THE X–MINIMAL PATTERNS AND THE FORCING RELATION

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1. Introduction

The aim of this note is to give rather negative result about forcing relation with respect to the set of X-minimal (green) patterns. These patterns have been recently discovered [4], [2] as the simpliest types of patterns forced by any ones. The terminology used here is mainly the same as in [1] which is a general reference as well.

Let $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}$ and $\varphi : P \to P$. Then (P, φ) is periodic orbit (or cycle) if $P = \{\varphi^i(p_1)\}_{i=1}^n$ (φ is cyclic permutation of P). We will usually omit φ and we simply say that P is a cycle. The period of the cycle P is $\text{per}(P) = n$.

Two periodic orbits (P, φ), (Q, ψ) are equivalent if there exists a homeomorphism $h : \text{conv}(P) \to \text{conv}(Q)$ such that $h(P) = Q$ and $\psi \circ h|_P = h \circ \varphi$. An equivalence class of this relation will be called pattern. If A is a pattern and $(P, \varphi) \in A$ we say that the cycle P has pattern A (P is representative of A) and we will use the symbol $[P]$ to denote pattern A.

Let $\mathcal{I}$ be the set of all closed real intervals. We consider the space $C(\mathcal{I})$ of all continuous maps $f : I \to I$ where $I \in \mathcal{I}$. A function $f \in C(\mathcal{I})$ has a cycle (P, φ) if $f|_P = \varphi$. In this case we shall say that f exhibits pattern $[P]$. A pattern A forces a

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pattern B if all maps in $C(\mathfrak{I})$ exhibiting A, exhibit also B. By f_P we mean P-linear map of P, i.e., $f_P \in C(\mathfrak{I})$ such that $I = \mathrm{conv}(P)$, $f_P|_P = \varphi$ and for any interval $J \subset I$ such that $J \cap P = \emptyset$ we have that $f_P|_J$ is linear. Obviously, f_P is a piecewise linear map. A modality of f_P (mod(P), mod([P])) is the number of local extremes of f_P in the interior of I.

A cycle (P, φ) has eccentricity m/n if for any map $f \in C(\mathfrak{I})$ with cycle P there exists a fixed point $c \in \mathrm{Fix}(f)$ such that

$$\frac{\#\{x \in P;\ x \le c\}}{\#\{x \in P;\ x \ge c\}} = \frac{m}{n}.$$

Note that cycle $(h(P), h^{-1} \circ \varphi \circ h)$ where $h(x) = -x$ has eccentricity n/m and so we can define eccentricity of pattern $[P]$ as eccentricity of its representative such that eccentricity is greater than or equal to 1. A pattern may have more different eccentricities. But as we have shown in [4] for investigating of forcing relation on patterns it is principal to consider only the patterns with a unique eccentricity m/n. Such a pattern is called an m/n-unipattern (m/n-unicycle for its representative). In what follows we shall always consider patterns (cycles) with eccentricity m/n greater than 1. Note that for an m/n-unipattern $[P]$ the relation $\mathrm{per}(P) = k(m+n)$ holds.

Definition. A pattern A with eccentricity m/n is X-minimal if it does not force any other pattern with the same eccentricity.

The X-minimality and its full description give us a powerful tool for deeper understanding of ideas connected to Sharkovskii's Theorem (see [4]). To recall the basic facts about X-minimality we need a definition of a green pattern. We suppose the spatial labeling of P, i.e., $P = \{p_1 < \cdots < p_{k(m+n)}\}$.

Definition. We shall say that an m/n-pattern $[P]$ is green if the map f_P has the following properties:

(i) f_P has a unique fixed point c ($p_{km} < c < p_{km+1}$),
(ii) if $p_i < p_j \le c$ and $f_P(p_i) > c$, $f_P(p_j) > c$ then $f_P(p_i) > f_P(p_j)$,
(iii) if $p_i < p_j < c$ and $f_P(p_i) < c$, $f_P(p_j) < c$ then $f_P(p_i) < f_P(p_j)$,
(iv) if $c < p_i < p_j$ then $c > f_P(p_i) > f_P(p_j)$.

Theorem 1.1. *Let $[P]$ be an X-minimal m/n-pattern. Then*
(v) *$[P]$ is green,*
(vi) $\mathrm{per}(P) = m + n$.

Note that the conditions from Theorem 1.1 are not sufficient for X-minimality. A complete characterization is not needed for our purpose and can be found in [4].

Considering a forcing relation on patterns the meaning of X-minimality seems to be principal. In [4] (see also [2]) one can find the following

Theorem 1.2. *Let A be a pattern with eccentricity greater than or equal to m/n. Then A forces an X-minimal pattern with eccentricity m/n.*

The previous Theorem says besides others that looking for a minimal entropy on the set of m/n-patterns, it is sufficient to check only X-minimal patterns. In [5], [3] the following result was independently proved.

Theorem 1.3. *A topological entropy on the set of m/n-patterns attains its minimum at a unimodal X-minimal m/n-pattern.*

Considering the theorems above a question arises. Under which condition we can say that an m/n-pattern forces an X-minimal m/n-pattern which is not unimodal. The main result of this note is Theorem 2.2 which shows that such a condition cannot be simply expressed using modality, eccentricity or entropy. The Theorem 2.1 completes our information on forcing on green patterns.

2. The Theorems

Theorem 2.1. *Let $u, v, s, t \in N$ and $u/v > 1, s/t > 1$. Then every green s/t-pattern is forced by a green u/v-pattern.*

Remark. Even if we assume that $u/v \geq s/t$ we cannot replace the green u/v-pattern by an X-minimal u/v-pattern (not even if s/t-pattern is X-minimal).

Theorem 2.2. *Let $m, n, p \in N$. There exists a pattern $[P]$ such that*:

(vii) *the modality of $[P]$ is greater than m,*
(viii) *a topological entropy of $[P]$ is greater than n,*
(ix) *an eccentricity of $[P]$ is greater than p,*
(x) *green patterns forced by $[P]$ have modality equal to one.*

3. The Lemmas and the Construction

In this part we will state the results which will be useful when proving the Theorems. The Lemma 3.1 is a folk knowledge.

We will say that a finite sequence $\{J_k\}_{k=0}^{m-1}$ of closed intervals is f-cyclic if $f(J_k) \supset J_{k+1}$ and $f(J_{m-1}) \supset J_0$.

Lemma 3.1. *Let $\{J_k\}_{k=0}^{m-1}$ be f-cyclic. Then there is a periodic point $x \in \mathrm{Per}(f)$ such that $f^k(x) \in J_k$ for $k = 0, \ldots, m-1$ and $f^m(x) = x$.*

In the proof of Theorem 2.2 we will use the orbits of the Lipschitz function F with the very uniform behaviour described in Lemma 3.2.

The Construction (see also [6]). *Let* $\alpha > 2$. *We will define a map* $F : [0, 1] \to [0, 1]$ *with infinitely many linear pieces with the slope* $\pm\alpha$. *Moreover,* F *will have only one limit point of the critical points and any critical point will be mapped after finite time into the set of fixed points* $\{0, \alpha/(\alpha + 1)\}$ (*see Figure* 1).

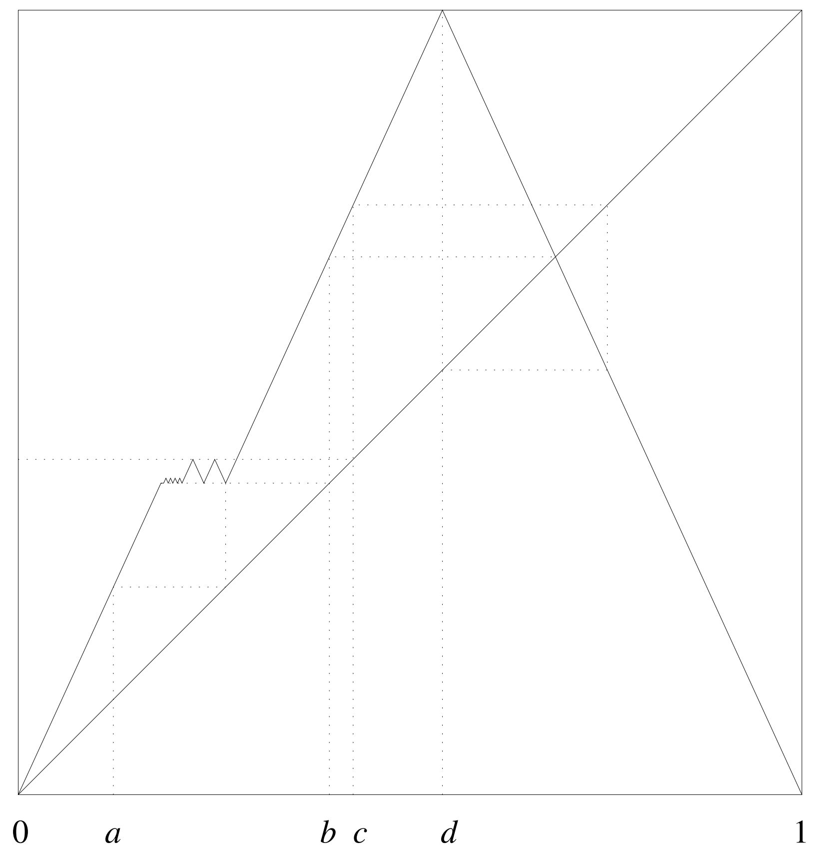

Fig. 1

Here is the formal definition of F. *Let*

$$
\begin{aligned}
x_0 &= 1, & F(x_0) &= 0,\\
x_1 &= \frac{\alpha - 1}{\alpha}, & F(x_1) &= 1,\\
x_2 &= \frac{\alpha}{\alpha + 1} - \frac{2}{\alpha^2}, & F(x_2) &= \frac{\alpha^2 - 2}{\alpha(\alpha + 1)},\\
x_\infty &= \frac{\alpha^2 - 2}{\alpha^2(\alpha + 1)}, & F(x_\infty) &= \frac{\alpha^2 - 2}{\alpha(\alpha + 1)}, \qquad F(0) = 0.
\end{aligned}
$$

Now for $n \geq 1$ let

$$p_n = \frac{1}{\alpha}\Big(\frac{1}{\alpha} - \frac{1}{\alpha+1} + \sum_{i=1}^{2n} \frac{(-1)^i}{\alpha^{i+1}}\Big)$$

and $\Pi(p_n) = p_{n+1}$. We will define the set $\{x_i\}_{i=3}^{\infty}$ and $F(x_i)$ inductivelly using following simple algorithm:

0. *Set $n = 1$ and $p = p_1$.*
1. *If $x_{2n} - 2p \leq x_\infty$ then set $p = \Pi(p)$,*
   else let
   $x_{2n+1} = x_{2n} - p, \quad F(x_{2n+1}) = (\alpha^2 - 2)/\alpha(\alpha+1) + \alpha p,$
   $x_{2n+2} = x_{2n} - 2p, \quad F(x_{2n+2}) = F(x_2).$
2. *Set $n = n + 1$ and goto step 1.*

Finally let function F be linear on the complementary intervals to the points $\{0, x_\infty\} \cup \{x_i\}_{i=0}^{\infty}$.

We have mentioned a uniform behaviour of the function F. Namely we have the following

Proposition. *Function F has the following properties*:

(xi) $F \in C(\mathfrak{I})$.

(xii) $x_i > x_{i+1}$ *for* $i \geq 0$ *and* $\lim_{i\to\infty} x_i = x_\infty$.

(xiii) *F has slopes $\pm\alpha$ on $[0, x_\infty]$ and $[x_{i+1}, x_i]$ for $i \geq 0$.*

(xiv) *$L(F) = \alpha$ ($L(F)$ is Lipschitz constant of F).*

(xv) *For every x_i either there is a $k_i \in N$ such that $F^{k_i}(x_i) = 0$ (i is odd) or $F^2(x_i) = \alpha/(\alpha+1)$ and $\alpha/(\alpha+1)$ is a fixed point.*

(xvi) *F^2 is transitive.*

(xvii) $h(F) = \log\alpha$.

(xviii) *For any $n \in N$ and $\varepsilon > 0$ there is a cycle P in F such that $h(P) > h(F) - \varepsilon$ and* $\mathrm{mod}(P) > n\}$.

(xix) *Green patterns exhibited by F are unimodal.*

Proof. Properties (xi)–(xv) and (xix) follow from the construction of the function F (see Figure 1).

In order to prove (xvi) let us show that for each interval $J \subset I$ there is $n > 0$ such that $F^n(J) = I$. It is clear if $J \cap \big(\{0, x_\infty, \alpha/(\alpha+1)\} \cup \{x_i\}_{i=0}^{\infty}\big) \neq \emptyset$. Assume that the previous intersection is empty. The definition of F implies that $|F(J)| = |J|\alpha$. Hence there is $m > 0$ such that $F^m(J) \cap \big(\{0, x_\infty, \alpha/(\alpha+1)\} \cup \{x_i\}_{i=0}^{\infty}\big) \neq \emptyset$ and (xvi) is proved.

Finally, it only remains to show (xvii),(xviii). These properties can be seen using Lemma 3.2. Using this lemma one can find a periodic orbit P of function F with modality greater than n such that the absolute value of the slopes of f_P is not less than $\alpha - 1/n$. Clearly $h(F) > h(P) = h(f_P) \geq \log(\alpha - 1/n)$ (cf. [7]). Finally we use the simple fact that $h(F) \leq \log L(F)$. □

Let $U_\varepsilon(x) = [x - \varepsilon, x + \varepsilon]$ (a closed ε-neighbourhood of a point x).

Lemma 3.2. *Let $m, l \in N$ and $M = \{m_i\}_{i=1}^{l}$ where $1 < m_i \in N$. Then there is an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there is a periodic orbit P of function F with eccentricity greater than m such that*

(xx) $P \subset \bigcup_{i=1}^{l} U_\varepsilon(x_{m_i}) \cup [0, a] \cup [b, c] \cup [d, 1]$,

(xxi) $P \cap U_\varepsilon(x_{m_i}) \neq \emptyset$ *for* $i = 1, \dots, l$,

(xxii) $P \cap [0, \varepsilon] \neq \emptyset$,

where

$$a = \frac{1}{\alpha}\Big(\frac{\alpha}{\alpha+1} - \frac{2}{\alpha^2}\Big), \qquad b = \frac{1}{\alpha}\Big(\frac{\alpha}{\alpha+1} + \alpha - 2\Big),$$
$$c = \frac{1}{\alpha}\Big(\alpha - 1 - \frac{1}{\alpha} + \frac{1}{\alpha^2}\Big), \quad d = 1 - \frac{1}{\alpha}$$

(*see Figure* 1).

Proof. Put $\omega = \max\{m_1, \dots, m_l\}$ and $\varepsilon_0 = \min\{a, x_\omega - x_{\omega+1}\}$. Fix $0 < \varepsilon < \varepsilon_0$. Obviously $U_\varepsilon(x_{m_j}) \cap \{x_i\}_{i=0}^{\infty} = \{x_{m_j}\}$ for each $j \in \{1, \dots, l\}$.

Denote k_j the least positive integer such that $F^{k_j}(U_\varepsilon(x_{m_j})) \supset [0, a]$. Such k_j exists for each $j \in \{1, \dots, l\}$ by (xv) (see the proof of (xvi)). It is easy to see that for all $j \in \{1, \dots, l\}$ there is interval $J_j \subset U_\varepsilon(x_{m_j})$ such that $F^{k_j}(J_j) = [0, a]$ and $F^i(J_j) \subset ([0, a] \cup [b, c] \cup [d, 1])$ for $i = 1, \dots, k_j$.

Let $L(U)$ denote the leftmost connected part of the preimage of U with respect to F and inductively $L^n(U) = L(L^{n-1}(U))$. For each $j \in \{1, \dots, l\}$ let $n_j \in N$ be such that $L^{n_j}(U_\varepsilon(x_{m_j})) \subset [0, \varepsilon]$ and $n_j > mk_j$. Denote $I_j = L^{n_j}(U_\varepsilon(x_{m_j}))$. It is clear that $F^i(I_j) \subset [0, a]$ for all $i \in \{0, \dots, n_j\}$.

Finally we have that

$$\begin{aligned}\{&F(I_1), \dots, F^{n_1}(I_1), F(J_1), \dots, F^{k_1}(J_1)\\ &F(I_2), \dots, F^{n_2}(I_2), F(J_2), \dots, F^{k_2}(J_2)\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &F(I_l), \dots, F^{n_l}(I_l), F(J_l), \dots, F^{k_l}(J_l)\}\end{aligned}$$

is F-cyclic. Using Lemma 3.1 and the sequence above we obtain cycle P that clearly satisfies (xx), (xxi) and (xxii). Moreover, the eccentricity of P is greater than $\sum n_i \big/ \sum k_i > m$. □

4. The Proofs

Proof of the Theorem 2.1. Let (P, φ) be a representative of a green s/t-pattern. For simplicity assume that $\mathrm{per}(P) = s + t$ (s, t do not have to be coprime). We will show how to extend it to some green u/v-pattern which forces $[P]$. Let l be the least positive integer such that $lv > t$ and $lu - s > lv - t$. Consider an orbit (Q, ψ) defined the following way: $P \subset Q$, $Q \setminus P = \{q_i\}_{i=1}^{l(u+v)-(s+t)}$ and put $Q = Q_3 \cup Q_1 \cup P \cup Q_2$ where $Q_3 < Q_1 < P < Q_2$ and

$$
\begin{aligned}
Q_1 &= \{q_{2(lv-t)-1} < q_{2(lv-t)-3} < \cdots < q_3 < q_1\},\\
Q_2 &= \{q_2 < q_4 < \cdots < q_{2(lv-t)-2} < q_{2(lv-t)}\},\\
Q_3 &= \{q_{2(lv-t)+1} < q_{2(lv-t)+2} < \cdots < q_{l(u+v)-s-t-1} < q_{l(u+v)-s-t}\},\\
P &= \{p_1 < p_2 < \cdots < p_{s+t}\}.
\end{aligned}
$$

Assuming that c is a fixed point of f_P, $p_s < c < p_{s+1}$ we have $\#(Q \cap [0, c]) = lu$, $\#(Q \cap [c, 1]) = lv$. Let

$$
\begin{aligned}
\psi^i(p_1) &= \varphi^i(p_1) && \text{for } i = 1, 2, \ldots, s + t - 1,\\
\psi^i(p_{s+t}) &= q_i, && \text{for } i = 1, 2, \ldots, 2(lv - t),\\
\psi^i(q_{2(lv-t)}) &= q_{2(lv-t)+i} && \text{for } i = 1, 2, \ldots, lu - lv - s + t,\\
\psi(q_{l(u+v)-s-t}) &= p_1.
\end{aligned}
$$

Clearly (Q, ψ) is green (note that $\psi^{s+t-1}(p_1) = p_{s+t}$) with eccentricity u/v, $\mathrm{per}(Q) = l(u + v)$. In order to show that pattern $[Q]$ forces $[P]$ consider an interval $[p_1, \alpha] \subset [p_1, p_2]$ such that $f_Q^i([p_1, \alpha])$ is a subset of the basic interval given by Q for each $i \in \{1, 2, \ldots, s + t - 1\}$ and $f_Q^{s+t-1}([p_1, \alpha]) = [p_{s+t-1}, p_{s+t}]$. Obviously there exists unique α satisfying these conditions. Since $[p_1, \alpha] \subset [q_1, p_2] \subset f_Q^{s+t}([p_1, \alpha])$ the conditions of Lemma 3.1 are satisfied. Hence f_Q has a cycle $R \subset \bigcup_{i=1}^{s+t} f_Q^i([p_1, \alpha])$ and it is clear that $[R] = [P]$. So $[Q]$ forces $[P]$. $\square$

Proof of the Theorem 2.2. Fix a triple of positive integers m, n, p. Let α be greater than n. Using Lemma 3.2 we can choose an orbit P of a map F such that the properties (vii)–(x) are satisfied.

By (xix) all green patterns exhibited by F are unimodal. To get a large modality we need a large number l in Lemma 3.2 and analogously an entropy will be close to α for sufficiently small ε (cf. [7]). So we can find a representative P of the pattern $[P]$ exhibited by F satisfying (vii)–(x). $\square$

Remark. It seems to be clear how to construct a pattern $[P]$ which satisfies the conditions (vii)–(ix) of Theorem 2.2 and f_P exhibits X-minimal patterns with modalities less than or equal to $p \in N$.

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