INDEXED FAMILIES OF BIJECTIONS AND ITERABLE FUNCTIONS

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ABSTRACT

Let $\mathrm{Bij}(X)$ denote the family of all bijections mapping a nonempty set X onto itself. Assume that we are given a subfamily $\{f_s : s \in S\}$ of $\mathrm{Bij}(X)$ indexed by an additive subsemigroup S of the positive half-line and fulfilling the condition $f_s \circ f_t = f_{s+t}$ for all $s, t \in S$ large enough. We show that, in such a case, there exists a uniquely determined map $g \in \mathrm{Bij}\,(X)$ which is embeddable into an iteration group $\{g^s : s \in \mathbb{R}\} \subset \mathrm{Bij}(X)$ such that $f_s = g^s$ for all $s \in S$.

Generalizations as well as some special cases are also discussed.

1. Introduction

Let $(S, +)$ be a (not necessarily commutative) semigroup with a distinguished nonzero element e. A self-mapping f of a nonempty set X is called *S-iterable* if and only if there exists a map $F : X \times S \longrightarrow X$ satisfying the *translation equation*

(T) $$F\,(F(x, t), s)) = F(x, s + t) \quad \text{for all } x \in X, s, t \in S,$$

and the *embeddability* condition

(E) $$F(x, e) = f(x) \quad \text{for all } x \in X.$$

The map F itself will be called an *S-iteration semigroup* (resp. *S-iteration group* provided that the semigroup $(S, +)$ happens to be a group). In the case where $S = (0, \infty)$, (resp. $S = \mathbb{R}$), $e = 1$ and $+$ stands for the usual addition, the corresponding S-iterable mapping is termed *iterable* whereas the S-iteration semigroup (resp. S-iteration group) in question is simply called *iteration semigroup* (resp. *iteration group*) (see, e.g., M.C. Zdun [6], Gy. Targonski [5] or W. Jarczyk [2]). Members of the given semigroup (group) $(S,+)$ will be referred to as *iteration indices* whereas the sections $f^s := F(\cdot, s)$, $s \in S$, of an S-iteration semigroup F will be named *iterations* of a function $f = f^e$.

In what follows we shall be considering the following question. Assume that we are given a semigroup $(S,+)$ and an indexed subfamily $\{f_s : s \in S\}$ of the family $\mathrm{Bij}(X)$ of all bijections of a nonempty set X onto itself. Suppose that a map $F : X \times S \longrightarrow X$ given by the formula

$$F(x,s) := f_s(x), \quad x \in X, s \in S,$$

satisfies the translation equation (T) merely for some (reasonably large) set of pairs of indices (s,t). Does there exist an S-iteration semigroup $\{g^s : s \in S\}$ such that $f_s = g^s$ for all s from some (reasonably large) set of indices belonging to S? For instance, assuming that a family $\{f_s : s \in (0,\infty)\}$ satisfies the relationship $f_{s+t} = f_s \circ f_t$ for all pairs (s,t) from the outside of some disc centered at the origin in $\mathbb{R}^2$ (i.e., from a vicinity of infinity), we may ask whether there exists an iterable mapping f whose iterations f^s coincide with the maps f_s for all s from some vicinity of $+\infty$ in $\mathbb{R}$. We shall see that this is really the case even in the situtation where the set of "good" pairs (s,t) of indices is considerably smaller than a vicinity of infinity in $\mathbb{R}^2$.

In the sequel, as the main tool, we shall be using the following result obtained by the author in [1] (see also M. Kuczma [3]).

Theorem A. *Let $(G,+)$ and $(H,+)$ be two (not necessarily commutative) groups and let $(S,+)$ be a subsemigroup of $(G,+)$ such that $S - S = G$. Let further $\mathfrak{J}$ be a nonempty collection of subsets of G closed under finite set-theoretical unions, hereditary with respect to descending inclusions and such that jointly with a set $E \subset G$ it contains the family $\{x - E : x \in G\}$. If $S \notin \mathfrak{J}$ and $M \subset G \times G$ has the property that for some $U \in \mathfrak{J}$ the sections $M_x := \{y \in G : (x,y) \in M\}$ belong to $\mathfrak{J}$ whenever $x \in G \setminus U$, then for every map $\varphi : S \longrightarrow H$ such that*

(1) $$\varphi(s+t) = \varphi(s) + \varphi(t), \qquad (s,t) \in (S \times S) \setminus M,$$

there exists exactly one homomorphism $\Phi : G \longrightarrow H$ such that

$$\varphi(s) = \Phi(s) \qquad \textit{for all } s \in S \setminus U.$$

2. Main results

Each nonempty set X (with no structure involved) generates automatically the group $(\mathrm{Bij}(X), \circ)$ of all bijections of X onto itself. From that point of view, bijections seem to be the most natural mappings to be considered in the context described in the Introduction. Nevertheless, in what follows, the group structure of the family of transformations investigated turns out to be essential and not the bijectivity itself. Therefore, we will be dealing in general with indexed subfamilies of an abstract group of transformations.

Theorem 1. *Given a nonempty set X and a subsemigroup $(S,+)$ of $((0,\infty),+)$ generating the additive group $(\mathbb{R},+)$, assume that $(\mathcal{F}(X),\circ)$ is a transformation group and λ is an arbitrary nonnegative mapping on S. Let $\{f_s : s \in S\}$ be an indexed subfamily of $\mathcal{F}(X)$ with the property that the relationship*

$$f_{s+t} = f_s \circ f_t \qquad \text{whenever } t > \lambda(s) \tag{2}$$

holds true for all $s, t \in S$. Then there exists exactly one iteration group $\{g^s : s \in \mathbb{R}\}$ of an element g from $\mathcal{F}(X)$ such that

$$f_s = g^s \qquad \text{for every } s \in S.$$

Proof. Define a map $\varphi : S \longrightarrow \mathcal{F}(X)$ by the formula

$$\varphi(s) = f_s, \qquad s \in S.$$

Let $\mathcal{J}$ stand for the totality of bounded subsets of $\mathbb{R}$. Clearly, $\mathcal{J}$ enjoys all the properties spoken of in Theorem A; moreover, $S \notin \mathcal{J}$ and $S - S = \mathbb{R}$. Finally, *every* section M_x, $x \in \mathbb{R}$, of the set

$$M := \{(s,t) \in S \times S : t \leq \lambda(s)\}$$

belongs to $\mathcal{J}$; in particular, the exceptional set $U \in \mathcal{J}$ occurring in Theorem A is just empty. Since, on account of (2) and the definition of M, we have (1), Theorem A applied for the group $(H,+) = (\mathcal{F}(X),\circ)$ states that there exists exactly one homomorphism $\Phi : \mathbb{R} \longrightarrow \mathcal{F}(X)$ such that $\varphi(s) = \Phi(s)$ for all $s \in S$. Setting $g^s := \Phi(s)$, $s \in \mathbb{R}$, we infer that

$$g^{s+t} = \Phi(s+t) = \Phi(s) \circ \Phi(t) = g^s \circ g^t, \qquad s, t \in \mathbb{R},$$

i.e., that $\{g^s : s \in \mathbb{R}\}$ yields an iteration group of a function $g := g^1 \in \mathcal{F}(X)$. Since

$$f_s = \varphi(s) = \Phi(s) = g^s \qquad \text{for all } s \in S,$$

the proof has been completed.

Corollary. *For an arbitrarily fixed $s_o \geq 0$ any indexed subfamily $\{f_s : s > s_o\}$ of transformation group $(\mathcal{F}(X),\circ)$ with the property*

$$f_{s+t} = f_s \circ f_t, \qquad s, t \in (s_o, \infty), \tag{3}$$

coincides with a collection $\{g^s : s \in \mathbb{R}\}$ of iterations of an iterable member g of $\mathcal{F}(X)$.

Proof. Take $S := (s_o, \infty)$ and $\lambda(s) := s_o$ for all $s \in S$.

Remark 1. In the case when $\mathcal{F}(X) = \mathrm{Bij}(X)$, the Corollary may also be proved in a direct way. Indeed, setting $f_{-s} := (f_s)^{-1}$, $s \in S := (s_o, \infty)$, note that

$$f_s \circ f_{-t} = f_{-t} \circ f_s, \qquad s, t \in S.$$

Actually, due to (3) and the commutativity of S, one has $f_s = f_s \circ f_t \circ (f_t)^{-1} = f_t \circ f_s \circ (f_t)^{-1}$, i.e., $f_{-t} \circ f_s = (f_t)^{-1} \circ f_s = f_s \circ (f_t)^{-1} = f_s \circ f_{-t}$, as claimed.

Consequently, for all elements $s, t, u, v \in S$ satisfying the equality $s - t = u - v$ we have

$$f_s \circ f_v = f_{s+v} = f_{t+u} = f_t \circ f_u$$

whence $f_s \circ f_{-t} \circ f_v = f_{-t} \circ f_s \circ f_v = f_{-t} \circ f_t \circ f_u = f_u$, i.e.,

$$f_s \circ f_{-t} = f_u \circ f_{-v},$$

which proves that the definition

$$f_{s-t} := f_s \circ f_{-t} = f_s \circ (f_t)^{-1} \qquad s, t \in S,$$

is correct. Since, obviously, $S - S = \mathbb{R}$ we have extended the family $\{f_s : s > s_o\}$ to a collection $\{f_x : x \in \mathbb{R}\} \subset \mathcal{F}(X)$. We omit a somewhat messy, although quite clear, calculations showing that such (unique) extension procedure leads to an iteration group of the transformation $f := f_1$.

The situation is no longer that simple in the case where the iteration indices are taken from a noncommutative semigroup. Nevertheless, the following counter-part of the Corollary may be proved.

Theorem 2. *Let* $(G, +)$ *be a* (*not necessarily commutative*) *nontrivial group and let* $(S, +)$ *be its subsemigroup such that* $S - S = G = -S + S$. *Assume that we are given an indexed subfamily* $\{f_s : s \in S\}$ *of a transformation group* $(\mathcal{F}(X), \circ)$ *consisting of self-mappings of a nonempty set* X. *If*

$$f_{s+t} = f_s \circ f_t \qquad \text{whenever } t \in S + s \text{ or } s \in S + t, \tag{4}$$

then there exists exactly one G*-iteration group* $\{g^s : s \in G\}$ *of an element* g *from* $\mathcal{F}(X)$ *such that*

$$f_s = g^s \qquad \text{for every } s \in S.$$

To prove this theorem we need the following

Lemma. *Let* $(G, +)$ *be a* (*not necessarily commutative*) *group and let* $(S, +)$ *be its subsemigroup such that* $S - S = G = -S + S$. *Then each finite collection of sets of the form* $s + S + t + S + u$, *where* $s, t, u \in S$, *has a nonempty intersection.*

Proof. Suppose the contrary: there exists a positive integer n and there exist elements $s_1, \ldots, s_n, t_1, \ldots, t_n, u_1, \ldots, u_n \in S$ such that

$$\bigcap_{i=1}^{n}(s_i + S + t_i + S + u_i) = \emptyset.$$

We may assume that n is the smallest possible positive integer admitting the existence of such elements. Clearly $n \geq 2$ whence

$$(s_1+S+t_1+S + u_1) \cap (s_2+S+t_2+S+u_2) \cap \bigcap_{i=3}^{n}(s_i+S+t_i+S+u_i) = \emptyset,$$

which implies that

$$(S+t_1+S) \cap (-s_1+s_2+S+t_2+S+u_2-u_1) \cap \bigcap_{i=3}^{n}(-s_1+s_i+S+t_i+S+u_i-u_1) = \emptyset.$$

Since $G = S - S = -S + S$ there exist elements $p_i, q_i, v_i, w_i \in S$ such that $-s_1 + s_i = p_i - q_i$ and $u_i - u_1 = -w_i + v_i$ for $i \in \{2, \ldots, n\}$ and, therefore,

$$(S+t_1+S) \cap (p_2-q_2+S+t_2+S-w_2+v_2) \cap \bigcap_{i=3}^{n}(p_i-q_i+S+t_i+S-w_i+v_i) = \emptyset.$$

Obviously, for each $s, t \in S$ one has

$$(5) \qquad S \subset (-s + S) \cap (S - t) \qquad \text{and} \qquad (s + S) \cup (S + t) \subset S,$$

which gives

$$(S + t_1 + S) \cap (p_2 + S + t_2 + S + v_2) \cap \bigcap_{i=3}^{n}(p_i + S + t_i + S + v_i) = \emptyset,$$

and, consequently,

$$(p_2 + S + t_1 + S + v_2) \cap (p_2 + S + t_2 + S + v_2) \cap \bigcap_{i=3}^{n}(p_i + S + t_i + S + v_i) = \emptyset.$$

Now,

$$(S + t_1 + S) \cap (S + t_2 + S) \cap \bigcap_{i=3}^{n}(-p_2 + p_i + S + t_i + S + v_i - v_2) = \emptyset.$$

Similar arguments lead to the existence of elements $p'_i, q'_i \in S, i \in \{3, \ldots, n\}$, such that

$$(S + t_1 + S) \cap (S + t_2 + S) \cap \bigcap_{i=3}^{n}(p'_i + S + t_i + S + q'_i) = \emptyset.$$

Finally, replacing the last summand in the first bracket by $t_2 + S$ and the first summand in the second bracket by $S + t_1$ we arrive at

$$(S + t_1 + t_2 + S) \cap (S + t_1 + t_2 + S) \cap \bigcap_{i=3}^{n}(p'_i + S + t_i + S + q'_i) = \emptyset,$$

i.e.,

$$(S + t_1' + S) \cap \bigcap_{i=3}^{n} (p_i' + S + t_i + S + q_i') = \emptyset,$$

where we have put $t_1' := t_1 + t_2 \in S$, which contradicts the minimality of n and completes the proof.

Proof of Theorem 2. Put $T := S \cup (-S)$ and

$$M := \{(s,t) \in S \times S : s - t \notin T\}.$$

Define a map $\varphi : S \longrightarrow \mathcal{F}(X)$ by the formula

$$\varphi(s) = f_s, \qquad s \in S.$$

Then, by means of (4), we get

$$\varphi(s+t) = \varphi(s) + \varphi(t), \qquad (s,t) \in (S \times S) \setminus M. \tag{1}$$

Let $\mathcal{J}$ stand for the totality of all sets of the form

$$\bigcup_{i=1}^{n} (x_i + (G \setminus T) + y_i), \qquad x_1, ..., x_n, y_1, ..., y_n \in G, \quad n \in \mathbb{N},$$

and all their subsets. Clearly, the family $\mathcal{J}$ is a nonempty collection of subsets of G closed under finite set-theoretical unions, hereditary with respect to descending inclusions and such that jointly with a set $E \subset G$ it contains the family $\{x - E : x \in G\}$. We shall show that

$$S \notin \mathcal{J}. \tag{6}$$

Indeed, otherwise there would exist an $n \in \mathbb{N}$ and elements $x_1, ..., x_n, y_1, ..., y_n \in G$ such that

$$S \subset \bigcup_{i=1}^{n} (x_i + (G \setminus T) + y_i),$$

whence, in view of the symmetry of $G \setminus T$, we would get

$$G = S - S = \bigcup_{i,j=1}^{n} (x_i + (G \setminus T) + y_i - y_j + (G \setminus T) - x_j),$$

i.e.,

$$\bigcap_{i,j=1}^{n} (x_i + T + y_i - y_j + T - x_j) = \emptyset.$$

Since $S \subset T$ and there exists elements

$$s_i, t_i, u_{i,j}, v_{i,j}, p_j, q_j \in S, \qquad i, j \in \{1, ..., n\},$$

such that

$$x_i = s_i - t_i, \quad y_i - y_j = u_{i,j} - v_{i,j}, \quad x_j = -q_j + p_j, \qquad i, j \in \{1, ..., n\},$$

we infer that

$$\bigcap_{i,j=1}^{n} (s_i - t_i + S + u_{i,j} - v_{i,j} + S - p_j + q_j) = \emptyset.$$

Since we have (5) for each $s, t \in S$ the latter equality yields

$$\bigcap_{i,j=1}^{n} (s_i + S + u_{i,j} + S + q_j) = \emptyset,$$

which contradicts the Lemma and finishes the proof of (6).

In order to apply Theorem A it remains to examine the largeness of the vertical sections of the set M. Clearly, $M_x = \emptyset \in \mathcal{J}$ provided that $x \notin S$. For $x \in S$ we have

$$M_x = \{t \in S : x - t \notin T\} = \{t \in S : t \notin T + x\} = (G \setminus T) + x \in \mathcal{J}.$$

This means that exceptional set $U \in \mathcal{J}$ occurring in Theorem A is just empty. By Theorem A applied for the group $(H, +) = (\mathcal{F}(X), \circ)$ we derive the existence of exactly one homomorphism $\Phi : G \longrightarrow \mathcal{F}(X)$ such that $\varphi(s) = \Phi(s)$ for all $s \in S$. Setting $g^s := \Phi(s)$, $s \in G$, we infer that

$$g^{s+t} = \Phi(s + t) = \Phi(s) \circ \Phi(t) = g^s \circ g^t, \qquad s, t \in G,$$

i.e., that $\{g^s : s \in G\}$ yields a G-iteration group of a function $g := g^e \in \mathcal{F}(X)$, where $e \in G \setminus \{0\}$ is arbitrarily fixed. Since

$$f_s = \varphi(s) = \Phi(s) = g^s \qquad \text{for all} \qquad s \in S,$$

the proof has been finished.

3. Concluding remarks

Iteration semigroups with iteration indices running over noncommutative semigroups appear naturally in the theory of automata (see, e.g., Z. Moszner [4]).

The restriction

$$t \in S + s \qquad \text{or} \qquad s \in S + t$$

upon the admissible indices in (4) states that we are assuming the fundamental relationship $f_{s+t} = f_s \circ f_t$ merely for *comparable* s, t from S in the sense that

$$s \preceq t \qquad \text{or} \qquad t \preceq s,$$

where the preordering relation $\preceq$ is defined as follows:

$$x \preceq y \qquad \text{if and only if} \qquad y - x \in S, \quad x, y \in G.$$

The semigroup $(S, +)$ serves then as a substitute of a positive cone in a real vector space.

Remark 2. There exist nontrivial noncommutative groups $(G, +)$ generated by proper subsemigroups $(S, +)$ such that

$$S - S = -S + S = G.$$

Consider, for instance, the group $(G, \circ)$ of all affine transformations of the real line onto itself with positive directional coefficients. It is a simple matter to check that its subsemigroup $(S, \circ)$ consisting of all functions of the form $\mathbb{R} \ni x \longmapsto ax + b \in \mathbb{R}$ where both a and b are positive, generates $(G, \circ)$ in a manner described above.

Remark 3. The assumption upon the target space $\mathcal{F}(X)$ to be a group is indispensible to apply Theorem A which played the crucial role in our approach. Unfortunately, such an assumption leaves numerous families of self-mappings of X beyond the scope of our considerations. This remark seems to be a reasonable motivation to look for analogues of Theorem A dealing with semigroups instead of groups. This, however, remains an open problem.

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