SOME CONTACT BIFURCATIONS IN TWO–DIMENSIONAL EXAMPLES

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ABSTRACT

In this paper we consider some properties of two-dimensional noninvertible maps, which possess a chaotic attractor. Contact bifurcations causing a qualitative change in the shape of chaotic attractors, or causing their destabilization, have been considered in several papers since the early work in 1978 [19]. Such contact bifurcations, due to the contact between the boundary of a chaotic attracting set or area, and the boundary of its basin of attraction, may involve a fuzzy (or chaotic) basin boundary. In the examples considered in this paper we shall see that such bifurcations correspond to homoclinic bifurcations of repelling cycles of the map (repelling nodes, foci or saddles).

1. Introduction

We deal with a noninvertible two-dimensional map $T : \mathbb{R}^2 \to \mathbb{R}^2$, defined by continuous functions, piecewise continuously differentiable, which possesses a chaotic set or area, and whose dynamics are considered as a function of a real parameter.

Nowadays one begins to recognize that critical curves of T are useful tools of analysis in the understanding and description of the bifurcation mechanisms, and transition to chaotic behaviour, in noninvertible maps, after the first results obtained since 1964 ([24, 25], [18, 19, 20] and references therein). In the next section we shall review some elementary definitions.

Besides critical curves, another important tool in the study of chaotic dynamics is that of homoclinic orbits of repelling cycles. The role played by a homoclinic orbit is well known in the study of the dynamics of a map with a unique inverse since the work of Birkhoff [5], Shilnikov [30, 31, 16, 17], Smale [32, 33]. The same role is played by a homoclinic orbit in endomorphisms, where we can have homoclinic orbits of fixed points (or k-cycles) not only of type saddle, but also of type node or focus. If the map T is continuously differentiable, in [21] it can be seen that in any neighbourhood of a point homoclinic to a fixed point of T (or of T^k), infinitely many cycles of a suitable power of T exist, with periodic points near the homoclinic points, and chaos in the sense by Li and Yorke [22]. In the case of orbits homoclinic to repelling nodes or foci, the first proof of the above statement can be found in [23]. It is very likely that the same result holds also for maps T which are only piecewise countinuously differentiable. In the case of repelling nodes or foci this conjecture is proved in [11], where is also shown the role of the critical curves in detecting the bifurcation, that is, it is proved that a homoclinic bifurcation (or explosion of infinitely many homoclinic orbits which did not exist before bifurcation) is caused by *critical homoclinic orbits* at the bifurcation value.

Now the natural question that arises is "when do occur such critical and non-critical homoclinic orbits of fixed points or k-cycles of T?"

In the case of one-dimensional endomorphisms such global bifurcations (which are the homoclinic bifurcations) are related to structural changes in some attracting set, which cannot be explained by use of local analysis, as eigenvalues, and are coupled with changes in the basins of attraction. For example a homoclinic bifurcation occurs at the closure of any box of the 1st kind or of the 2nd kind in the "box-within-a-box" bifurcation structure described in [26]. Such bifurcations are contact bifurcations of an attracting set with the boundary of its basin of attraction. The fundamental differences between the two kinds of bifurcations are briefly recalled in the Appendix (and it may also be interpreted in terms of differences in the type of homoclinic bifurcations occurring [11]). At the closure of a box of the 2nd kind of a k-cycle, an immediate basin involved in the contact also has a contact with another immediate basin. This occurs by pairs, causing the transition from $2k$-cyclic invariant attracting intervals into k-cyclic ones. At the closure of a box of the 1st kind of a k-cycle, a contact with the immediate basin takes place in points which do not belong to another immediate basin, but in points which are limit points of components of the total basins (which have a fractal, or chaotic, structure). This causes the transition from k disjoint intervals into a unique interval (chaotic in a nonstrict sense) which includes all the previous ones, inside which, soon after the bifurcation, it is possible to see regions with high density of iterated points, related to the old

k intervals, and regions with low density, related to the old regions of immediate basins and some (but not all) of the fractal components of the old total basins.

At such homoclinic bifurcations, an invariant interval involved has a contact with its basin in a periodic point which is also a critical point (of some rank) of T. At the bifurcation value the preimages of that critical-periodic point appear (infinitely many) inside the invariant intervals, and after bifurcation the preimages of that repelling cycle (no longer critical) belong to infinitely many “holes,” made up of points whose orbits escape the old invariant intervals.

It comes natural to conjecture that the role of a global, homoclinic, bifurcation is almost the same also in the case of two-dimensional endomorphisms, in particular when we have chaotic areas (in a nonstrict sense) bounded by arcs of critical curves (to be recalled in the next section). That is, the qualitative changes in the structure of an invariant attracting set, with chaotic dynamics inside, which do not depend on the local character of T (i.e., on the jacobian and eigenvalues, at the points of the attracting set), take place when a contact between the invariant set and its immediate basin occurs, and are marked by homoclinic bifurcations of some cycle of T belonging to the boundary.

We may also introduce a sort of classification, among contact bifurcations due to the contact between the boundary of a chaotic attracting set or area, and the boundary, or frontier, of its basin of attraction. Clearly, with respect to the one-dimensional case now the spectrum of possibilities is wider, but we may distinguish among two different situations. A contact bifurcation is said to be of *type (I)* if it involves isolated points of the frontier (a point of the frontier is an isolated point if a neighbourhood of it exists which does not contain other points of the frontier). A contact bifurcation is said to be of *type (II)* if it involves non-isolated points of the frontier. Then, a contact bifurcation of type (II) is said to be of the 2*nd kind* if it causes a qualitative change in the shape of a chaotic attractor of T as the reunion of a finite number of chaotic sets into a smaller number of chaotic sets, which persist attracting (i.e., belongs to an absorbing area) after the bifurcation. A contact bifurcation of type (II) is said to be of the 1*st kind* if it causes a sudden change in the shape of the attracting set as the transition occurs into a wider attractor or chaotic area, which strictly includes the preexisting ones, involved in the contact.

We note that the contact bifurcations of the 2nd kind and of the 1st kind defined above refer to qualitative changes of a bounded chaotic area, which persists also after bifurcation. However, among the contact bifurcations of type (II) we have to include also the so called “final bifurcation,” which changes the chaotic attractor into a chaotic repeller, that is, whose effect is to make disappear the chaotic attractor.

In the next section we shall further characterize such bifurcations, and in Section 3 we shall give several examples using the double logistic map presented in [13], and a map considered in [27].

2. Some definitions, and properties of contact bifurcations of chaotic areas

Let T be a two-dimensional endomorphism. The critical curve of T of rank 1, denoted by LC (or by LC_0), is the locus of points of the plane having at least two coincident rank 1 preimages. We shall assume that the map T is such that LC is a curve of the plane made up of a finite number of disjoint branches. The locus of points which are coincident preimages of points of LC is denoted by LC_{-1}. The critical curves of T of rank $(k+1)$ are the rank k images of the critical curve LC, that is, $LC_k = T^k(LC)$. It is clear that the critical curves of rank k of T are the generalization of critical points of one-dimensional endomorphisms. As in one-dimensional endomorphisms, where critical points define the boundary of absorbing intervals and invariant intervals, and characterize the global bifurcations leading to chaotic dynamics, also in two-dimensional endomorphisms critical curves may be useful to determine the boundary of trapping areas or of invariant areas, at least in the simplest cases, and thus to characterize the global bifurcations of invariant sets. And also in more general cases, as introduced by Barugola and Cathala [3], where the boundary of the chaotic area is of "mixed type" (which means that it includes also arcs of some unstable set issuing from some cycle saddle belonging to the boundary itself), the role of critical points is probably the same as that in the simpler situations, as it will appear in [29].

Besides the references given in the introduction, we recall that properties of critical curves may be found in [1,2,4,6,7,8]. Some more specific properties of invariant areas bounded by critical points are given in [15], while the "mechanism of holes," which is a particular effect of a contact bifurcation between an invariant area (a chaotic area or not) and the boundary of its basin of attraction, is described, besides in [19], by the examples in [14] and in [28]. In [14] it is pointed out that the "dynamic effect" of a contact between chaotic area d, and the boundary of its basin $\partial \mathbf{D}$, depends on what kind of dynamics exist "on the other side" of the boundary, at the contact. We shall deal with this in the present work. More precisely, several types of dynamical behaviour may be observed and we have classified them in the introduction. Moreover, it is a conjecture of us that such bifurcations (involving the boundary of chaotic areas) are always homoclinic bifurcations (and it was, in its essence, already suggested in [20]).

We note that when an invariant absorbing area involved in a contact bifurcation with its basin boundary is not a chaotic area, then we do not have a homoclinic bifurcation. In such a case the contact bifurcation is a global bifurcation causing a change in the structure and shape of the basin of attraction, but not a change in the invariant attracting set. We note also that there are some other global bifurcations (due to a contact between the boundary of a basin and the critical curve LC) which cause a change in the basin structure, but not to the attracting sets existing inside it (which belong to absorbing areas not involved in the contact) [28].

We recall that chaotic area d is an invariant area, $T(d) = d$, such that d is a "minimal absorbing area" [14], with chaotic dynamics or chaotic transients in d (i.e., d may be chaotic in a nonstrict sense, and the generic trajectory appears as chaotic in d). Our assumption that the boundary of d is made up of a finite number of arcs of critical curves is only for the sake of simplicity. d is attracting if a neighbourhood U of d exists, whose points have trajectories which enter d after a finite number of iterations (the number depends on the point). Its basin of attraction $\mathbf{D}$ is the open set of points having this property. It is given by $\mathbf{D} = \bigcup_{n\geq 0} T^{-n}(U)$, where T^{-1} denotes all the inverses of T. A different notation makes use of the immediate basin D_0, which is the widest simply connected component of the total basin $\mathbf{D}$ containing d, then $\mathbf{D} = \bigcup_{n\geq 0} T^{-n}(D_0)$. We say that d is k-cyclic, or cyclic of period k, if d is made up of k disjoint areas, $d = \bigcup_{i=1}^{k} d_i$, such that $d_{i+1} = T(d_i)$, $T(d_k) = d_1$. That is, the map T^k possesses k disjoint invariant areas d_i, each of which has its own basin $\mathbf{D}_i = \bigcup_{n\geq 0} T^{-nk}(D_{0i})$, being D_{0i} the immediate basin of d_i.

A contact bifurcation between chaotic area d and its basin occurs when the boundaries ∂d and $\mathbf{F}_0 = \partial D_0$ have a non-void intersection. In the case of a k-cyclic chaotic area we refer to the map T^k. We recall that the boundary of a set A is given by $\overline{A} \cap \overline{\mathbf{C}(A)}$ where overline denotes the closure and $\mathbf{C}(A)$ the complementary set of A. The boundary of the immediate basin may be fractal or not, and it may contain isolated points (repelling nodes or foci).

A contact bifurcation of type (I), involving an isolated fixed point P of the frontier $\mathbf{F}_0$ is probably the snap-back-repeller bifurcation of that point, i.e., it denotes the appearance of the first homoclinic points of the cycle. In fact, if P is an isolated point of the frontier $\mathbf{F}_0$ then a neighbourhood U of P exists made up of points (apart from P) whose trajectories will escape U and never return in U, and thus without points homoclinic to P. If the contact bifurcation involves P then infinitely many preimages of P appear inside the chaotic area, and preimages of P can be found in any neighbourhood of P, and thus are homoclinic points.

Contact bifurcations of type (II) involving non-isolated points of $\mathbf{F}_0$ are also, generally, homoclinic bifurcations. In fact, if p is such a contact point, then the images of p, which necessarily belong to $\mathbf{F}_0$, generally belong to the stable set of some cycle of T, such that, after bifurcation, the unstable set of the cycle (entering the "old" chaotic area d) and the stable set of the cycle (belonging to the "old" frontier) have intersection points. This is the justification of our conjecture. And what are the effects of a contact bifurcation? That is, when shall we have a contact bifurcation of the 2nd kind or of the 1st kind or a final bifurcation? We state the following proposition, for the contact bifurcations of type (II) defined in the introduction, relative to an invariant chaotic area d of T, non-cyclic (if d is k-cyclic then the statement is to be referred to each invariant set of T^k) (see also [19], [13]):

Proposition. *When the contact between the boundary ∂d of d, and the boundary ∂D_0 of its immediate basin D_0 occurs in non-isolated points of ∂D_0 in one of the situations*:

(i) *the contact points belong to the* immediate basin *of another bounded attracting set,*
(ii) ∂D_0 *is a limit set of non-connected components of the total basins of other bounded attractors,*
(iii) *the contact points are limit points of the region whose points have unbounded trajectories,*

then we have, respectively,

(i) *a contact bifurcation of the* 2*nd kind,*
(ii) *a contact bifurcation of the* 1*st kind,*
(iii) *a final bifurcation.*

Some remarks are useful in order to characterize the dynamics of T.

(1) At the bifurcation value, the contacts between ∂d and ∂D_0 may be points or arcs of invariant curves.

(2) The boundary ∂D_0 may be a fractal set, so that it is difficult to detect the contacts with ∂d.

(3) The dynamics occurring in all the three kinds of contact bifurcations introduced above, can be explained by use of the same basic mechanism, which is, roughly speaking: inside the "old" immediate basin D_0 holes are created made up of points belonging to "old" different basins, that is, whose asymptotic behaviour changes after the bifurcation.

(4) It is not necessary to know exactly which are the contact points at the bifurcation value and what is the basin on the other side, in order to understand if a contact bifurcation of the 2nd kind or of the 1st kind is going to occur. The proposition states that it is enough to see if the contact occurs with an *immediate basin* or a *non-immediate basin* of another bounded attracting set.

(5) Even if we do not know the shape of the immediate basin D_0, and the structure of the other basins, we can understand which contact bifurcation occurred from the dynamical behaviour of the computed trajectories. This is obvious when case (iii) occurs. But also the distinction between the cases (i) and (ii) is simple. Indeed, we have called them the 2nd kind or the 1st kind considering their different dynamic effects. Let us consider, to simplify the exposition, two chaotic areas of T, d_1 and d_2, disjoint and invariant before a bifurcation value, which are no longer invariant after, then:

– when the new invariant chaotic attractor d is obtained as the "reunion" of the previous ones ($d = d_1 \cup d_2$) then a bifurcation of the 2nd kind has occurred (and in this case the immediate basin of d can be considered as given by the reunion of the two previous immediate basins, of d_1 and d_2);

– when the new invariant chaotic attractor d strictly includes d_1 and d_2 ($d \supset d_1 \cup d_2$) then a bifurcation of the 1st kind has occurred (and in this case the two old immediate basins of d_1 and d_2 belong to d, as well as some of the other components of the old total basins of d_1 and d_2, giving rise to the phenomenon of regions ($d \backslash (d_1 \cup d_2)$) with low density of iterated points).

Otherwise stated, the characteristic of a contact bifurcation of the 1st kind is that soon after the bifurcation the chaotic area d is wider then those existing before the bifurcation, and such that the iterated points of a generic trajectory visit very often the old chaotic areas, which are regions of high density of points; while it visits less frequently the remaining parts, which are regions of low density of iterated points. For example, rare points of a trajectory may be observed in the regions belonging to the immediate basins of the old areas, and to some of the regions of d which were in non-immediate basins before the bifurcation. This phenomenon is due to the fact the at the bifurcation value the invariant chaotic areas have contacts with the frontier of their immediate basins which are accumulation sets of connected components of different total basins, in a fractal way.

(6) The knowledge of the contacts (arcs or points) at the bifurcation value is usefull to recognize homoclinic bifurcations. In particular, the contact of two nearby immediate basins occurring at a bifurcation of the 2nd kind implies that the first homoclinic bifurcation of some cycle will occur (because before the bifurcation the unstable sets entering the immediate basins on opposite sides have no intersection with the stable sets belonging to the frontier). While the contact occurring at a bifurcation of the 1st kind implies that the homoclinic bifurcation of some cycle will occur which is generally not the first (because before the bifurcation the unstable set of some cycle on the frontier of an immediate basin enters a region of fractal shape of connected components, and frontiers exist containing stable sets of the same cycle, so that homoclinic points on one side of the immediate frontier already exist before the bifurcation).

(7) The bifurcations of type (II), characterized by the proposition given above, and in particular the bifurcations of the 2nd kind or of the 1st kind, and the above remarks, explain the different phenomena observed by several authors, and ofted called "crises."

(8) It is worth noting that other bifurcations may often be observed in the qualitative change of the shape of attracting sets, as, for example, the sudden disappearance of a chaotic region and appearance of an attracting cycle inside the chaotic region. This phenomenon of "order within chaos" is often observed coupled with the phenomenon of "intermittency," which suggests the appearance of an attracting cycle with points in specific parts of the chaotic attractor. This kind of bifurcation is not different from what is already known to occur in one-dimensional endomorphisms or in maps of the circle into itself. For example, we observe this phenomenon when

the parameter of the logistic map is increased through the bifurcation value (fold bifurcation) which creates a couple of 3-cycles. That is, the (numerically observed) aperiodic trajectories inside an invariant interval shows (before the bifurcation) intermittent passages (with smaller steps) in three narrow zones, where the cycles will appear at the fold-bifurcation. Similar behaviours can be observed on closed invariant curves or inside invariant absorbing areas. This kind of bifurcation is not considered in this work. It may be considered among the "global effects" of local bifurcations. However, it seems important to us to recall this kind of qualitative change because it belongs to the basic mechanisms with which can be explained the transition from regular to chaotic regimes in two-dimensional endomorphisms as well as in one-dimensional ones.

In the following section we shall see several examples of contact bifurcations of chaotic areas, of both the 2nd kind and the 1st kind, as well as of final bifurcations, which are homoclinic bifurcations. Other examples have been shown in [9, 10, 12], [28, 29].

3. Examples of contact bifurcations of chaotic areas

Example 1. The first example we consider is the double-logistic map T partially studied in [13], given by:

$$(x, y) \to ((1-\lambda)x + 4\lambda y(1-y), (1-\lambda)y + 4\lambda x(1-x))$$

for $0.6 \le \lambda \le 1$. All the interesting dynamics occur in the square $[0, 1]\times[0, 1]$ which is mapped into itself by T. The restriction of T to the line $x = y$, we shall call it Δ, reduces to the dynamics of a one-dimensional map homeomorphic to the standard logistic map, or Myrberg's map, and due to symmetry properties, points symmetric with respect to the line Δ possess symmetric trajectories. T has four fixed points, two on the line Δ, which are the origin and the fixed point $S^* = (\frac{3}{4}, \frac{3}{4})$, plus P_1^* and P_2^* which are symmetric with respect to Δ.

(E1) At $\lambda = 0.641$ T possesses a 2.7-cyclic chaotic attractor (which belongs to an annular absorbing area), see Fig. 1a, which becomes a 7-cyclic chaotic attractor at $\lambda = 0.643$ (Fig. 1b). The bifurcation value belongs to the interval [0.64218, 0.64219] and it corresponds to a contact bifurcation of the 2nd kind between two areas by pairs of the map T^{14}, involving a 7-cycle saddle. See the point V among two nearby chaotic areas on Fig. 2a, the enlargements near V in the figures 2b and 2c, and the chaotic set which crosses the stable set of V on Fig. 2d. As the unstable set issuing from V (see Fig. 2a) has approximately the same shape of the chaotic set, we can state that the stable and unstable sets of V crossess transversally after the contact bifurcation (i.e., the first homoclinic bifurcation of the saddle V occurs at this contact bifurcation).

(E2) The 7-cyclic chaotic set shown on Fig. 1b bifurcates into a wider annular chaotic area, shown on Fig. 1c. The transition occurs at a value of λ in the interval [0.64392, 0.64393]. The symmetric attractor with respect to Δ is shown on Fig. 3a, together with the seven total basins of the map T^7, before the bifurcation. That is, on Fig. 3a each different tonality of grey corresponds to the basin of one of the 7 disjoint attractors of the map T^7, showing that both the immediate basins and the total basins have a complex structure. In the enlargement of Fig. 3b a saddle cycle of T of period 3.7, i.e., a 3-cycle of the map T^7, is shown on the boundary of the immediate basin. The enlargement of Fig. 3c is near one of these saddle points, called X_1, and we are very close to the bifurcation value. The tonality of the colour on the other side of the contact points cannot be stated with precision, the arc of $\mathbf{F}_0$ is a limit set of areas belonging to all the seven basins. Thus a contact bifurcation of the 1st kind occurs. On Fig. 3d, after bifurcation, the stable and unstable sets of X_1 intersect transversally (i.e., a homoclinic bifurcation of the saddle X_1 occurs at this contact bifurcation, which is not the first, as homoclinic points already exist before on one side of the immediate boundary $\mathbf{F}_0$).

Some more remarks on this example. Before the bifurcation of the 1st kind, the immediate basins of the map T^7 are not simply connected. As it can be seen from Fig. 3a, they are multiply connected, with complex lakes (in the terminology introduced in [28]), i.e., in a fractal structure (containing infinitely many islands, which contain infinitely many lakes, which ... etc. The contact with ∂D_0 occurs on the boundaries of three of such lakes, which are accumulation of areas in the situation (ii) of the proposition.

(E3) The annular chaotic area shown on Fig. 1c persists up to $\lambda = 0.702$ (see Fig. 4a), and T also possesses a symmetric invariant area on the other side of Δ. The two immediate basins of T are separated by a segment of Δ, stable set of a 2-cycle saddle Q_1–Q_2 (which exists now on Δ, besides the fixed points of T). At $\lambda =$ 0.703 (see Fig. 4b) T has a unique chaotic area, symmetric with respect to Δ. At the bifurcation value, contact bifurcation of the 2nd kind, as two immediate basins are involved, the two chaotic areas have in common the 2-cycle Q_1–Q_2, which is critical (i.e., made up of critical points). Fig. 4c shows the unstable set of the 2-cycle after bifurcation, crossing the local stable set on Δ (i.e., the first homoclinic bifurcation of the 2-cycle saddle Q_1–Q_2 occurs at this contact bifurcation).

The three holes appearing in the chaotic area of Fig. 4b (bounded by arcs of critical curves) are around the repelling node S^* and around the repelling nodes P_1^* and P_2^*. The holes will disappear at contact bifurcations of type (I) which are the first homoclinic bifurcation of those cycles. This is shown in [13], together with several other similar bifurcations, up to the “final” one, which is the homoclinic bifurcation of the origin occurring at $\lambda = 1$.

Example 2. Our second example is taken from [27]. We consider the map defined by

$$(x, y) \to (x^2 - y^2 + \lambda + \epsilon x, 2xy - 2.5\epsilon y)$$

Also this map has symmetric dynamics, now with respect to the line $y = 0$ (i.e., the x-axis), and the restriction of T to the line $y = 0$ reduces to a one-dimensional map homeomorphic to the Myrberg's map.

(E4) Let $\lambda = -0.525$. At $\epsilon = -0.9$ an invariant chaotic area d_1 in the half-plane $y > 0$ exists, see Fig. 5a, and a symmetric one, d_2, also exists in $y < 0$. The two immediate basins are separated by a segment on the line $y = 0$, belonging to the stable set of a 2-cycle saddle, with periodic points α_1–α_2 on this line. As ϵ approaches the contact bifurcation value, several "tongues" in the chaotic area appear, approaching the 2-cycle saddle and at the bifurcation value the qualitative picture of the contact near the periodic point α_2 is shown on Fig. 5b. After the bifurcation (contact bifurcation of the 2nd kind), we observe the reunion of the two chaotic areas into a single one (see Fig. 5c). Reasoning as in the example (E1) we can conclude that this is the first homoclinic bifurcation of the 2-cycle saddle α_1–α_2 .

(E5) Let $\lambda = -0.8788$, $\epsilon = -0.95126$. An invariant chaotic area d in the half-plane $y > 0$ exists, see Fig. 6a, and a symmetric one, d', also exists in $y < 0$. The total basins of the two chaotic areas are shown on Fig. 6b. We remark that before the contact bifurcation which causes the appearance of a unique chaotic area of T, shown on Fig. 6c, the two immediate basins D_0 and D'_0 (of d and d' respectively) are simply connected, as shown on Fig. 6b, differently from what occurs in the previous example. Moreover, these two immediate basins are not disjoint. In fact, they have a contact point in a fixed point repelling node on the line $y = 0$, see the point N on Fig. 6b, which is a limit point of fractal components of the two total basins $\mathbf{D}$ and $\mathbf{D}'$. This implies that the repelling node N is already a snap-back-repeller, i.e., homoclinic points of N already exist. In fact, all the "black and white" components of the total basins are obtained by taking the preimages of any rank of the two immediate basins D_0 and D'_0, each of which has on the boundary a preimage of the point N. As in any neighbourhood of N we can find infinitely many of such areas, it follows that in any neighbourhood of N we can find preimages of finite rank, and thus homoclinic points, of N. We note also that clearly all the homoclinic points of N are outside the two immediate basins D_0 and D'_0.

At the bifurcation value, the contact of the chaotic areas d_i ($i = 1, 2$) with the frontier of their immediate basins occurs at points which are far from the point N, and are limit points of fractal components of the total basins. Thus the bifurcation is of the 1st kind. We note that the bifurcation value is difficult to detect, and so are the points of contact between the chaotic area and the immediate boundary. However, the contact points are limit points (from the outside of the immediate basin) of fractal components of the total basins. Thus this bifurcation is a further homoclinic bifurcation of the repelling node N, which creates an explosion of new homoclinic points of N, for example inside the regions previously belonging to the immediate basins D_0 and D_0'.

Moreover, it may also be possible that this bifurcation is also the homoclinic bifurcation of some different cycle of T. For example, if the contact points belong to the stable set of a cycle of T on the frontier of the immediate basin (a 2-cycle of T, as it seems that the contact occurs in two places), then this is probably a homoclinic bifurcation, not the first one, also of that cycle (not the first one because the branch of the unstable set not entering the immediate basin intersects the stable set on the frontier of other components).

As this example shows, when a bifurcation of the 1st kind occurs and the immediate basins are not disjoint, then the contact points of the such immediate basins are limit points of fractal components of the total basins.

As shown in the examples of the 1st kind bifurcations (E2) and (E5), the generic trajectory, soon after the bifurcation (see Fig. 1c and Fig. 6c), shows regions of high density of iterated points (which correspond to the old chaotic areas) and regions with a low density of iterated points (which correspond to regions previously belonging to the immediate basins and parts of the total basins).

Appendix

The “box-within-a-box” bifurcation structure is described in [26] for a one-dimensional unimodal smooth map, but it may be considered at the basis of the bifurcation mechanisms of any smooth one-dimensional multimodal map.

A box of the 1st kind of a k-cycle opens (or starts) at the fold bifurcation giving rise to a couple of k-cycles (i.e., two cycles of period k), one attracting and one repelling. While a box of the 2nd kind of a k-cycle opens (or starts) at a flip bifurcation of a k-cycle, giving rise to an attracting $2k$-cycle.

The closure of a box of the 2nd kind corresponds to the first homoclinic bifurcation of the k-cycle whose flip bifurcation started the box. Just before the bifurcation, $2k$-cyclic attracting invariant intervals exist (inside which the dynamics seem chaotic), whose immediate basins (considering the map T^{2k}) are such that two consecutive ones are bounded by the same periodic point; that is, these immediate basins may be coupled into pairs of two consecutive ones, which are cyclic for the map T^k, and separated by a repelling point of the k-cycle whose flip bifurcation started the box. At the bifurcation value and after, k-cyclic invariant intervals are attracting, that is, the closure of a box of the 2nd kind causes the transition from $2k$- to k-cyclic attracting invariant intervals (inside which the dynamics seem chaotic), and the new immediate basins are the reunion, by pairs, of the old immediate basins.

The closure of a box of the 1st kind of a k-cycle occurs at the homoclinic bifurcation of the k-cycle born repelling at the fold bifurcation which started the box. Just before the bifurcation, k-cyclic attracting invariant intervals exist, whose immediate basins, considering the map T^k, are such that any two of them are not consecutive. Each point on the boundary of an immediate basin is a limit point of disjoint components of the total basins. The k immediate basins are separated by connected components of the total basins, which are intertwined in a chaotic way. Soon after the bifurcation, the k intervals are no longer invariant, and a wider invariant absorbing interval generally exists, with complex dynamics, which include not only the old k intervals, but also some components of their old total basins. Such an absorbing interval includes generally an attracting cycle of high period, and the computed trajectory seems chaotic into this invariant interval. That is, this bifurcation causes (apparently) a sudden increase of the chaotic (in a nonstrict sense) attracting set, from k disjoint intervals into a unique interval which includes all the previous ones. Soon after the bifurcation the iterated points of a generic trajectory visits more often the old k invariant intervals and less frequently the remaining parts, so that we have regions with high density of iterated points (related to the old k intervals) and regions with low density (related to the old regions of immediate basins and fractal components of the old total basins).

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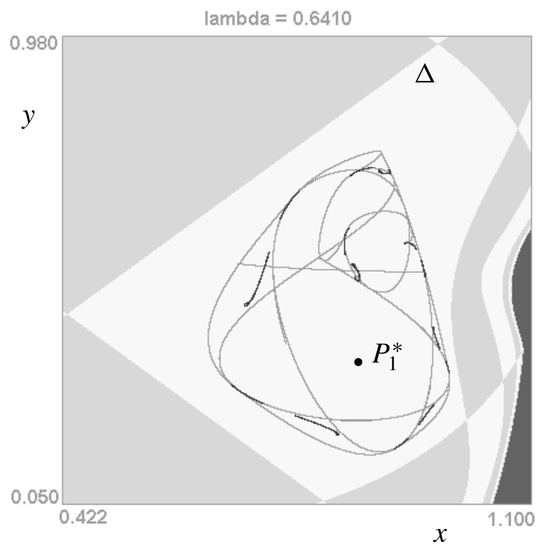

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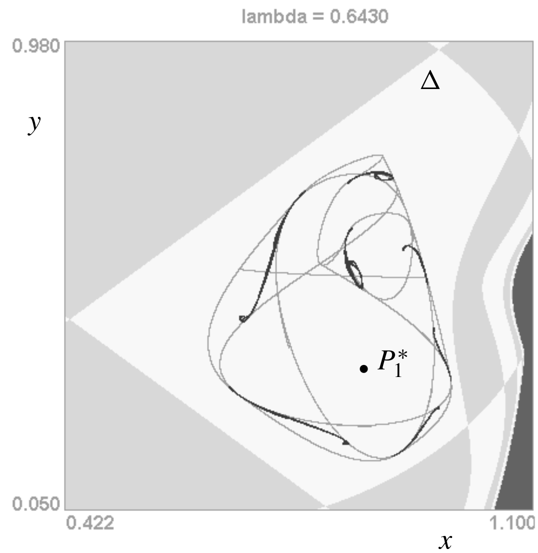

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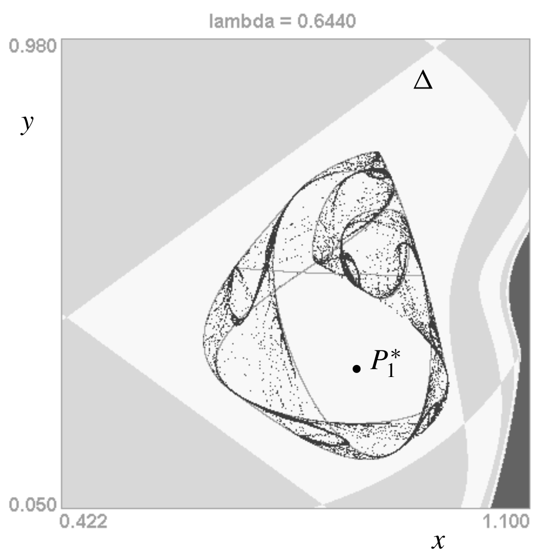

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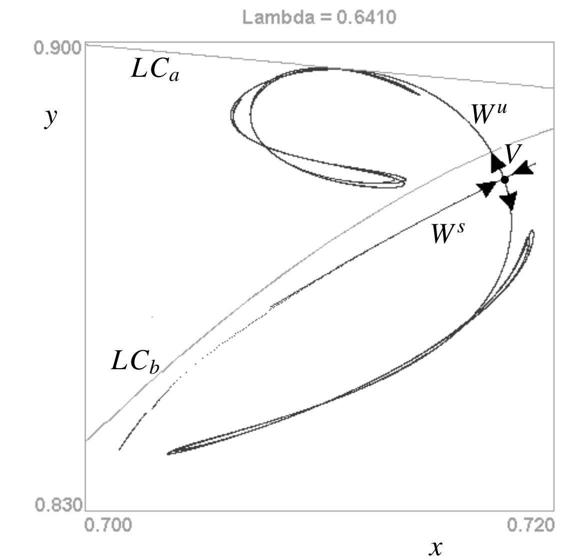

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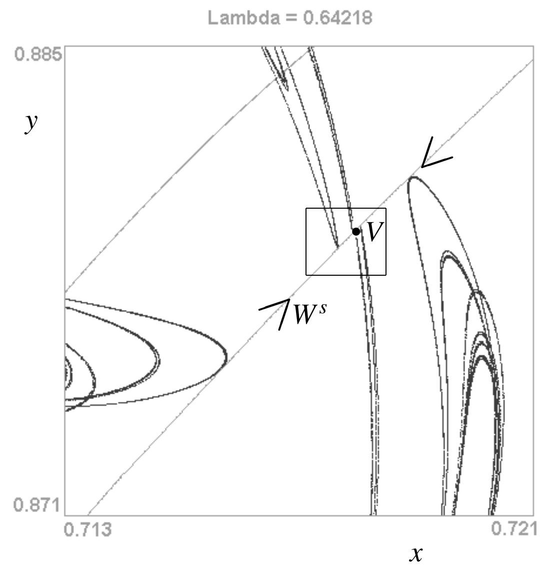

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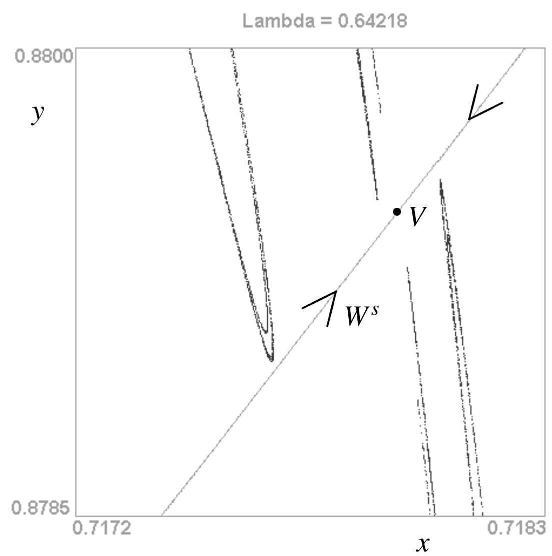

Fig. 2c
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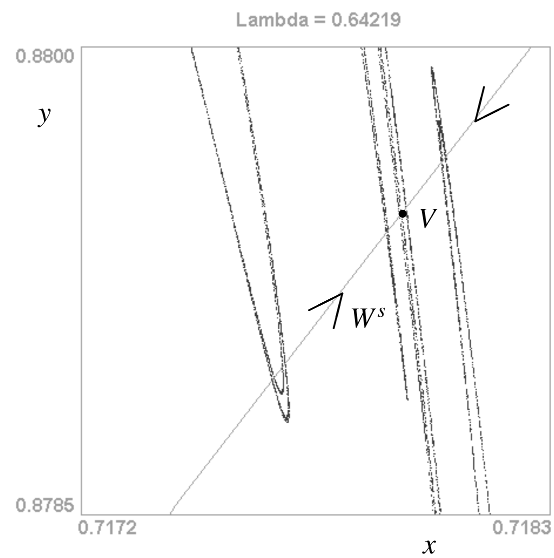

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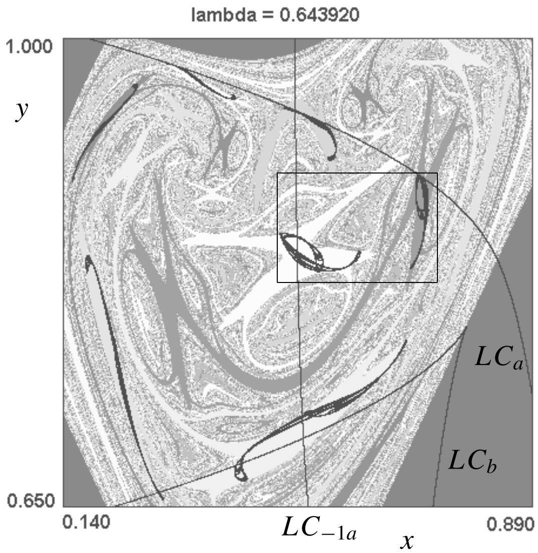

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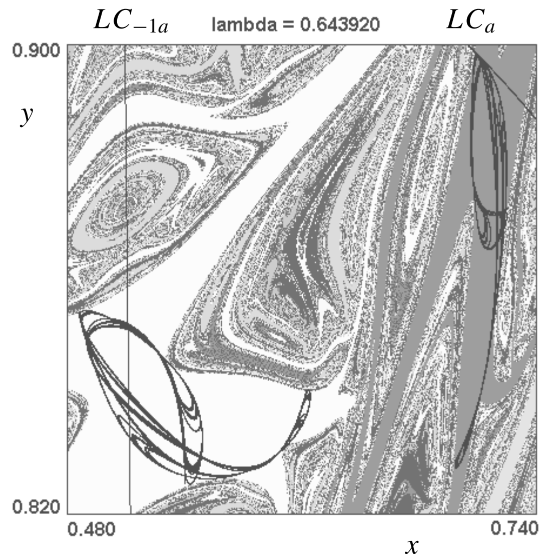

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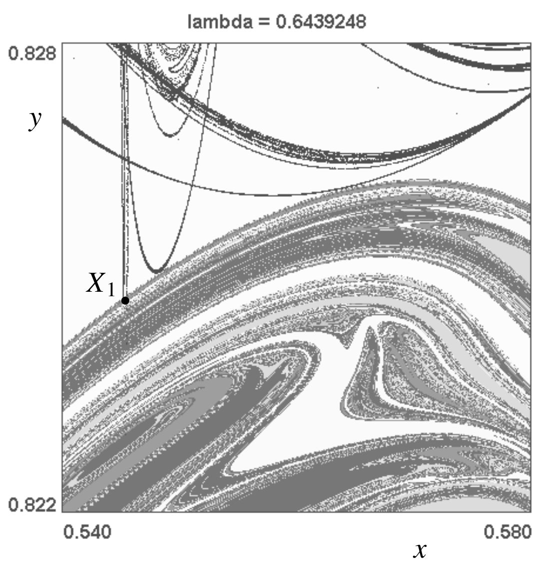

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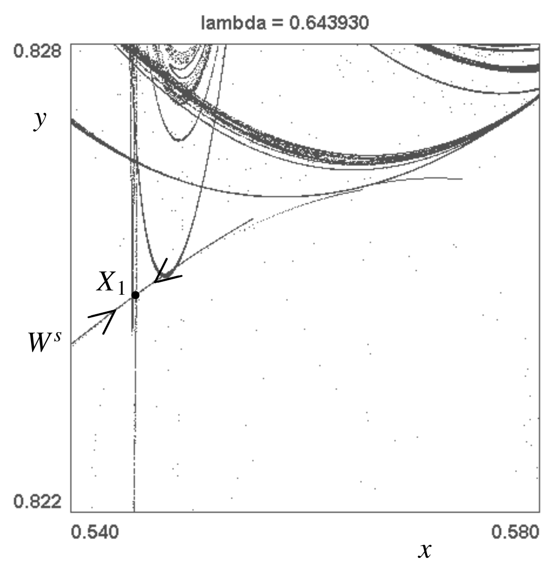

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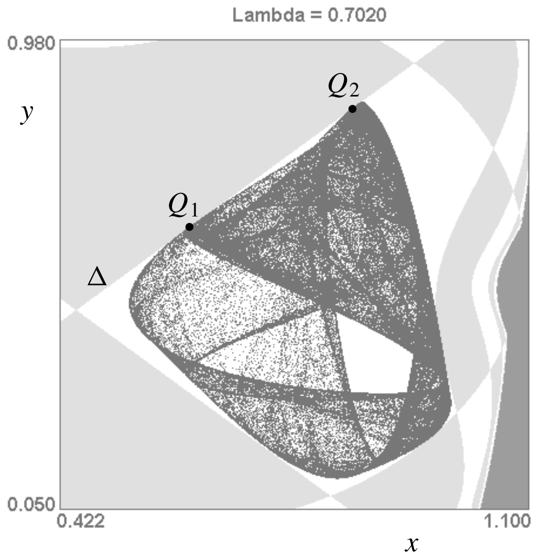

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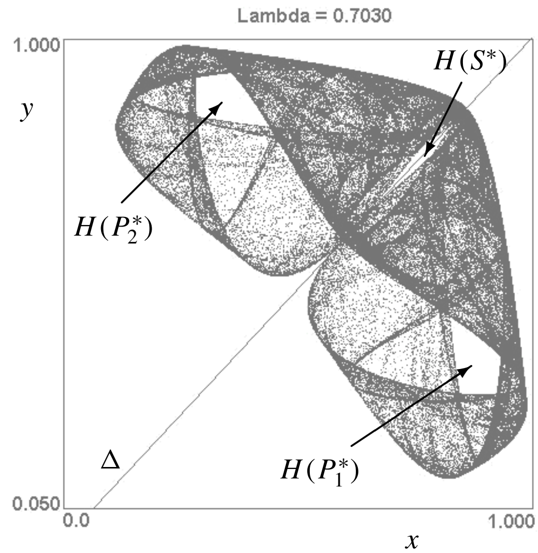

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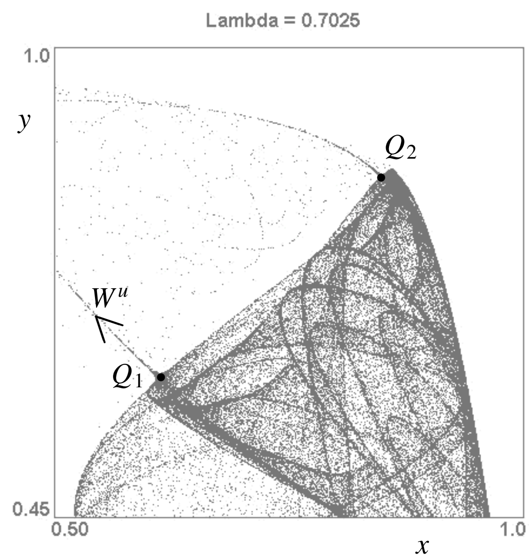

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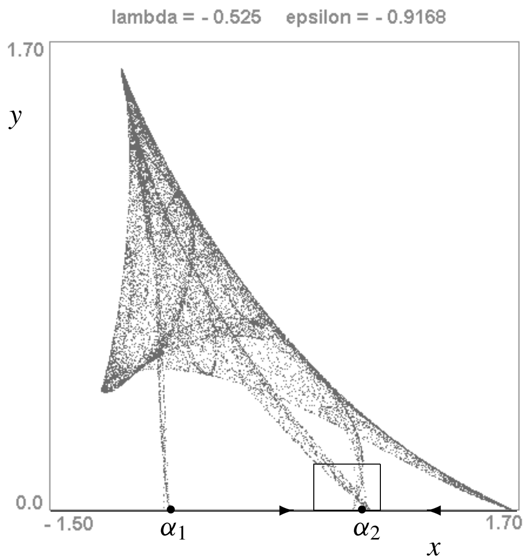

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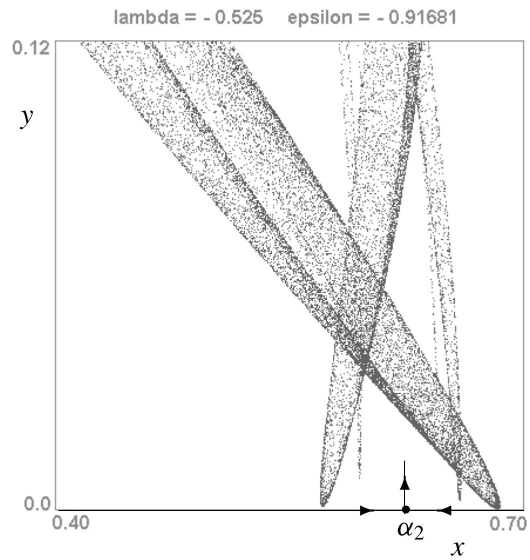

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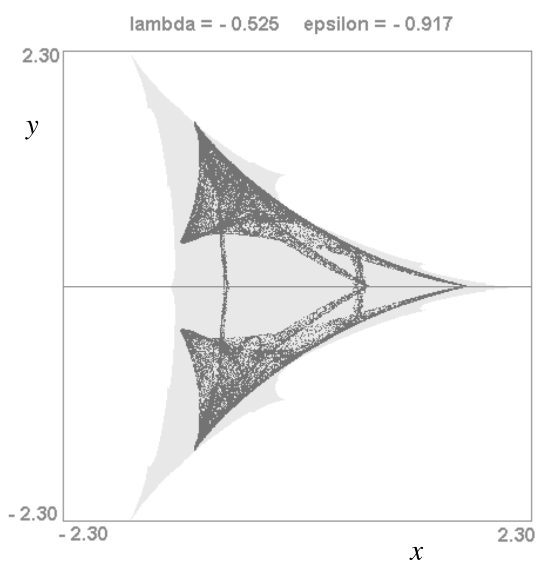

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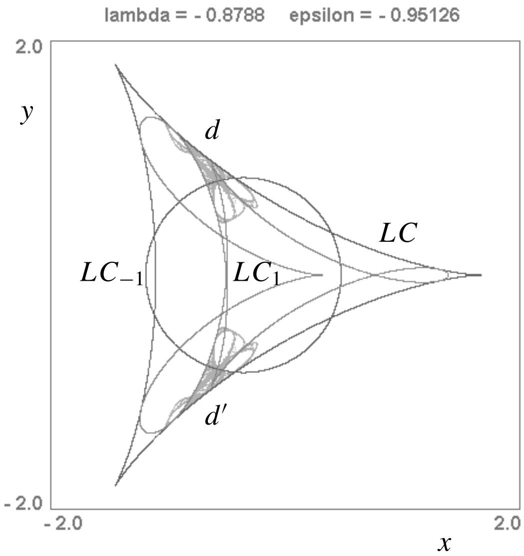

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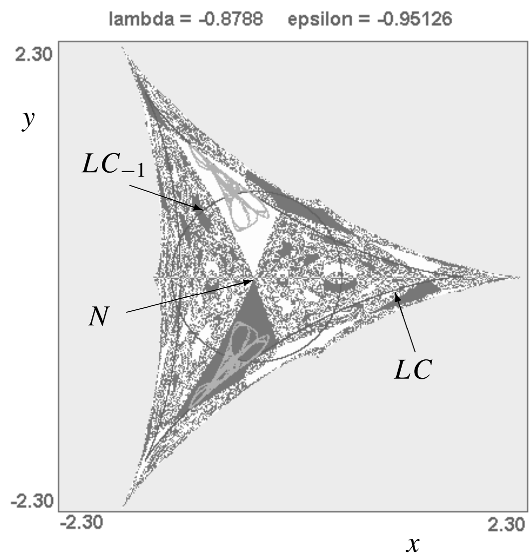

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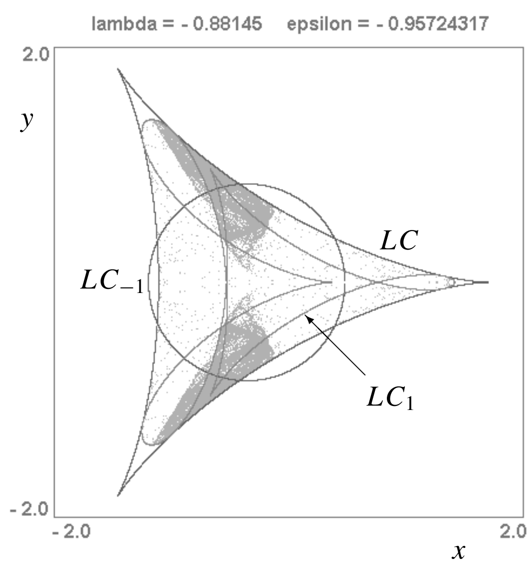

Fig. 6c