{2}z$, Lemma 3.6 gives that C_a is compact, and therefore the essentially characteristic polynomial of C_a is $q_a(z) = z$. That's why we can restrict ourselves to the case $k \geq 2$. We prepare the definition of a by several steps.

The first step is to choose an open region G with the following properties:

1. G is bounded by an infinitely smooth curve ∂G, simply connected and connected,
2. $G \subset \boldsymbol{D}$, and
3. $\overline{G} \cap \boldsymbol{T} = \widehat{1i} = \{e^{i\theta} : \theta \in [0, \frac{1}{2}\pi]\}$

Figure 1 shows an example of such a set G.

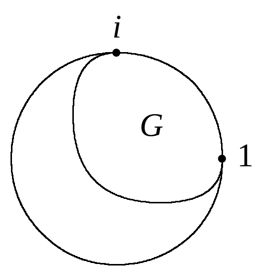

Fig. 1.

By virtue of the Riemann mapping theorem there is a conformal mapping $\tilde{\varphi}$: $\boldsymbol{D} \to G$ which extends continuously to $\tilde{\varphi} : \overline{\boldsymbol{D}} \to \overline{G}$ due to the following theorem of Caratheodory (see, e.g., [5, p. 88, no. 345]):

> If there is a bijective continuous mapping with continuous inverse between the boundary of some simply connected area G and the unit circle, then every conformal mapping which maps G onto $\boldsymbol{D}$ realizes such a mapping of the boundaries.

Now it is possible to find a linear fractional mapping μ such that for $\varphi(z) = \tilde{\varphi}(\mu(z))$ the equalities

$$\varphi(1) = 1, \qquad \varphi(i) = i \qquad \text{and} \qquad \varphi(\widehat{1i}) = \widehat{1i}$$

hold. Our strategy is as follows: For each fixed $k \geq 2$ we wish to find some angle $\theta \in [0, 2\pi)$ such that

$$\varphi_\theta(z) = \varphi(e^{i\theta} z)$$

gives a composition operator with essentially characteristic polynomial z^k. In order to do this we parametrize ∂G by

$$g : [0, 2\pi) \to \partial G \qquad g(\theta) = \begin{cases} e^{i\theta}, & \theta \in [0, \frac{1}{2}\pi], \\ \varphi(e^{i\theta}), & \theta \in (\frac{1}{2}\pi, 2\pi). \end{cases}$$

If we use on $[0, 2\pi)$ the topology obtained by identifying $[0, 2\pi)$ with the complex unit circle $\boldsymbol{T}$, then the function g is continuous, bijective and has a continuous inverse. Further we denote by φ_θ^k the k-th iterates of φ_θ,

$$\varphi_\theta^k(z) = \varphi_\theta(\varphi_\theta^{k-1}(z)) \qquad (\varphi_\theta^0(z) = \mathrm{id}),$$

and we define functions f^k by

$$f^k(\theta) = g^{-1}(\varphi_\theta^k(1))$$

for those $\theta \in [0, 2\pi)$ with $\varphi_\theta^k(1) \in \partial G$. Our aim is to show that there is a strongly decreasing sequence $\{\theta_0, \theta_1, \theta_2, \ldots\}$ with the following properties: For every $j \in \mathbb{N}$ one has

1. $\theta_j \in (0, 2\pi)$,
2. f^j is a continuous, bijective, strongly increasing function defined on $[0, \theta_{j-1})$ and mapping into $[0, 2\pi)$,
3. $f^j(\theta_j) = \frac{1}{2}\pi$,
4. $f^j(\theta) \in (\frac{1}{2}\pi, 2\pi) \qquad \forall \theta \in (\theta_j, \theta_{j-1})$,
5. $f^j(\theta) \in [0, \frac{1}{2}\pi) \qquad \forall \theta \in [0, \theta_j)$.

The existence of such a sequence will now be proved by induction. Defining $\theta_0 = 2\pi$ and $\theta_1 = \frac{1}{2}\pi$ we have $\varphi_\theta^1(1) = \varphi(e^{i\theta}) \in \partial G\ \forall\theta \in [0, 2\pi)$. The function f^1 is defined on $[0, \theta_0) = [0, 2\pi)$, is continuous and bijective as a composition of continuous and bijective functions and hence strongly monotonic. Obviously f is not decreasing but increasing. Further we have

$$f(\theta_1) = f(\tfrac{1}{2}\pi) = g^{-1}(\varphi(i)) = g^{-1}(i) = \tfrac{1}{2}\pi,$$
$$f(0) = g^{-1}(\varphi(1)) = g^{-1}(1) = 0,$$

and because of monotony,

$$f(\theta) \in (\tfrac{1}{2}\pi, 2\pi) \qquad \forall\theta \in (\tfrac{1}{2}\pi, 2\pi) = (\theta_1, \theta_0),$$
$$f(\theta) \in [0, \tfrac{1}{2}\pi) \qquad \forall\theta \in [0, \tfrac{1}{2}\pi) = [0, \theta_1).$$

Now we assume these properties for $j - 1$ and wish to prove them for j. For all $\theta \in [0, \theta_{j-1})$ we know that $f^{j-1}(\theta) \in [0, \frac{1}{2}\pi)$ and hence we have $\varphi_\theta^{j-1}(1) \in \widehat{1i}$, $e^\theta \varphi_\theta^{j-1}(1) \in \boldsymbol{T}$ and

$$\varphi_\theta^j(1) = \varphi(e^\theta \varphi_\theta^{j-1}(1)) \in \partial G.$$

So f^j is defined on $[0, \theta_{j-1})$ and maps into $[0, 2\pi)$. Being a composition of continuous, bijective mappings, f^j is continuous and bijective and hence strongly monotoneously increasing. Further we see

$$\begin{aligned} f^j(\theta_{j-1}) &= g^{-1}(\varphi_{\theta_{j-1}}(g(g^{-1}(\varphi_{\theta_{j-1}}^{j-1}(1))))) = g^{-1}(\varphi_{\theta_{j-1}}(g(f^{j-1}(\theta_{j-1})))) \\ &= g^{-1}(\varphi_{\theta_{j-1}}(g(\tfrac{1}{2}\pi)) = g^{-1}(\varphi(e^{\theta_{j-1}}i)), \end{aligned}$$

and because $\varphi(e^{\theta_{j-1}}i) \in \boldsymbol{T} \setminus \widehat{1i}$ we obtain $f^j(\theta_{j-1}) \in (\frac{1}{2}\pi, 2\pi)$. Together with $f^j(0) = 0$ the “Zwischenwertsatz” for the continuous function f^j implies that there is a $\theta_j \in (0, \theta_{j-1})$ with $f^j(\theta_j) = \frac{1}{2}\pi$. Using the monotony of f^j we get $f^j(\theta) \in (\frac{1}{2}\pi, 2\pi)\ \forall\theta \in (\theta_j, \theta_{j-1})$ and $f^j(\theta) \in [0, \frac{1}{2}\pi)\ \forall\theta \in [0, \theta_j)$ which finishes our induction step. We remark that $0 < \theta_j < \theta_{j-1}$ gives that the sequence $\{\theta_0, \theta_1, \ldots\}$ is strongly decreasing.

Now we come to the last step of the construction. Let’s fix a $k \geq 2$. Then we choose a $\theta < \pi$ from the interval $(\theta_{k-1}, \theta_{k-2})$ and are going to prove that C_{φ_θ} has the essentially characteristic polynomial z^k. In order to do this we first show that $C_{\varphi_\theta}^{k-1}$ is not compact. If $k = 2$, then $\varphi_\theta(\boldsymbol{T}) = \partial G$, $\lambda(\varphi_\theta(\boldsymbol{T}) \cap \boldsymbol{T}) = \lambda(\widehat{1i}) = \frac{1}{4}$ and so Lemma 3.9 implies that C_{φ_θ} is not compact. If $k > 2$, then because of the inclusions $\varphi_\theta(\bar{\boldsymbol{D}}) \subseteq \bar{\boldsymbol{D}}$ and $\varphi_\theta(\boldsymbol{D}) \subseteq \boldsymbol{D}$ we have

$$\varphi_\theta^{k-1}(\boldsymbol{T}) \cap \boldsymbol{T} = \varphi_\theta^{k-2}(\varphi_\theta(\boldsymbol{T}) \cap \boldsymbol{T}) \cap \boldsymbol{T} = \varphi_\theta^{k-2}(\widehat{1i}) \cap \boldsymbol{T}.$$

Using that $f^{k-2}(\theta) < \frac{1}{2}\pi$ we obtain

$$\varphi_\theta^{k-2}(\widehat{1i}) \cap \boldsymbol{T} = g([f^{k-2}(\theta), \tfrac{1}{2}\pi]) \tag{10}$$

and hence,

$$\lambda(\varphi_\theta^{k-1}(\boldsymbol{T}) \cap \boldsymbol{T}) = \lambda(g([f^{k-2}(\theta), \tfrac{1}{2}\pi])) = \frac{\frac{1}{2}\pi - f^{k-2}(\theta)}{2\pi} > 0$$

Lemma 3.9 gives that $C_{\varphi_\theta}^{k-1}$ is not compact.

The last step of our proof is to show that $C_{\varphi_\theta}^k$ is compact, which will give that z^k is the essentially characteristic polynomial of C_{φ_θ}. Taking into consideration that $f^{k-1}(\theta) > \frac{1}{2}\pi$ and using some ideas of the previous step of the proof we obtain

$$\varphi_\theta^k(\boldsymbol{T}) \cap \boldsymbol{T} = \varphi_\theta^{k-1}(\varphi_\theta(\boldsymbol{T}) \cap \boldsymbol{T}) \cap \boldsymbol{T} = \varphi_\theta^{k-1}(\widehat{1i}) \cap \boldsymbol{T}. \tag{11}$$

For $k = 2$ the preimage of $\widehat{1i}$ under φ_θ is the arc

$$\widehat{e^{-i\theta}e^{i(\pi/2-\theta)}} = \{e^{\tau i} : \tau \in [-\theta, \tfrac{1}{2}\pi - \theta]\},$$

which does not intersect with $\widehat{1i}$ for $\theta \in (\frac{1}{2}\pi, \pi)$. Hence we get $\varphi_\theta^2(\boldsymbol{T}) \cap \boldsymbol{T} = \emptyset$.

For $k > 2$ from (10) and (11) we obtain that

$$\begin{aligned}\varphi_\theta^k(\boldsymbol{T}) \cap \boldsymbol{T} &= \varphi_\theta^{k-1}(\widehat{1i}) \cap \boldsymbol{T} = \varphi_\theta(\varphi_\theta^{k-2}(\widehat{1i}) \cap \boldsymbol{T}) \cap \boldsymbol{T} \\ &= \varphi_\theta(g([f^{k-2}(\theta), \tfrac{1}{2}\pi])) \cap \boldsymbol{T} = g([f^{k-1}(\theta), g^{-1}(\varphi_\theta(i))]) \cap \boldsymbol{T}.\end{aligned}$$

Since $g^{-1}(\varphi_\theta(i)) > f^{k-1}(\theta) > \frac{1}{2}\pi$, the last intersection is empty. So we have for all $k \geq 2$ that $\varphi_\theta^k(\boldsymbol{T}) \cap \boldsymbol{T} = \emptyset$, which means $\varphi_\theta^k(\overline{\boldsymbol{D}}) \subseteq \boldsymbol{D}$. By virtue of the continuity of φ_θ^k the image $\varphi_\theta^k(\overline{\boldsymbol{D}})$ is a compact set $M \subseteq \boldsymbol{D}$. Hence there is an $r \in [0, 1)$ with $M \subseteq r\boldsymbol{D}$, and because $r\boldsymbol{D}$ is contained in some polygon inscribed in the unit circle, Lemma 3.9 gives that $C_{\varphi_\theta^k} = C_{\varphi_\theta}^k$ is compact. □

4.2. Theorem. *Every composition operator C_a which is an essential Carleman shift is an irreducible Carleman shift.*

Proof. If C_a is an essential Carleman shift then there is a polynomial $q(z) = z^k - 1$ with $q(C_a) \in \mathcal{K}$. Because $q(0) = 1$, Lemma 3.11 proves the theorem. □

4.3. Theorem. *For an inner analytic mapping $a : \boldsymbol{D} \to \boldsymbol{D}$ the following statements are equivalent*:

1. *C_a is algebraic.*
2. *C_a is essentially algebraic.*
3. *C_a has coinciding essentially characteristic and characteristic polynomials $z^k - 1$ (i.e., C_a is an irreducible Carleman shift).*
4. *The mapping a is a generalized rotation with period k.*

Proof. We consider the analytic functions $a^0 = \mathrm{id}$, $a^1 = a$, $a^2, \ldots$. If these functions are pairwise distinct, then Lemma 3.8 and the fact that a is inner (implying that $\lambda(E(a^j)) = 1$) give

$$\|p(C_a)\|_{\mathrm{ess}}^2 \geq \sum_{j=0}^{n} |p_j|^2 \lambda(E(a^j)) = \sum_{j=0}^{n} |p_j|^2$$

for any non-vanishing polynomial $p(z) = \sum_{j=0}^{n} p_j z^j$. This means, that $p(C_a)$ cannot be compact. So none of the four statements is valid in this case.

We are left with the case of at least two coinciding iterates of a. Let's assume that $a^m = a^{m+k}$ $(k > 0)$. We get $a^k = \mathrm{id}$, because $a^m(\boldsymbol{D}) = \boldsymbol{D}$. Theorem 3.4 gives that a must be a generalized rotation, otherwise we would have that $a^n \to \alpha$ uniformly on compact subsets of $\boldsymbol{D}$, which contradicts $a^k = \mathrm{id}$. Because $C_a^k - I = 0$ we obtain from Theorem 4.2 that C_a must be an irreducible Carleman shift. So we proved all the four statements in this case. □

4.4. Theorem. *If a is a linear fractional mapping which maps $\boldsymbol{D} \to \boldsymbol{D}$, then we are faced with exactly one of the following five cases*:

1. *C_a is neither algebraic nor essentially algebraic,*
2. *C_a is algebraic and essentially algebraic with coinciding characteristic and essentially characteristic polynomials, which are equal to $z^k - 1$ (i.e., C_a is in fact an irreducible Carleman shift),*
3. *C_a is algebraic and essentially algebraic with the characteristic polynomial $z^2 - z$ and the essentially characteristic polynomial z,*
4. *C_a is essentially algebraic but not algebraic, the essentially characteristic polynomial being z, or*
5. *C_a is essentially algebraic but not algebraic, the essentially characteristic polynomial being z^2.*

All these cases occur and can be characterized as follows:

- *Case 2 $\Leftrightarrow$ a is a generalized rotation having the period $k \in \mathbb{N}$.*
- *Case 3 $\Leftrightarrow$ a is constant.*
- *Case 4 $\Leftrightarrow$ a is not constant and maps the unit circle $\boldsymbol{T}$ into a circle contained in $\boldsymbol{D}$.*
- *Case 5 $\Leftrightarrow$ a maps the unit circle $\boldsymbol{T}$ into a circle which touches $\boldsymbol{T}$ at exactly one point, which is not a fixed point of a.*

Proof. First we assume that a is inner. Then Theorem 4.3 gives that C_a belongs either to case 1 or to case 2, and 2 can be characterized as given in the theorem.

So let's come to non-inner functions. Then case 2 does not occur as Lemma 3.11 shows. A linear fractional mapping $a : \boldsymbol{D} \to \boldsymbol{D}$ is analytic on some $G \supset \bar{\boldsymbol{D}}$, because either a is analytic on $\mathbb{C}$ or is analytic on $\mathbb{C} \setminus \{z_0\}$ for some pole z_0 of first order, where $z_0 \notin \bar{\boldsymbol{D}}$ because $a(\boldsymbol{D}) \subseteq \boldsymbol{D}$.

Now we look at the Denjoy–Wolff point α of a. If $\alpha \in \boldsymbol{D}$ then Lemma 3.5 gives that either C_a is not essentially algebraic or the essentially characteristic polynomial is of form z^k. If $|\alpha| = 1$ and $a'(\alpha) < 1$ then again Lemma 3.5 shows that C_a is not essentially algebraic.

Due to Theorem 3.4 it remains to to consider the case where $|\alpha| = 1$ and $a'(\alpha) = 1$. As the next step we show that then C_a is not essentially algebraic. Without loss of generality we can assume that $\alpha = 1$. Let $a(z) = (cz + d)/(ez + f)$. If $a(1) = 1$ then $c + d = e + f$, and from $a'(1) = 1$ we get $cf - de = (e + f)^2$. Inserting $c = e + f - d$ in the second condition and using that $e + f = c + d = 0$ would contradict our assumtions, we obtain $d = -e$ and hence $c = 2e + f$. Because $|a(0)| < 1$ we have $|-e/f| < 1$ and therefore we can introduce a new parameter $p = e/f \in \boldsymbol{D}$ to obtain the representation

$$a(z) = \frac{(2p + 1)z - p}{pz + 1}$$

for our linear fractional mapping. Now we distinguish three cases: $|p + \frac{1}{2}|$ is less, equal, or greater than $\frac{1}{2}$. If $p = -\frac{1}{2} + re^{i\varphi}$ with $r < \frac{1}{2}$, we set $t = -2p/(p + 1)$ which gives $p = -t/(t + 2)$ and get

$$t = \frac{1 - 2re^{i\varphi}}{\frac{1}{2} + re^{i\varphi}} = \frac{(2 - 4re^{i\varphi})(1 + 2re^{-i\varphi})}{1 + 4r^2 + 2r(e^{i\varphi} + e^{-i\varphi})} = \frac{2 - 8r^2 - 8i\sin\varphi}{1 + 4r\cos\varphi + 4r^2}.$$

The denominator is real and bounded below by $(1 - 2r)^2$ which is positive for $r < \frac{1}{2}$ and the real part of the numerator is $2 - 8r^2$ and therefore positive. That means we have the case $a(z) = ((2 - t)z + t)/(-tz + 2 + t)$ with $\mathrm{Re}(t) > 0$. From Cowen (see [7, p.152, Example 5] or [6, Cor. 6.2]) we know that the spectra of C_a is a spiral connecting 0 and 1 and therefore C_a cannot be essentially algebraic.

If $p = -\frac{1}{2} + \frac{1}{2}e^{i\varphi}$ then a is of the form $a(z) = e^{i\varphi}(z - \lambda)/(-\bar{\lambda}z + 1)$ with $\lambda = \frac{1}{2} - \frac{1}{2}e^{-i\varphi} = -\bar{p}$ and therefore inner. Finally if $p = -\frac{1}{2} + \frac{1}{2}q$ we have $a(-1) = (-3q + 1)/(-q + 3) \in \bar{\boldsymbol{D}}$, and $|-3q + 1| \leq |-q + 3|$ leads to $|q| \leq 1$. That means, if $|p + \frac{1}{2}| > \frac{1}{2}$ then $a(\boldsymbol{D}) \not\subset \boldsymbol{D}$.

Next we show that for a non-inner self-mapping a which defines an essentially algebraic composition operator exactly one of the cases 3, 4 or 5 occurs, and we describe the essentially characteristic polynomial q_a. We already know that the essentially characteristic polynomial is of the form z^k and distinguish between several cases in dependence on the location of the image $a(\boldsymbol{T})$ (which is always a circle).

If $a(\boldsymbol{T})$ is a point, which happens if and only if a is constant, then obviously $q_a(z) = z$ and C_a is algebraic with the characteristic polynomial $z^2 - z$; so we are in case 3. If $a(\boldsymbol{T})$ is contained in $\boldsymbol{D}$ then C_a is compact by Lemma 3.6 and not algebraic due to Lemma 3.10 and the fact that the essentially characteristic polynomial would be a divisor of the characteristic polynomial. So we are in case 4.

If $a(\mathbf{T})$ touches $\mathbf{T}$ at one point which is not a fixed point of a then, by Lemma 3.6, C_a is not compact. However, $a^2(\mathbf{T})$ is contained in $\mathbf{D}$ and so $C_{a^2} = C_a^2$ is compact. Hence we are faced with case 5.

If $a(\mathbf{T})$ touches $\mathbf{T}$ at a fixed point then $a^k(\mathbf{T})$ does so for all k and the only candidates z^k for the essentially characteristic polynomial drop out as one can check using Lemma 3.9 part 3) for the mapping a^k, i.e., C_a is not essentially algebraic. If $a(\mathbf{T}) = \mathbf{T}$ then a would be inner.

Finally we see that $a_1(z) = e^i z$, $a_2(z) = e^{2\pi i/k}$, $a_3(z) = 0$, $a_4(z) = \frac{1}{2}z$ and $a_5(z) = -\frac{1}{2}z - \frac{1}{2}$ respectively give examples for the five cases. □

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