GOTTLOB FREGE, A PIONEER IN ITERATION THEORY

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ABSTRACT

This is a historical note on the habilitation thesis of the famous logician Gottlob Frege (1848–1925): "Rechnungsmethoden, die sich auf eine Erweiterung des Grössenbegriffes gründen," Jena 1874. In this paper of Frege one can find astonishingly rich results on iteration theory. Frege investigates the translation equation in explicit form and he uses differential equations and the infinitesimal generator to determine solutions of the translation equation.

Introduction

This is a historical note on the habilitation thesis [6] of Gottlob Frege (born Nov. 8, 1848 in Wismar, died July 26, 1925 in Bad Kleinen) which appeared in Jena, Verlag Friedrich Frommann, 1874 as a 27 page booklet with the title "*Rechnungsmethoden, die sich auf eine Erweiterung des Grössenbegriffes gründen*" (Methods of calculations, based on an extension of the notion of magnitudes). It is available as a reprint from a collection of Frege's work ([7], pp. 50–84). We will follow this reprint, which is a slightly modified version (mainly in spelling) of the original [6], and we will quote it in the sequel as "Rechnungsmethoden." There is also an English word for word

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translation of the "Rechnungsmethoden" ([8], p. 56–92)[1] but, generally, I preferred to follow my own translations and interpretions of Frege's text. In both the reprint [7] and the English translation [8], the pages of the original booklet [6] are indicated, starting with page 1. Therefore in the sequel, referring to a text of the Rechnungsmethoden, it will be quoted in the following way: "[7], p. $m[n]$," where m denotes the page number in the German reprint [7] and n is the page number referring to the original [6].

When one reads historical works in mathematics, one has to keep in mind the historical background and the state of mathematical knowledge and way of thinking of the time when the work was written. Let us therefore recall, in this connection, that the "theory of proportions," one can also say "theory of magnitudes," of the ancient Greeks (Euclid, book No. V) was an extension of the notion of the rationals, made to deal with the ratio of magnitudes (length of a straight line, etc.), even in cases when they are incommensurable. This theory of proportions was, up to Frege's time, the basic theory of real numbers (the reals considered as magnitudes or as ratio of magnitudes). Let us further remember that the complex numbers were considered as a generalization of the real numbers, having a geometrical realization as magnitudes. Both items were generalizations based on the law of addition and comparison of mathematical objects.

Also other generalizations of mathematical operations have to be mentioned in connection with Frege's Rechnungsmethoden: the raising of a number to its n-th power, even when n is a real noninteger number (which finally was worked out by Newton) or the evaluation of recursion formulas like that for the n-factorials, considered valid not just for natural numbers which lead to the gamma function, introduced by Euler. The gamma function was a starting point of several investigations, subsumed under the title "interpolation problem," to evaluate functions defined on the integers by recursive formulas with real arguments. These kinds of generalizations are often called by historians "mathematics à la mode," as they were very common, especially in the eighteenth and nineteenth century.

So, at first glance, Frege followed this (not very new) trend in his habilitation thesis, making fashionable mathematics. But if one goes through his paper, one will find many methods and results on functional equations and especially on the so-called "translation equation" (see equation (1) below) and related differential equations. The style of Frege's text is, compared to other contemporary mathematical papers, relatively clear. On the other hand, Frege disregards special cases, nondifferentiable solutions etc. In particular he does not specify the domain of definition of the solutions of the considered functional equation, one of the most complicated things in functional equations. So, one can find several gaps and incorrectness in his statements. Nevertheless, it is worthwhile for a mathematician to read Frege's Rechnungsmethoden.

In his preface to the Rechungsmethoden, Frege writes about the difficulties in the

[1] I owe this information to William Beyer (Los Alamos). I also thank him and János Aczél for their helpful comments.

problem of visual perception of the magnitude of geometrical objects, like the length of a straight line or the magnitude of an angle. Then he writes:

> "*Wenn wir den Größenbegriff, wie gezeigt, nicht in der Anschauung vorfinden, sondern selber schaffen, so ist der Versuch gerechtfertigt, diese Definition so einzurichten, daß sie eine möglichst mannigfaltige Anwendung gestattet, damit der Arithmetik ein möglichst großes Gebiet untertan wurde.*" [7], *p.* 51[2]. [2]

Then Frege claims, vaguely, that one can ascribe a magnitude (*Größe*) to each operation. The operation f can be iterated: ff, fff ..., and each of these "iterates" yields a single operation. Also these iterates of f can be composed again, and this again yields a single iterate of f, so he claims the associativity law for the composition of operations. Then Frege raises the question of inverses of operations and iterative roots of operations.

Frege concludes his Introduction with:

> "*Die Anwendungen, welche wir machen wollen, haben nicht den Zweck, bisher ungelöste Probleme zu bewältigen, sondern sollen teils auf eine Verbindung aufmerksam machen, welche sich durch den Begriff der Größe der Funktionen zwischen verschiedenen Gebieten der Arithmetik herstellen läßt, teils die Klassen von Problemen kenntlich machen, zu deren Lösung eine weiter ausgeführte Theorie beitragen könnte.*" [7], *p.* 52[3]. [3]

In what follows, I will give a brief sketch of the mathematical content of Frege's Rechnungsmethoden, together with my comments and remarks. The more or less free and shortened transcriptions of the original mathematical text of Frege are written in *italics*, using Frege's *original mathematical notation*. My comments and remarks are written in roman letters.

Frege, after the introduction, continues in his Rechnungsmethoden ([7], p. 52[3]) with the following chapter.

[2] Since, as it has been shown, we cannot find the notion of magnitudes by perception, but create it by ourselves, the attempt will be justified to state this definition in such a way, that it allows manifold applications, so that a field, as large as possible, will be subjected to arithmetic.

[3] The subsequent applications do not have the purpose of mastering up to now unsolved problems. They have the aim partly to call attention to connections between different fields of arithmetic which can be stated by the notion of magnitude of functions, and partly to show classes of problems to whose solution a further developed theory could contribute.

1. Functions in a single variable

§ 1. *After what has been mentioned above, it will be understandable that we assign to the functions* $\varphi\,(\varphi(x))$, $\varphi\,(\varphi\,(\varphi(x)))$ *the twofold, threefold magnitude of that of the* (real) *function* $\varphi(x)$. *It is no less clear that* $\psi(x)$ *is to be assigned with one fourth of the magnitude of* $\varphi(x)$ *if* $\varphi(x)$ *is identical with* $\psi\,(\psi\,(\psi\,(\psi(x))))$; $\chi(x)$ *has the reciprocal of the magnitude of* $\varphi(x)$ *if* $\varphi\,(\chi(x)) = x$; *and finally, that the magnitude of* the identity function *has to be designated as the zero magnitude. We distinguish therefore between the magnitude of a function and the value that this function takes for a value of the argument. It is obvious, that all possible functions cannot build one single "Größengebiet"* (domain of magnitudes), *but they are split off in infinitely many different ones. ...*

[4] *One can pose the following questions*:
What is the function, whose magnitude has a given ratio to that of the magnitude of a given function?
Given two functions, are they in the same Größengebiet and, if yes, what is the ratio of their magnitudes?

The answer to these questions is closely connected with the knowledge of the general form of a function which is the n-fold of a given one. More precisely, one has to have a function of n and x which, for $n = 1$ *turns into the given function of* x, *and for which, generally, the functional equation* (Frege writes in [6] indeed *Functionalgleichung*)

$$f(n_0, f(n_1, x)) = f(n_0 + n_1, x) \tag{1}$$

holds.

If one denotes the value of this function by X, *then one can also say*: *one has to have an equation connecting the magnitude* n, *the value* X *and the arguments* x *of a function. Such an equation we will call a "Größengleichung"* (magnitude equation). *If* n *is given by* X *and* x *via*

$$n = \psi(X, x),$$

then the function ψ *must have the property that by elimination of* x_0 *from* $n_0 = \psi(X, x_0)$ *and* $n_1 = \psi(x_0, x_1)$ *we get the equation*

$$n_0 + n_1 = \psi(X, x_1)$$

or

$$\psi(X, x_0) + \psi(x_0, x_1) = \psi(X, x_1). \tag{2}$$

[4] [7], p. 53[4].

Frege continues ([7], p. 54[5]):

We now come to the methods of solving the problem of the Größengleichungen.

Method of substitution

§ 2. *This method is based on the obvious fact that if* (2) *holds for any* X, x_0, x_1 *then the equation*

$$\psi\,(\vartheta(X), \vartheta(x_0)) + \psi\,(\vartheta(x_0), \vartheta(x)) = \psi\,(\vartheta(X), \vartheta(x))\,,$$

holds too [for any function ϑ]. *Thus,* $\psi\,(\vartheta(X), \vartheta(x)) = n$ *constitutes a Größengleichung of a new function. We have by this a means to derive other Größengleichungen from a known one. It can be shown* [claims Frege erroneously] *that one can obtain each Größengleichung by this method. If we set* $x_1 = X$ *then* (2) *becomes*

$$\psi(x_1, x_0) + \psi(x_0, x_1) = 0.$$

Subtracting this from (2) *we get*

$$\psi(X, x_0) - \psi(x_1, x_0) = \psi(X, x_1).$$

If we consider x_0 *as a constant, we can write the Größengleichung* $n = \psi(X, x)$ *in the form*

$$n = \vartheta(X) - \vartheta(x),$$

where $\vartheta(x) = \psi(x, x_0)$. *This Größengleichung now, it can be said, is obtained by the method of substitution from the simplest one*

$$n = X - x,$$

which is the Größengleichung of addition.

Despite of the fact that Frege's calculations are incomplete, e.g., he ignores that one cannot eliminate a variable from a function in every case, one can say the following. The above calculations of Frege in connection with iteration problems are important and were rediscovered at a later time by several mathematicians in dealing with the solutions of the translation equation and are now standard technique.

We recognize now the exact meaning of "Größengebiet" and "Größengleichung" in the sense of Frege. Größengebiet is a family of functions obtained by iteration, also in the generalized sense, from a given one, say f. So, in order to be mathematically exact, it is nothing else then a family $f(n, x)$, which satisfies the translation equation (1) and $f(1, x) = f(x)$. Two functions are compatible, if they are in the same Größengebiet. A "Größengleichung" is a relation for n, x and $X = f(n, x)$ which will be satisfied by a specific solution of (1), and (hopefully) determines such a solution. If one gets a new

Größengleichung, then one gets another solution of (1) and, which was important in the aim of Frege, an iterated (also in the generalized sense) of another new function.

To illustrate this method of substitution, Frege gives some examples:

(i) The general power of a real number, say b, is obtained by determining a solution of (1) with $f(1, x) = b \cdot x$. This solution is given due to his method by substituting $\vartheta(X) = \lg X / \lg b$ and $\vartheta(x) = \lg x / \lg b$, which yields

$$n = \frac{\lg X - \lg x}{\lg b} \quad \text{or,} \quad X = b^n x.$$

(ii) A generalization of the geometric sum $c + cb + cb^2 + \cdots + cb^n$, as an iterate of the function $f(x) = c + b \cdot x$.

(iii) Using the substitution

$$\vartheta(x) = \operatorname{arctg}\left(\frac{x+b}{a}\right)$$

one gets from $X - x = n$ [5]

$$\operatorname{arctg}\left(\frac{X+b}{a}\right) - \operatorname{arctg}\left(\frac{x+b}{a}\right) = n \quad \textit{or,}$$

$$X = \frac{(a^2 + b^2)\operatorname{tg} n + (a + b \operatorname{tg} n)x}{a - b \operatorname{tg} n - \operatorname{tg} n \cdot x}$$

We here have the Größengleichung of the fractional function with linear nominator and denomiator:

$$X = \frac{A + Bx}{C + Dx}$$

and it is

$$b = \frac{C - B}{2D}, \quad a = \frac{\sqrt{-(C-B)^2 - 4AD}}{2D}$$

$$\operatorname{tg} n = -\frac{\sqrt{-(C-B)^2 - 4AD}}{B + C}.$$

In these examples Frege calculates from the Größengleichung $n = \vartheta(X) - \vartheta(x)$ the representation of the solutions of the translation equation (1) in the form

$$f(n, x) = \vartheta^{-1}\left(\vartheta(x) + n\right),$$

but he does not write down explicitly this formula, which became later a very important and often used formula in the theory of functional equations.

Then Frege continues ([7], p. 57[6]).

[5] [7] p. 56[6].

The method of substitution gives us unlimited possibilities to find new Größengleichungen, but it possesses the inconvenience ("Übelstand"), that often it is very difficult to find such a substitution that may lead to the prescribed end.

So Frege introduces [6]

The method of integration

§ 3. *A function of the infinitesimally small magnitude δ has to be of the form*

$$X_\delta = x + \delta\varphi(x),$$

if, for vanishing δ, X comes to be x. In the case of the Größengleichung $X_n = f(n, x)$ we get for $n = \delta$

$$X_\delta = f(0, x) + \delta\Big(\frac{\partial f(n, x)}{\partial n}\Big)_{n=0} = x + \delta\Big(\frac{\partial f(n, x)}{\partial n}\Big)_{n=0},$$

which shows the connection between f and φ. We can ignore the exceptional case where $f(n, x)$ for $n = 0$ would be discontinuous, since this almost never happens.

By this what is nowadays called the "infinitesimal generator" of $f(n, x)$,

$$\varphi(x) = \frac{\partial f(n, x)}{\partial n}\Big|_{n=0}$$

is introduced. Calculating with infinitesimal magnitudes, as in former times, he continues.

We will now assume, that the function φ is known, and that we will have to find from this the Größengleichung. Substituting $X_n = f(n, x)$ in $x + \delta\varphi(x)$ we get

$$f(n + \delta, x) = f(n, x) + \delta\varphi\,(f(n, x))\,.$$

Subtracting now $f(n, x)$ and dividing by δ we get

$$\frac{\partial f(n, x)}{\partial n} = \varphi\,(f(n, x))\,.$$

This differential equation, together with two others, are called, in connection with iteration problems, the *Jabotinsky equations* (see Jabotinsky [12], Targonski [18], [19] and also Aczél–Gronau [3]). In particular the above equation has been called by

[6] [7] p. 57[7].

Targonski the 2nd Jabotinsky equation. In what follows we also shall find the two other Jabotinsky equations.[7]

[8] *Integrating the equation*

$$\frac{\partial X}{\partial n} = \varphi(X),$$

where x has to be considered as a constant we get

$$n = \int \frac{dX}{\varphi(X)} + C = \vartheta(X) + C.$$

The constant C is determined by the fact that for $X = x$ *we have to have* $n = 0$. *Therefore we get*

$$n = \vartheta(X) - \vartheta(x).$$

So we come back to the method of substitution. We also get to know a new connection between the functions f *and* ϑ. *We also know now how to determine* ϑ *as soon as* φ *is known. So the knowledge of* φ *is most essential.*

Here Frege emphasizes the importance of the infinitesimal generator. After this, he gives a heuristic method (involving the fixed points of the given function $f(x)$) for finding the infinitesimal generator of $f(n, x)$ for $f(1, x) = f(x)$, f a given function.

[9] *To check whether a function is the right* φ*-function*, *check the following equation. If* $f(x)$ *is the given function*, *then*

$$f(x) + \delta\varphi\,(f(x)) = f\,(x + \delta\varphi(x))$$

must hold; *i.e.*,

$$f(x) + \delta\varphi f(x) = f(x) + \delta\varphi(x)\frac{df(x)}{dx}.$$

$$\varphi f(x) = \varphi(x)\frac{df(x)}{dx}.$$

This last equation is the so-called 3rd Jabotinsky equation, satisfied by any differentiable solution of (1) (which automatically satisfies $f(0, x) = x$). One can get this equation from the commutativity of iteration

$$f\,(t, f(n, x)) = f\,(n, f(t, x))$$

[7] These equations are also called *Aczél–Jabotinsky equations*, since János Aczél has introduced these equations in connection with the translation equation in different papers already from 1949 on (see [3]). To be historically correct one should now introduce the name *Frege–Aczél–Jabotinsky equations*. For the sake of simplicity the author prefers the notation *Jabotinsky equations*.

[8] [7], p. 58[7].

[9] [7], p. 59[8].

by differentiating this equation with respect to t and putting $t = 0$ afterwards. This was also done by Frege (see above), but only for the case $n = 1$, because he did not need for his purposes the general case.

After an example and an (insufficiently exact) method to compute the Taylor series for the infinitesimal generator φ, Frege shows how to

[10] *represent the Größengleichung in form of a Taylor series. It is*

$$f(n, x) = x + \frac{\varphi_1(x)}{1} n + \frac{\varphi_2(x)}{1 \cdot 2} n^2 + \cdots$$

where $\varphi_k(x) = (\partial^k f(n, x)/\partial n^k)_{n=0}$. *We now have*

$$f\,(n, x + \delta\varphi_1(x)) = f(n + \delta, x).$$

If one transforms this last equation in the same way as above in the case of the 2nd Jabotinsky equation, one can read the so-called 1st Jabotinsky equation

$$\frac{\partial f(n, x)}{\partial x} \cdot \varphi(x) = \frac{\partial f(n, x)}{\partial n}.$$

This equation is used by Frege to establish the recursion formulas

$$\varphi_{k+1}(x) = \varphi_1(x)\varphi_k'(x).$$

By this the $\varphi_2, \varphi_3, \ldots$ are uniquely determined by $\varphi_1 = \varphi$.

2. Functions in several variables

§ 4. Now Frege considers iterates of multidimensional functions and states for those again the translation equation, in his words:

[11] *Funktionsgleichungen*

$$f_k(n_0, f_1(n_1), f_2(n_1) \ldots f_m(n_1)) = f_k(n_0{+}n_1, x_1, x_2 \ldots x_m), \quad k = 1, 2, 3 \ldots m,$$

where $f_i(n_1)$ *stands for* $f_i(n_1, x_1, x_2 \ldots x_m)$.[12]

Here again he uses the

[10] [7], p. 60[9].
[11] [7], p. 62[10].
[12] Also in the original [6] and [7] the threefold dots are written in this way: $f_2(n_1) \ldots f_m(n_1)$, without commas.

The methods of integration and substitution[13]

§5. *In a system of functions of magnitude* 0 *all the values are equal to the arguments.* [This means $f_k(0, x_1, \dots, x_m) = x_k$ for $k = 1, \dots, m$]. *An infinitesimally small system of functions* [This means $f_k(\delta, x_1, \dots, x_m)$, where δ is infinitesimally small] *has the form*

$$\begin{aligned} X_1 &= x_1 + \delta\varphi_1(x_1, x_2 \dots x_m) \\ X_2 &= x_2 + \delta\varphi_2(x_1, x_2 \dots x_m) \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ X_m &= x_m + \delta\varphi_m(x_1, x_2 \dots x_m), \end{aligned}$$

where

$$\varphi_k(x_1, x_2 \dots x_m) = \left(\frac{\partial f_k(n, x_1, x_2 \dots x_m)}{\partial n}\right)_{n=0}$$

[the infinitesimal generator]. *Therefore we have*

$$f_k(n + \delta, x_1 \dots x_m) = f_k(n, x_1 \dots x_m) + \delta\varphi_k(f_1, f_2 \dots f_m)$$

and [here comes again the (2nd Jabotinsky) differential equation]

$$\frac{\partial f_k(n, x_1 \dots x_m)}{\partial n} = \frac{\partial X_k}{\partial n} = \varphi_k(X_1, X_2 \dots X_m).$$

We have to integrate the simultaneous system of differential equations

$$\frac{\partial X_k}{\partial n} = \varphi_k(X_1, X_2 \dots X_m), \quad k = 1, 2 \dots m, \tag{1}$$

where the $x_1, x_2 \dots x_m$ *have to be considered as integrating constants to be introduced.*

...

We will first divide the equations (1) *by the first one of these to eliminate* n, *and get*

$$\frac{dX_k}{dX_1} = \frac{\varphi_k}{\varphi_1}, \quad k = 2, 3, \dots m.$$

After some (more or less vague) calculations Frege gets from (1) the formula[14] (*Größengleichung*)

$$n = \vartheta_k(X_1, X_2 \dots X_m) - \vartheta_k(x_1, x_2 \dots x_m). \tag{4}$$

These formulas show (as Frege claims) *that*, *in the same way as in the case of one variable*, *all Größengleichungen can be found by substitution from the simple system*

$$X_k - x_k = n, \; k = 1, 2 \dots m.$$

[13] [7], p. 62[10].
[14] [7], p. 63[11].

Conversely, as it can be easily shown, if one takes arbitrarily chosen functions $\vartheta_1, \vartheta_2 \dots \vartheta_m$, *by* (4), *one always obtains a system of Größengleichungen.*

...

One could now proceed with similar considerations like those for the functions in one variable; but, since this would lead to many repetitions, we will consider here, as main subject of our investigations, a special kind of functions and these are the simplest and most important ones.

Linear homogeneous functions[15]

§ 6. *Since linear homogeneous functions do not change their form when they are substituted into themselves, we also know the form of its infinitesimally small function system and of the functions* φ (i.e., its infinitesimal generator). *Let*

$$\varphi_1 = c_{11}x_1 + c_{12}x_2 + c_{13}x_3 \dots c_{1m}x_m$$
$$\varphi_2 = c_{21}x_1 + c_{22}x_2 + c_{23}x_3 \dots c_{2m}x_m$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\varphi_m = c_{m1}x_1 + c_{m2}x_2 + c_{m3}x_3 \dots c_{mm}x_m.$$

One therefore has to integrate the differential equations

$$\frac{dX_1}{dn} = c_{11}X_1 + c_{12}X_2 + c_{13}X_3 \dots c_{1m}X_m$$
$$\frac{dX_2}{dn} = c_{21}X_1 + c_{22}X_2 + c_{23}X_3 \dots c_{2m}X_m$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\frac{dX_m}{dn} = c_{m1}X_1 + c_{m2}X_2 + c_{m3}X_3 \dots c_{mm}X_m.$$

What follows is a tedious and complicated calculation, integrating the linear differential system using their characteristic equation. Furthermore, Frege shows a method to solve the iteration problem for linear homogeneous (and also inhomogeneous) functions in m variables. This leads to calculating the logarithm of the matrix $A = (a_{ij})$, where the system of linear functions to be iterated is given by[16]

$$X_1 = \sum_{i=1}^{i=m}\{a_{1i}x_i\}, \quad X_2 = \sum_{i=1}^{i=m}\{a_{2i}x_i\} \dots . \; X_m = \sum_{i=1}^{i=m}\{a_{mi}x_i\}.$$

But Frege did not use matrix calculus and therefore could not use the expression "logarithm of a matrix." Frege writes down for the case $n = 2$ concrete formulas

[15] [7], p. 65[12].
[16] [7], p. 67[14].

for the solution of the translation equation, which gives the iterates of a given linear homogeneous function (see [7], p. 71–74[18–20]).

Then some applications on continued fractions and also for linear difference equations follow.

[17] *To show how to solve functional equations by calculating the Größengleichungen, we consider the equation*

$$f(n) = \sum_{i=1}^{i=m} \{a_i f(n-i)\},$$

where f *has to be determined. We set*

$$f(k) = X_1, \quad f(k-i) = X_{i+1} = x_i.$$

This yields the system of functions

$$\begin{aligned} X_1 &= a_1x_1 + a_2x_2 + a_3x_3 \dots a_mx_m \\ X_2 &= \quad x_1 \\ X_3 &= \qquad\quad x_2 \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ X_m &= \qquad\qquad\qquad x_{m-1} \end{aligned}$$

We now need to know m special values, say

$$f(0), f(1) \dots f(m-1).$$

If one replaces $x_m, x_{m-1} \dots x_1$ *by these values, one gets* $X_1 = f(m)$. *The values* $X_1, X_2 \dots X_m$, *replacing* $x_1, x_2 \dots x_m$, *yields* $f(m+1)$. *This shows that n-fold iteration of the function system yields the value* $f(m+n-1)$. *Whence one has to establish the Größengleichung of our function system, which requires the solution of the equation*

$$\zeta^m - a_1\zeta^{m-1} - a_2\zeta^{m-2} \dots - a_m = 0$$

Consider the Schimper's series (*Schimpersche Reihe*)[18] *as example*:

$$0, 1, 1, 2, 3, 5, 8 \dots$$

[17] [7], p. 75[20].

[18] Most probably Karl-Friedrich Schimper, 1803–1867, a well known German botanist. There are other wellknown botanists named Schimper, related to him. K.F. Schimper also stayed in Jena in 1854–1855. He principally was concerned formulating a theory of phyllotaxis. He pointed out that leaves that grow in spiral formations are arranged in regular cyclic patterns and that each species has a characteristic pattern ([11]). The numbers quoted here are nothing else than the Fibonacci numbers.

The general term can be obtained by solving the functional equation

$$f(n) = f(n-1) + f(n-2).$$

One therefore has to establish the Größengleichung of

$$X_1 = x_1 + x_2,$$
$$X_2 = x_1.$$

The equation

$$\zeta^2 - \zeta - 1 = 0$$

has the roots

$$\zeta_0 = \frac{1+\sqrt{5}}{2}, \quad \zeta_1 = \frac{1-\sqrt{5}}{2}.$$

We set $x_1 = 1$, $x_2 = 0$, *counting* x_1 *as* 0*th element and* x_2 *as* (-1)*st element, we get the general element* $a_{11}(n)$ *owing to a formula developed above*:

$$a_{11}(n) = \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^{n+1} - \Big(\frac{1-\sqrt{5}}{2}\Big)^{n+1}\Big].$$

This representation of the Fibonacci numbers was also given by Lucas in 1876 (for references and more on history of the Fibonacci numbers see [5] and [4]).

After this Frege shows two other examples of the summation of trigonometric functions. The examples at the end of section 1 and also the examples in this section are related to the interpolation problem, discussed and treated by several mathematicians like John Wallis (1616–1703), Isaac Newton (1643–1727), Christian Goldbach (1690–1764), Daniel Bernoulli (1700–1782) and finally in several works by Euler[19]. Especially Goldbach was concerned with this interpolation problem and he was the one who inspired Euler to pursue this problem (see [13]). The works of Euler should have been known to Frege.

An example of transformations of rigid bodies follows, and then Frege concludes the Rechnungsmethoden with the sentence:

[20] *Apart from some general theorems that we have proved for all functions and systems of functions, we only dealt with linear functions. It would perhaps be of interest to do similar investigations on function systems, like quadratic and Cremona transformations.*

[19] e.g., in: *Institutiones calculi differentialis*, chapter 17.
[20] [7], p. 83[26].

3. Final remarks

Frege's contribution to iteration theory has been unknown up to now. One cannot find any quotation in relevant books on this topic, e.g., [1], [2], [14], [15], [18] etc. The reason is, on the one hand, that the Rechnungsmethoden have been published as a small booklet in a publishing house in Jena, but not in an internationally distributed mathematical journal, and on the other hand, that Frege did not pursue this field in mathematics, as far as one can see from [7], [9] and [10]. Actually Frege was engaged in explaining the "Größenbegriff" (notion of magnitudes) also later on, e.g., in his work *Grundgesetze der Arithmetik I*, Jena 1893, and in his correspondence, e.g, with David Hilbert (see [10], p. 55 f). Here especially the footnote 16 on page 74 is interesting to read in connection with the Rechnungsmethoden. But in none of the accessible works, from and about Frege, could I find any reference to the Rechnungsmethoden.

I personally became acquainted with Frege's Rechnungsmethoden in 1992, during a meeting on history of mathematics, where Karl-Heinz Schlote from Leipzig gave a talk with the title *Freges Erweiterung des Größenbegriffs – eine Sackgasse*? From this talk, I first heard about Frege's contributions to functional equations.

Schlote gives in [16] also some historical remarks about the habilitation of Frege: Gottlob Frege submitted his petition for the habilitation, together with his habilitation thesis "Rechnungsmethoden" to the philosophical faculty of the university of Jena, on March 16, 1874... This thesis was refereed by Ernst Abbe in a very positive sense. It was seen as an attempt to make further advances in the classification of functions. In addition to this approval, Abbe makes the critical remark that, after more detailed studies than that of the presented thesis, one could decide whether these ideas are sufficiently important to enter the annals of mathematics or whether, in essence, they are exhausted by this thesis. On the basis of this opinion of Abbe, the "durchlauchtigsten Erhalter der Gesamt-Universität Jena" approved on May 1, 1874 the habilitation of Frege as "Privatdozent." For further interpretations of Frege's Rechnungsmethoden see [16].

Historically (see, e.g., [15] or [18]), Ernst Schröder (1841-1902), also a famous logician, was one of the first who systematically dealt with iterations of functions. In 1871 the paper [17] Schröder raised the same kind of questions as posed in the Rechnungsmethoden of Frege. Frege knew Schröder, since Frege wrote "Critical remarks to some points in E. Schröders Lectures on the Algebra of Logic," ([7], p. 193), but I could not figure out from [7]–[10], whether Frege had knowledge of Schröder's mathematical work on iteration theory. Curiously Frege himself complained in the preface to the *Grundgesetze der Arithmetik I* that his own works were almost ignored by the mathematicians, especially by Dedekind, Otto Stolz, von Helmholtz and Kronecker. It would be of interest to compare the paper [17] of Schröder with Frege's Rechnungsmethoden. Schröder, for example, has not written explicitly the translation equation as a functional equation. He also did not use differential equations.

As I pointed out previously, the Rechnungsmethoden are full of ideas and new methods. In my opinion also Frege's Rechnungsmethoden are worth being mentioned in reference books on iteration theory, despite the fact that they had, as far as I know, no influence on later mathematicians.

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