DYNAMICAL PROPERTIES OF A PIECEWISE MONOTONIC INTERVAL MAP AND ITS SMALL PERTURBATIONS

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ABSTRACT

Consider a piecewise monotonic map $T : X \to \mathbb{R}$, where X is a finite union of intervals, and set $R(T) = \bigcap_{n=0}^{\infty} \overline{T^{-n}X}$. Using the Markov diagram the entropy, the pressure and the structure of the nonwandering set of T can be described. Investigating random perturbations of T leads to the study of the Perron–Frobenius operator. This is used to characterize T-invariant Borel probability measures on [0, 1], which are absolutely continuous with respect to the Lebesgue measure. Each of these measures has a density, which is an eigenfunction of the Perron–Frobenius operator to the eigenvalue 1. If a certain condition is satisfied then these eigenfunctions are stable under small perturbations of the Perron–Frobenius operator. Finally deterministic perturbations of T are discussed. The entropy, the pressure and the Hausdorff dimension are lower semi-continuous, and upper bounds for the jumps up can be given. Furthermore the behaviour of the decomposition of the nonwandering set of T into maximal topologically transitive subsets with positive entropy is investigated.

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Summary

Let $T : X \to \mathbb{R}$ be a piecewise monotonic map, where X is a finite union of intervals. Set $R(T) = \bigcap_{n=0}^{\infty} \overline{T^{-n}X}$. We discuss the Markov diagram of a piecewise monotonic map, which was introduced by F. Hofbauer (cf. [3], [4], [5] and [9]). In particular we describe how the Markov diagram can be used to "compute" entropy, pressure and the structure of the nonwandering set.

Next we discuss random perturbations of piecewise monotonic maps. Given a fixed expanding piecewise monotonic map $T : [0, 1] \to [0, 1]$, randomly perturbing the initial value leads to the investigation of the Perron–Frobenius operator. It was proved by F. Hofbauer and G. Keller in [6] (cf. also [17]) that this operator essentially behaves like a finite Markov chain. In particular there exists a T-invariant probability measure μ on $[0, 1]$, which is absolutely continuous with respect to the Lebesgue measure. Now assume that μ is unique, and $([0, 1], T, \mu)$ is weak mixing. If T is continuous, and no turning point of T is periodic, then V. Baladi and L.-S. Young could recently show in [1] that the density of the invariant measure for certain small random perturbations of T is close to the density of μ in the L^1-norm.

Finally we discuss deterministic perturbations of piecewise monotonic maps. It was shown by M. Misiurewicz and W. Szlenk in [12] that the entropy is lower semi-continuous, and upper bounds of the jumps up were given by M. Misiurewicz ([10] and [11]). Also the pressure and the Hausdorff dimension are lower semi-continuous, and upper bounds for the jumps up can be given (cf. [15] and [18]). In particular, if T is continuous, and no turning point of T is periodic, then entropy, pressure and Hausdorff dimension are continuous. On the other hand, the decomposition of the nonwandering set into maximal topologically transitive subsets behaves very unstably. However, it can be shown that a maximal topologically transitive subset with positive entropy cannot be destroyed completely by an arbitrary small perturbation (cf. [16]).

1. Dynamical properties of a piecewise monotonic map

In this section we consider a piecewise monotonic map. In order to study dynamical properties of a piecewise monotonic map T, Franz Hofbauer introduced in [3] the Markov diagram, which describes the orbit structure of $(R(T), T)$ (cf. also [4] and [5]). In [9] Gerhard Keller improved this method introducing the Markov extension. This Markov extension makes every piecewise monotonic map to a Markov map.

After describing how we can make $(R(T), T)$ to a topological dynamical system, we introduce the Markov diagram. Then we describe how it can be used to describe dynamical properties, as the topological entropy or the structure of the nonwandering set.

1.1. Piecewise monotonic maps

Let X be a finite union of closed intervals. We call $\mathcal{Z}$ a *finite partition* of X, if $\mathcal{Z}$ consists of finitely many pairwise disjoint open intervals with $\bigcup_{Z\in\mathcal{Z}} \overline{Z} = X$. A function $f : X \to \mathbb{R}$ is called *piecewise continuous* with respect to the finite partition $\mathcal{Z}(f)$ of X, if $f|_Z$ can be extended to a continuous function on the closure of Z for all $Z \in \mathcal{Z}(f)$. For every $x \in X$ at least one of the numbers $f(x^+) := \lim_{y\to x^+} f(y)$ and $f(x^-) := \lim_{y\to x^-} f(y)$ exist, and we always assume, that $f(x) = f(x^+)$ or $f(x) = f(x^-)$. A function $f : X \to \mathbb{R}$ is called *piecewise constant* with respect to the finite partition $\mathcal{Z}(f)$ of X, if $f|_Z$ is constant for all $Z \in \mathcal{Z}(f)$. A piecewise continuous function $f : X \to \mathbb{R}$ is called *piecewise continuous of class R^n*, where $n \in \mathbb{N} \cup \{0, \infty\}$, if there exists a finite partition $\mathcal{Z}(f)$ of X, such that f is piecewise continuous with respect to $\mathcal{Z}(f)$, and for every $j \in \mathbb{N}$, $j \leq n$ the map $f|_Z$ is j times differentiable and $(f|_Z)^{(j)}$ can be extended to a continuous function on the closure of Z for all $Z \in \mathcal{Z}(f)$. Note that if f is of class R^n and $k \leq n$ then f is of class R^k.

Definition 1. A map $T : X \to \mathbb{R}$ is called *piecewise monotone* with respect to the finite partition $\mathcal{Z}$ of X, if $T|_Z$ is bounded, strictly monotone and continuous for all $Z \in \mathcal{Z}$.

For a piecewise monotonic map $T : X \to \mathbb{R}$ we define

$$R(T) := \bigcap_{j=0}^{\infty} \overline{T^{-j}X}. \tag{1.1}$$

A piecewise monotonic map $T : X \to \mathbb{R}$ is called *piecewise monotone of class R^n*, if T is piecewise continuous of class R^n. We call a piecewise monotonic map T, which is at least of class R^1, an *expanding* piecewise monotonic map, if there exists a $j \geq 1$, such that $(T^j)'$ is (more exactly: can be extended to) a piecewise continuous function on $X_j := \bigcap_{l=0}^{j-1} \overline{T^{-l}X}$ and $\inf_{x\in X_j} |(T^j)'(x)| > 1$. If $T : X \to \mathbb{R}$ is piecewise monotone with respect to the finite partition $\mathcal{Z}$ of X, and if $\mathcal{Y}$ is a finite partition of X refining $\mathcal{Z}$ then $\mathcal{Y}$ is called a *generator*, if for every sequence $(Y_n)_{n\in\mathbb{N}_0}$ the set $\bigcap_{j=0}^{\infty} T^{-j} Y_j$ contains at most one point. Note that for every expanding piecewise monotonic map T every finite partition $\mathcal{Y}$ refining $\mathcal{Z}$ is a generator.

A *topological dynamical system* (X, T) is a continuous map T of a compact metric space X into itself.

If $T : X \to \mathbb{R}$ is a piecewise monotonic map then $(R(T), T)$ (this is an abbreviation for $\big(R(T), T|_{R(T)}\big)$) need not be a topological dynamical system, since $T|_{R(T)}$ need not be continuous. Hence, in order to get a topological dynamical system, we modify (X, T) as in [15] via a standard doubling points construction. We shortly describe this construction. Let $\mathcal{Y}$ be a finite partition of X, which refines $\mathcal{Z}$.

We assume throughout this paper, that $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_N\}$ with $Y_1 < Y_2 < \cdots < Y_N$. Set $E(T) := \{\inf Z, \sup Z : Z \in \mathcal{Z}\}$, $E_1(T) := E(T) \setminus (\overline{\mathbb{R} \setminus X})$ and $E := \{\inf Y, \sup Y : Y \in \mathcal{Y}\} \setminus (\overline{\mathbb{R} \setminus X})$. We have that $E(T)$ is the set of endpoints of elements of $\mathcal{Z}$, $E_1(T)$ is the set of all elements of $E(T)$, which are inner points of X (call these points *inner endpoints* of $\mathcal{Z}$), and E is the set of all inner endpoints of $\mathcal{Y}$. Now define $W := \left(\bigcup_{j=0}^{\infty} T^{-j} E\right) \setminus (\overline{\mathbb{R} \setminus X})$. Set $\mathbb{R}_{\mathcal{Y}} := (\mathbb{R} \setminus W) \cup \{x^-, x^+ : x \in W\}$, and define $y < x^- < x^+ < z$, if $y < x < z$ holds in $\mathbb{R}$. This means, that we have doubled all inner endpoints of $\mathcal{Y}$, and we have also doubled all inverse images of doubled points. For $x \in \mathbb{R}_{\mathcal{Y}}$ define $\pi_{\mathcal{Y}}(x) := y$, where $y \in \mathbb{R}$ satisfies either $x = y$ or $y \in W$ and $x \in \{y^-, y^+\}$. We have that $x, y \in \mathbb{R}_{\mathcal{Y}}$, $\pi_{\mathcal{Y}}(x) < \pi_{\mathcal{Y}}(y)$ implies $x < y$.

Now we define a metric on $\mathbb{R}_{\mathcal{Y}}$. For $x, y \in \mathbb{R}_{\mathcal{Y}}$ with $x < y$ let $n_{\mathcal{Y}}(x, y)$ be the minimum of all $k \in \mathbb{N}_0$, such that there exists a $z \in \left(\bigcup_{j=0}^{k} T^{-j} E\right) \setminus (\overline{\mathbb{R} \setminus X})$ with $x \leq z^+$ and $z^- \leq y$ (set $n_{\mathcal{Y}}(x, y) := \infty$, if such a k does not exist). Then define

$$d_{\mathcal{Y}}(x, y) := |\pi_{\mathcal{Y}}(x) - \pi_{\mathcal{Y}}(y)| + \frac{1}{n_{\mathcal{Y}}(x, y) + 1}$$

(set $d_{\mathcal{Y}}(x, y) := |\pi_{\mathcal{Y}}(x) - \pi_{\mathcal{Y}}(y)|$, if $n_{\mathcal{Y}}(x, y) = \infty$). This gives rise to a metric $d_{\mathcal{Y}}$ on $\mathbb{R}_{\mathcal{Y}}$. The topology generated by $d_{\mathcal{Y}}$ is exactly the order topology on $\mathbb{R}_{\mathcal{Y}}$.

For a perfect subset $A \subseteq \mathbb{R}$ denote by $\hat{A}$ the closure of $A \setminus W$ in $\mathbb{R}_{\mathcal{Y}}$. Set $X_{\mathcal{Y}} := \hat{X}$, $R_{\mathcal{Y}} := \{x \in X_{\mathcal{Y}} : \pi_{\mathcal{Y}}(x) \in R(T)\}$, $\hat{\mathcal{Z}} := \{\hat{Z} : Z \in \mathcal{Z}\}$, and $\hat{\mathcal{Y}} := \{\hat{Y} : Y \in \mathcal{Y}\}$. Then there exist $a_1, a_2, \ldots, a_{2N} \in \mathbb{R}_{\mathcal{Y}}$ with $a_1 < a_2 < \cdots < a_{2N}$, where $N := \operatorname{card} \mathcal{Y}$, such that $\hat{\mathcal{Y}} = \{[a_{2j-1}, a_{2j}] : j = 1, 2, \ldots, N\}$, where $[a, b] := \{x \in \mathbb{R}_{\mathcal{Y}} : a \leq x \leq b\}$.

The map $T|_{X\setminus(W\cup E(T))}$ can be extended to a unique continuous piecewise monotonic map $T_{\mathcal{Y}} : X_{\mathcal{Y}} \to \mathbb{R}_{\mathcal{Y}}$. For a function $f : X \to \mathbb{R}$, which is piecewise continuous with respect to $\mathcal{Z}$, let $f_{\mathcal{Y}} : X_{\mathcal{Y}} \to \mathbb{R}$ be the unique continuous function with $f_{\mathcal{Y}}|_{X\setminus(W\cup E(T))} = f|_{X\setminus(W\cup E(T))}$. Hence $(R_{\mathcal{Y}}, T_{\mathcal{Y}})$ is a topological dynamical system, and $f_{\mathcal{Y}} : R_{\mathcal{Y}} \to \mathbb{R}$ is a continuous function.

1.2. The Markov diagram

Now we define an at most countable oriented graph $(\mathcal{D}, \to)$, called Markov diagram, which describes the orbit structure of $(R(T), T)$ (cf. [5]). Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, and let $\mathcal{Y}$ be a finite partition of X, which refines $\mathcal{Z}$. Construct $T_{\mathcal{Y}}$ as in Section 1.1. Let $Y_0 \in \hat{\mathcal{Y}}$ and let D be a perfect subinterval of Y_0. A nonempty $C \subseteq X_{\mathcal{Y}}$ is called *successor* of D, if there exists a $Y \in \hat{\mathcal{Y}}$ with $C = T_{\mathcal{Y}} D \cap Y$, and we write $D \to C$. We get that every successor C of D is again a perfect subinterval of an element of $\hat{\mathcal{Y}}$. Let $\mathcal{D}$ be the smallest set with $\hat{\mathcal{Y}} \subseteq \mathcal{D}$ and such that $D \in \mathcal{D}$ and $D \to C$ imply $C \in \mathcal{D}$. Then $(\mathcal{D}, \to)$ is called the *Markov diagram* of T with respect to $\mathcal{Y}$. The set $\mathcal{D}$ is at most countable and its elements are perfect subintervals of elements of $\hat{\mathcal{Y}}$.

Set $\mathcal{D}_0 := \hat{\mathcal{Y}}$, and for $r \in \mathbb{N}$ set $\mathcal{D}_r := \mathcal{D}_{r-1} \cup \{D \in \mathcal{D} : \exists\, C \in \mathcal{D}_{r-1} \text{ with } C \to D\}$. Then we have $\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \cdots$ and $\mathcal{D} = \bigcup_{r=0}^{\infty} \mathcal{D}_r$.

Let $(\mathcal{H}, \to)$ be an oriented graph. For $n \in \mathbb{N}$ we call $c_0 \to c_1 \to \cdots \to c_n$ a *path of length n in* $\mathcal{H}$, if $c_j \in \mathcal{H}$ for $j \in \{0, 1, \ldots, n\}$ and $c_{j-1} \to c_j$ for $j \in \{1, 2, \ldots, n\}$. We call $c_0 \to c_1 \to c_2 \to \cdots$ an *infinite path in* $\mathcal{H}$, if $c_j \in \mathcal{H}$ for all $j \in \mathbb{N}_0$ and $c_{j-1} \to c_j$ for all $j \in \mathbb{N}$. A subset $\mathcal{C}$ of $\mathcal{H}$ is called *closed*, if $c \in \mathcal{C}$, $d \in \mathcal{H}$ and $c \to d$ imply $d \in \mathcal{C}$. If $\mathcal{C} \subseteq \mathcal{H}$ then define

$$\overline{\mathcal{C}} := \{d \in \mathcal{H} : \text{there exists a finite path } c_0 \to c_1 \to \cdots \to c_n \text{ in } \mathcal{H} \text{ with } c_0 \in \mathcal{C} \text{ and } c_n = d\}$$

and $\tilde{\mathcal{C}} := \overline{\mathcal{C}} \setminus \mathcal{C}$. The oriented graph $\mathcal{H}$ is called *irreducible*, if for every $c, d \in \mathcal{H}$ there exists a finite path $c_0 \to c_1 \to \cdots \to c_n$ in $\mathcal{H}$ with $c_0 = c$ and $c_n = d$. If $\mathcal{H}$ is irreducible and finite then $\mathcal{H}$ is called *finite irreducible*. An irreducible subset $\mathcal{C}$ of $\mathcal{H}$ is called *maximal irreducible* in $\mathcal{H}$, if every $\mathcal{C}'$ with $\mathcal{C} \subsetneqq \mathcal{C}' \subseteq \mathcal{H}$ is not irreducible. If $\mathcal{C}$ is maximal irreducible then $\overline{\mathcal{C}}$ and $\tilde{\mathcal{C}}$ are closed subsets of $\mathcal{H}$.

If $(\mathcal{D}, \to)$ is the Markov diagram of T with respect to $\mathcal{Y}$ then we say an infinite path $D_0 \to D_1 \to D_2 \to \cdots$ *represents* $x \in R_{\mathcal{Y}}$, if $T_{\mathcal{Y}}{}^n x \in D_n$ for all $n \in \mathbb{N}_0$. The following correspondence between Markov diagram and orbits can be shown easily (cf. [5]).

Lemma 1. *For every $x \in R_{\mathcal{Y}}$ there exists a path $D_0 \to D_1 \to D_2 \to \cdots$ in $\mathcal{D}$, which represents x. On the other hand, for every infinite path $D_0 \to D_1 \to D_2 \to \cdots$ in $\mathcal{D}$ there exists an $x \in R_{\mathcal{Y}}$, such that $D_0 \to D_1 \to D_2 \to \cdots$ represents x. The Markov diagram has the property that two different elements x and y of $R_{\mathcal{Y}}$ cannot be represented by the same path $D_0 \to D_1 \to D_2 \to \cdots$ in $\mathcal{D}$, if and only if $\mathcal{Y}$ is a generator.*

Now we want to associate a matrix to the Markov diagram. Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, let $f : X \to \mathbb{R}$ be piecewise constant with respect to the finite partition $\mathcal{Z}(f)$ of X, let $\mathcal{Y}$ be a finite partition of X, which refines both $\mathcal{Z}$ and $\mathcal{Z}(f)$, and let $(\mathcal{D}, \to)$ be the Markov diagram of T with respect to $\mathcal{Y}$. If $C \in \mathcal{D}$ then let f_C be the unique real number with $f_{\mathcal{Y}}(x) = f_C$ for all $x \in C$. For $C, D \in \mathcal{D}$ define

$$F_{C,D}(f) := \begin{cases} e^{f_C} & \text{if } C \to D, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Set $F(f) := \big(F_{C,D}(f)\big)_{C,D \in \mathcal{D}}$, and for $\mathcal{C} \subseteq \mathcal{D}$ set $F_{\mathcal{C}}(f) := \big(F_{C,D}(f)\big)_{C,D \in \mathcal{C}}$. Simple calculations show the following result (cf. [14]).

Lemma 2. *The map $u \mapsto uF_{\mathcal{C}}(f)$ is continuous and linear $\ell^1(\mathcal{C}) \to \ell^1(\mathcal{C})$, and the map $v \mapsto F_{\mathcal{C}}(f)v$ is continuous and linear $\ell^\infty(\mathcal{C}) \to \ell^\infty(\mathcal{C})$. These two operators* *have the same norm* $\|F_{\mathcal{C}}(f)\|$ *and the same spectral radius* $r\big(F_{\mathcal{C}}(f)\big)$. *Furthermore we have*

$$\|F_{\mathcal{C}}(f)\| = \sup_{C\in\mathcal{C}} \sum_{D\in\mathcal{C}} F_{C,D}(f), \tag{1.3}$$

$$\|F_{\mathcal{C}}(f)^n\| = \sup_{C\in\mathcal{C}} \sum_{C_0=C\to C_1\to\cdots\to C_n} \prod_{j=0}^{n-1} e^{f_{C_j}} \qquad \text{for every } n\in\mathbb{N}, \tag{1.4}$$

where the sum is taken over all paths $C_0 \to C_1 \to \cdots \to C_n$ *of length* n *in* $\mathcal{C}$ *with* $C_0 = C$, *and*

$$r\big(F_{\mathcal{C}}(f)\big) = \lim_{n\to\infty} \|F_{\mathcal{C}}(f)^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|F_{\mathcal{C}}(f)^n\|^{\frac{1}{n}}. \tag{1.5}$$

Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, let $\mathcal{Y}$ be a finite partition of X refining $\mathcal{Z}$, and let $(\mathcal{D}, \to)$ be the Markov diagram of T with respect to $\mathcal{Y}$. For $\mathcal{C} \subseteq \mathcal{D}$ define

$$H(\mathcal{C}) := \bigcup_{C\in\mathcal{C}} C. \tag{1.6}$$

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed subsets of $\mathcal{D}$ with $\mathcal{C}_2 \subseteq \mathcal{C}_1$, then set

$$\psi(\mathcal{C}_1, \mathcal{C}_2) := \bigcap_{j=0}^{\infty} \overline{H(\mathcal{C}_1) \setminus T_{\mathcal{Y}}^{-j} H(\mathcal{C}_2)}.$$

For $x \in R_{\mathcal{Y}}$ and $n \in \mathbb{N}_0$ define $Y_n(x) := \bigcap_{j=0}^{n} T_{\mathcal{Y}}^{-j} Y_j$, where $Y_j \in \hat{\mathcal{Y}}$ and $T_{\mathcal{Y}}^{j} x \in Y_j$ for $j \in \{0, 1, 2, \ldots, n\}$. Now define $Y_\infty(x) := \bigcap_{n=0}^{\infty} Y_n(x)$, and set

$$L(\mathcal{C}_1, \mathcal{C}_2) := \psi(\mathcal{C}_1, \mathcal{C}_2) \setminus \Bigg(\bigcup_{x\in\psi(\mathcal{C}_1,\mathcal{C}_2)} \operatorname{int} Y_\infty(x) \Bigg), \tag{1.7}$$

if $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed subsets of $\mathcal{D}$ with $\mathcal{C}_2 \subseteq \mathcal{C}_1$. For a maximal irreducible $\mathcal{C} \subseteq \mathcal{D}$ define

$$L(\mathcal{C}) := L(\overline{\mathcal{C}}, \tilde{\mathcal{C}}). \tag{1.8}$$

The proofs in [5] show the following result.

Lemma 3. *Let* $\mathcal{C}$ *be a maximal irreducible subset of the Markov diagram* $(\mathcal{D}, \to)$ *of* T *with respect to* $\mathcal{Y}$. *Then for every* $x \in L(\mathcal{C})$ *there exists a path* $C_0 \to C_1 \to C_2 \to \cdots$ *in* $\mathcal{C}$, *which represents* x, *and for every path* $C_0 \to C_1 \to C_2 \to \cdots$ *in* $\mathcal{C}$ *there exists an* $x \in L(\mathcal{C})$, *such that* $C_0 \to C_1 \to C_2 \to \cdots$ *represents* x. *If* $\mathcal{Y}$ *is a generator then* $L(\mathcal{C})$ *equals the set of all points* $x \in R_{\mathcal{Y}}$, *which are represented by a path in* $\mathcal{C}$.

1.3. Topological entropy and pressure

Definition 2. Let (X, T) be a topological dynamical system. If $\varepsilon > 0$ and $n \in \mathbb{N}$ then a set $E \subseteq X$ is called (n, ε)-*separated*, if for every $x \neq y \in E$ there exists a $j \in \{0, 1, \dots, n-1\}$ with $d(T^j x, T^j y) > \varepsilon$. For a continuous function $f : X \to \mathbb{R}$ the *topological pressure* $p(X, T, f)$ is defined by

$$p(X, T, f) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_E \sum_{x \in E} \exp\left(\sum_{j=0}^{n-1} f(T^j x) \right), \tag{1.9}$$

where the supremum is taken over all (n, ε)-separated subsets E of X. The *topological entropy* $h_{\text{top}}(X, T)$ is defined by

$$h_{\text{top}}(X, T) := p(X, T, 0). \tag{1.10}$$

Let $\mathcal{Z}$ be a finite partition of X, let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to $\mathcal{Z}$, let $f : X \to \mathbb{R}$ be a piecewise continuous function with respect to $\mathcal{Z}$, and let $\mathcal{Y}$ be a finite partition of X refining $\mathcal{Z}$. Using the variational principle (see, e.g., Theorem 9.10 in [19]) we get the following result (see also Lemma 2 in [14]).

Lemma 4. *If $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are finite partitions of X refining $\mathcal{Z}$, then*

$$p(R_{\mathcal{Y}_1}, T_{\mathcal{Y}_1}, f_{\mathcal{Y}_1}) = p(R_{\mathcal{Y}_2}, T_{\mathcal{Y}_2}, f_{\mathcal{Y}_2}).$$

Now we define

$$p(R(T), T, f) := p(R_{\mathcal{Y}}, T_{\mathcal{Y}}, f_{\mathcal{Y}}). \tag{1.11}$$

Using Lemma 4 we see, that this definition does not depend on the partition $\mathcal{Y}$. Furthermore, for $n \in \mathbb{N}$ we define

$$S_n(R(T), f) := \sup_{x \in R_{\mathcal{Y}}} \sum_{j=0}^{n-1} f_{\mathcal{Y}}({T_{\mathcal{Y}}}^j x). \tag{1.12}$$

Observe that this definition does not depend on the partition $\mathcal{Y}$. We remark, that the condition

$$p(R(T), T, f) > \lim_{n \to \infty} \frac{1}{n} S_n(R(T), f),$$

which will be used in this paper is a generalization of the condition $p(R(T), T, f) > \sup_{x \in R(T)} f(x)$.

The following result relates the pressure with the matrix $F(f)$ introduced in Section 1.2. It is proved in [5] and [14] (cf. Theorem 7 and Theorem 10 in [5], and Lemma 6 in [14]).

Theorem 1. *Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, let $f : X \to \mathbb{R}$ be a piecewise constant function with respect to the finite partition $\mathcal{Z}(f)$ of X, and let $\mathcal{Y}$ be a finite partition of X refining both $\mathcal{Z}$ and $\mathcal{Z}(f)$. Denote by $(\mathcal{D}, \to)$ the Markov diagram of T with respect to $\mathcal{Y}$.*

(1) *We have*

$$p(R(T), T, f) = \log r\big(F(f)\big).$$

(2) *If $\mathcal{C} \subseteq \mathcal{D}$ is maximal irreducible then*

$$p(L(\mathcal{C}), T, f) = \log r\big(F_{\mathcal{C}}(f)\big).$$

(3) *If $\mathcal{C} \subseteq \mathcal{D}$ is maximal irreducible, and $\mathcal{C}' \subseteq \mathcal{C}$ is finite irreducible then*

$$p(L(\mathcal{C}'), T, f) = \log r\big(F_{\mathcal{C}'}(f)\big),$$

where $L(\mathcal{C}')$ is the set of all elements in $L(\mathcal{C})$, which can be represented by a path in $\mathcal{C}'$.

Finally we define the Hausdorff dimension. Let (X, d) be a metric space. For an $A \subseteq X$, $A \neq \emptyset$ define $\operatorname{diam} A := \sup_{x,y\in A} d(x, y)$. Let $Y \subseteq X$. For $t \geq 0$ and $\varepsilon > 0$ set

$$m(Y, t, \varepsilon) := \inf \Big\{\textstyle\sum_{A\in\mathcal{A}} (\operatorname{diam} A)^t : \mathcal{A} \text{ is an at most countable cover of } Y \text{ with } \operatorname{diam} A < \varepsilon \text{ for all } A \in \mathcal{A}\Big\}.$$

Then $\varepsilon \mapsto m(Y, t, \varepsilon)$ is decreasing, and hence

$$m(Y, t) := \lim_{\varepsilon\to 0} m(Y, t, \varepsilon) = \sup_{\varepsilon>0} m(Y, t, \varepsilon)$$

exists (but note that it may be $m(Y, t) = \infty$). Furthermore there exists a unique $t_0 \in [0, \infty]$ with $m(Y, t) = \infty$ for all $t < t_0$ and $m(Y, t) = 0$ for all $t > t_0$. This t_0 is called the Hausdorff dimension of Y.

Definition 3. Let Y be a nonempty subset of a metric space (X, d). Then

$$\operatorname{HD}(Y) := \inf \Big\{t \geq 0 : \lim_{\varepsilon\to 0} m(Y, t, \varepsilon) = 0\Big\}, \tag{1.13}$$

where we set $\operatorname{HD}(Y) := \infty$, if $\lim_{\varepsilon\to 0} m(Y, t, \varepsilon) > 0$ for all $t \geq 0$, is called the *Hausdorff dimension* of Y.

Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, and let $\mathcal{Y}$ be a finite partition of X refining $\mathcal{Z}$. In [14] the definition of the Hausdorff dimension is slightly modified, which allows to define the Hausdorff dimension on $\mathbb{R}_{\mathcal{Y}}$, such that B and $\pi_{\mathcal{Y}}(B)$ have the same Hausdorff dimension whenever $\pi_{\mathcal{Y}}(B)$ is a closed subset of X.

1.4. The structure of the nonwandering set

We begin this section with some definitions for dynamical systems. Then we describe a well known result on the structure of the nonwandering set of $(R(T), T)$.

Definition 4. Let (X, T) be a topological dynamical system, and let $x \in X$. Then

$$\omega(x) := \big\{ y \in X : \text{there exists a strictly increasing sequence } (n_k)_{k\in\mathbb{N}} \text{ in } \mathbb{N}_0 \text{ with } \lim_{k\to\infty} T^{n_k}x = y \big\} \tag{1.14}$$

is called the *ω-limit set* of x.

For every $x \in X$ the set $\omega(x)$ is a nonempty, closed and T-invariant subset of X.

Definition 5. Let (X, T) be a topological dynamical system, and let $R \subseteq X$. Then R is called *topologically transitive*, if there exists an $x \in R$ with $\omega(x) = R$. If R is topologically transitive and every R' with $R \subsetneqq R' \subseteq X$ is not topologically transitive then R is called *maximal topologically transitive*. Finally R is called *minimal*, if $\omega(x) = R$ for every $x \in R$.

Every topologically transitive subset R of X is closed and T-invariant.

Definition 6. Let (X, T) be a topological dynamical system. Then the *nonwandering set* $\Omega(X, T)$ of (X, T) is defined by

$$\Omega(X, T) := \big\{ x \in X : \text{for every open } U \text{ with } x \in U \text{ there exists an } n \in \mathbb{N} \text{ with } T^n U \cap U \neq \emptyset \big\}. \tag{1.15}$$

The nonwandering set is always a closed, T-invariant subset of X, and

$$\bigcup_{x\in X} \omega(x) \subseteq \Omega(X, T).$$

If $f : X \to \mathbb{R}$ is a continuous function then a well known result (see, e.g., Corollary 9.10.1 in [19]) says that

$$p(X, T, f) = p\big(\Omega(X, T), T|_{\Omega(X,T)}, f|_{\Omega(X,T)}\big). \tag{1.16}$$

This result implies that $h_{\text{top}}(X, T) = h_{\text{top}}\big(\Omega(X, T), T|_{\Omega(X,T)}\big)$.

Now we consider again a piecewise monotonic map $T : X \to \mathbb{R}$. The next result says, that $L(\mathcal{C})$ is topologically transitive, if $\mathcal{C}$ is a maximal irreducible subset of the Markov diagram. It is proved in [5] (see Theorem 4 in [5]). The main idea of this proof is Lemma 3 and the fact, that there exists an infinite path $C_0 \to C_1 \to C_2 \to \cdots$ in $\mathcal{C}$, which contains every finite path in $\mathcal{C}$, since $\mathcal{C}$ is irreducible.

Theorem 2. *Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, and let $\mathcal{Y}$ be a finite partition of X refining $\mathcal{Z}$. If $\mathcal{C}$ is a maximal irreducible subset of the Markov diagram of T with respect to $\mathcal{Y}$, then $L(\mathcal{C})$ is a topologically transitive subset of $(R_{\mathcal{Y}}, T_{\mathcal{Y}})$.*

In the case of an expanding T a formula analogous to (1.16) is also true for the Hausdorff dimension. The proof of our next result can be found in [14] (see Theorem 2 in [14]).

Theorem 3. *Let $T : X \to \mathbb{R}$ be an expanding piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, and let $\mathcal{Y}$ be a finite partition of X refining $\mathcal{Z}$.*

(1) *The function $t \mapsto p(R(T), T, -t \log |T'|)$ defined on $\mathbb{R}$ is continuous, strictly decreasing, and has a unique zero $t_R \in [0, 1]$. This value t_R satisfies*

$$\mathrm{HD}\big(R(T)\big) = \mathrm{HD}\big(\pi_{\mathcal{Y}}(\Omega(R_{\mathcal{Y}}, T_{\mathcal{Y}}))\big) = t_R.$$

(2) *If $\mathcal{C}$ is a maximal irreducible subset of the Markov diagram of T with respect to $\mathcal{Y}$ then the function $t \mapsto p(L(\mathcal{C}), T_{\mathcal{Y}}, -t \log |T'|_{\mathcal{Y}})$ defined on $\mathbb{R}$ is continuous, strictly decreasing, and has a unique zero $t_{\mathcal{C}} \in [0, 1]$. Furthermore this value $t_{\mathcal{C}}$ satisfies*

$$\mathrm{HD}\big(\pi_{\mathcal{Y}}(L(\mathcal{C}))\big) = t_{\mathcal{C}}.$$

Now we present a result, which describes the structure of the nonwandering set of a piecewise monotonic map. It is proved in [4], [5], [7] and [13].

Theorem 4. *Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, and let $\mathcal{Y}$ be a finite partition of X, which refines $\mathcal{Z}$. Then*

$$\Omega(R_{\mathcal{Y}}, T_{\mathcal{Y}}) = \bigcup_{\mathcal{C}\in\Gamma} L(\mathcal{C}) \cup \bigcup_{j\in J} N_j \cup P \cup W, \tag{1.17}$$

where Γ is the at most countable set of maximal irreducible subsets of the Markov diagram $(\mathcal{D}, \to)$ of T with respect to $\mathcal{Y}$, J is an at most finite index set, and the intersection of two different sets in the decomposition is at most finite. Furthermore:

(1) *For every $\mathcal{C} \in \Gamma$ the periodic points of $(L(\mathcal{C}), T_{\mathcal{Y}})$ are dense in $L(\mathcal{C})$. Furthermore either $L(\mathcal{C})$ consists only of one single periodic orbit (in this case for every $C \in \mathcal{C}$ there exists exactly one $D \in \mathcal{C}$ with $C \to D$), or $L(\mathcal{C})$ is an uncountable, maximal topologically transitive subset of $R_{\mathcal{Y}}$ with $h_{\mathrm{top}}(L(\mathcal{C}), T_{\mathcal{Y}}) > 0$ (in this case there exists at least one $C \in \mathcal{C}$, which has more than one successor in $\mathcal{C}$).*

(2) *For every $j \in J$ the set N_j is an uncountable, minimal subset of $R_{\mathcal{Y}}$, which contains no periodic points. Furthermore N_j is maximal topologically transitive, $h_{\mathrm{top}}(N_j, T_{\mathcal{Y}}) = 0$, and there exist only finitely many ergodic, $T_{\mathcal{Y}}$-invariant Borel probability measures on $(N_j, T_{\mathcal{Y}})$.*

(3) *The set P is closed and $T_{\mathcal{Y}}$-invariant, and consists of periodic points, which are contained in nontrivial intervals K with the property that ${T_{\mathcal{Y}}}^n$ maps K monotonically into K for an $n \in \mathbb{N}$.*

(4) *The set W consists of nonperiodic points, which are isolated in $\Omega(R_{\mathcal{Y}}, T_{\mathcal{Y}})$, and therefore are not contained in $\Omega\big(\Omega(R_{\mathcal{Y}}, T_{\mathcal{Y}}), T_{\mathcal{Y}}\big)$.*

Remark. If T is expanding then it follows from Theorem 1 in [14] that

$$\mathrm{HD}\big(\pi_{\mathcal{Y}}(N_j)\big) = 0$$

for all $j \in J$.

This result implies that every at most countable maximal topologically transitive subset of $R_{\mathcal{Y}}$ is a single periodic orbit. Hence every uncountable maximal topologically transitive subset of $R_{\mathcal{Y}}$ is one of the sets $L(\mathcal{C})$ or N_j. Define

$$\mathcal{T}_{\mathcal{Y}}(T) := \big\{L : L \text{ is an uncountable maximal topologically transitive subset of } R_{\mathcal{Y}}\big\}. \tag{1.18}$$

As a consequence of Theorem 4 we get that $\mathcal{T}_{\mathcal{Y}}(T)$ is essentially independent of $\mathcal{Y}$.

Corollary 4.1. *If $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are both finite partitions of X refining $\mathcal{Z}$ then there exists a bijective function $\varphi : \mathcal{T}_{\mathcal{Y}_1}(T) \to \mathcal{T}_{\mathcal{Y}_2}(T)$, such that $\pi_{\mathcal{Y}_2}\big(\varphi(L)\big) = \pi_{\mathcal{Y}_1}(L)$ for every $L \in \mathcal{T}_{\mathcal{Y}_1}(T)$.*

This motivates the following definition. Set

$$\mathcal{T}(R(T), T) := \{\pi_{\mathcal{Y}}(L) : L \in \mathcal{T}_{\mathcal{Y}}(T)\}. \tag{1.19}$$

By Corollary 4.1 $\mathcal{T}(R(T), T)$ does not depend on $\mathcal{Y}$. We can consider the elements of $\mathcal{T}(R(T), T)$ as the uncountable maximal topologically transitive subsets of $(R(T), T)$. The most interesting part of the dynamics takes place on the at most countable union of maximal topologically transitive subsets with positive entropy. Hence we define

$$\mathcal{M}(R(T), T) := \{L \in \mathcal{T}(R(T), T) : h_{\mathrm{top}}(L, T) > 0\}. \tag{1.20}$$

Observe that the elements of $\mathcal{M}(R(T), T)$ are of the form $\pi_{\mathcal{Y}}\big(L(\mathcal{C})\big)$. The following result is proved in [5] (in the case of the Hausdorff dimension we use also Theorem 3). Note that Theorem 3 implies that $h_{\mathrm{top}}(R(T), T) > 0$ is equivalent to $\mathrm{HD}\big(R(T)\big) > 0$, if T is expanding.

Theorem 5. *Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X.*

(1) *If $h_{\mathrm{top}}(R(T), T) > 0$ then*

$$h_{\mathrm{top}}(R(T), T) = \max_{L \in \mathcal{M}(R(T), T)} h_{\mathrm{top}}(L, T).$$

(2) *If* $f : X \to \mathbb{R}$ *is a piecewise continuous function with respect to the finite partition* $\mathcal{Z}(f)$ *of* X *with*

$$p(R(T), T, f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f)$$

then

$$p(R(T), T, f) = \max_{L\in\mathcal{M}(R(T),T)} p(L, T, f).$$

(3) *If* T *is expanding and* $h_{\text{top}}(R(T), T) > 0$ *then*

$$\text{HD}\big(R(T)\big) = \max_{L\in\mathcal{M}(R(T),T)} \text{HD}(L).$$

2. Random perturbations of a piecewise monotonic map

In this section we discuss random perturbations of a piecewise monotonic map $T : [0, 1] \to [0, 1]$. At first we consider the case, where the system is perturbed randomly at the time $n = 0$. Then we consider also the case, when our system is perturbed at each time n.

For the system, which is perturbated at time 0 we present the Perron–Frobenius operator as in [6] and [17]. The techniques of [17] apply also in the case of an expanding piecewise monotonic map with a countable number of intervals of monotonicity. All results in Sections 2.1 and 2.2 below hold also in this more general case.

In this section we always assume that $T : [0, 1] \to [0, 1]$ is an expanding piecewise monotonic map. By (1.1) we have that $R(T) = [0, 1]$.

2.1. The Perron–Frobenius operator

In Section 1 we mainly dealed with an initial value problem (x, T). This means we had a fixed piecewise monotonic map $T : [0, 1] \to [0, 1]$ (in fact we had a situation, which was a bit more general) and for an element $x \in [0, 1]$ we were interested in the asymptotic behaviour of the orbit $x, Tx, T^2x, T^3x, \dots$ of x. Now we assume that the initial value x is perturbed randomly (or, in other words, the exact value of x is not known), that means there exists a Borel probability measure μ on $[0, 1]$ such that $P(x \in A) = \mu(A)$ for every Borel set $A \subseteq [0, 1]$. We suppose that μ is absolutely continuous with respect to the Lebesgue measure. Hence there exists a density $f \in L^1([0, 1])$ ($f \geq 0$ and $\int_0^1 f(x)\, dx = 1$, where dx denotes the integration with respect to the Lebesgue measure), that means $\mu(A) = \int_A f(x)\, dx$. Now we are interested in the sequence $P(x \in A), P(Tx \in A), P(T^2x \in A), P(T^3x \in A), \dots$ for $x \in [0, 1]$ and a Borel set $A \subseteq [0, 1]$. In order to calculate $P(Tx \in A)$ we define the Perron–Frobenius operator.

Definition 7. Let $T : [0, 1] \to [0, 1]$ be an expanding piecewise monotonic map, and let $f : [0, 1] \to \mathbb{R}$ be a Borel measurable function. Then the *Perron–Frobenius operator* of T is defined by

$$Pf(x) := \sum_{y \in T^{-1}x} \frac{f(y)}{|T'(y)|}. \tag{2.1}$$

We have that $Pf : [0, 1] \to \mathbb{R}$ is again a Borel measurable function. Furthermore, if $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ are Borel measurable then

$$P\big((f \circ T)g\big) = f\, Pg. \tag{2.2}$$

By standard approximation arguments we get that

$$\int_0^1 Pf(x)\, dx = \int_0^1 f(x)\, dx \tag{2.3}$$

for every integrable function $f : [0, 1] \to \mathbb{R}$. It is clear that $f \geq 0$ implies $Pf \geq 0$.

Now suppose again that $P(x \in A) = \int_A f(x)\, dx$ for every Borel set $A \subseteq [0, 1]$. Then we have

$$\begin{aligned} P(Tx \in A) &= \int_0^1 (1_A \circ T)(x) f(x)\, dx \\ &= \int_0^1 P\big((1_A \circ T) f\big)(x)\, dx = \int_0^1 1_A(x) Pf(x)\, dx = \int_A Pf(x)\, dx, \end{aligned}$$

hence Pf is the density of $\mu \circ T^{-1}$. Therefore the density of the distribution at time n is $P^n f$, if the distribution at time 0 was f. This means that we are now interested in the behaviour of the sequence $f, Pf, P^2 f, P^3 f, \ldots$ for a suitable initial density f.

We have to look for a suitable Banach space, such that the Perron–Frobenius operator restricted to this Banach space behaves well. In [6] the Banach space of functions of bounded variation is considered. This Banach space turned out to be very good for our problem. Now we shall define functions of bounded variation.

Definition 8. Let $(A, \leq)$ be a totally ordered space. If $f : A \to \mathbb{R}$ is a function then

$$\mathrm{Var}(f) := \sup\Big\{\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| : x_0 < x_1 < \cdots < x_n \in A\Big\} \tag{2.4}$$

is called the *variation* of f. A function $f : A \to \mathbb{R}$ is called *function of bounded variation*, if $\mathrm{Var}(f) < \infty$.

As an element f in $L^1([0, 1])$ is a class of functions, which are equal Lebesgue almost everywhere, different representatives of f can have different variation. Hence we introduce the following definition.

Definition 9. For $f \in L^1([0,1])$ set

$$\text{Var}(f) := \inf\bigl\{\text{Var}(\tilde{f}) : \tilde{f} \text{ is a representative of } f\bigr\}. \tag{2.5}$$

Define

$$\text{BV} := \{f \in L^1([0,1]) : \text{Var}(f) < \infty\}, \tag{2.6}$$

and for $f \in \text{BV}$ define

$$\|f\|_{\text{BV}} := \|f\|_1 + \text{Var}(f). \tag{2.7}$$

Now we assume that $1/|T'|$ is a function of bounded variation. It is shown in [6] and [17] that $P : \text{BV} \to \text{BV}$ is a bounded linear operator.

Lemma 5. *Suppose that* $1/|T'|$ *is a function of bounded variation. Then the space* $(\text{BV}, \|\cdot\|_{\text{BV}})$ *is a Banach space. With respect to the norm* $\|\cdot\|_1$, BV *is dense in* $L^1([0,1])$. *Furthermore the Perron–Frobenius operator* $P : \text{BV} \to \text{BV}$ *is a nonnegative continuous linear operator with respect to the norm* $\|\cdot\|_{\text{BV}}$.

2.2. A spectral theorem for the Perron–Frobenius operator

We are interested in $P^n f$ for large n. To this end a spectral theorem is shown in [6] (Theorem 1 in [6]) and [17] (Theorem 1 in [17]). According to [17] the main ideas in the proof of this spectral theorem are the following. If $s \in \mathbb{R}$ is large enough then there exists a finite partition $\mathcal{A}$ of $[0,1]$ into pairwise disjoint intervals and a constant $D \geq 0$, such that

$$\text{Var}(Pf) \leq s\,\text{Var}(f) + D \sum_{A \in \mathcal{A}} \left| \int_A f(x)\,dx \right|$$

for all $f \in \text{BV}$. Furthermore there exist $r < 1$ and $C \geq 0$, such that

$$\|P^n f\|_{\text{BV}} \leq C\bigl(\|f\|_1 + r^n\,\text{Var}(f)\bigr)$$

for every $f \in \text{BV}$ and every $n \in \mathbb{N}$. These results imply that $\lim_{n\to\infty} \|P^n/n\|_{\text{BV}} = 0$, and that there exists a positive compact linear operator $K : \text{BV} \to \text{BV}$ and there exists an $n \in \mathbb{N}$ ($Kf = P^n E(f \mid \mathcal{A})$ for a suitable partition $\mathcal{A}$, where $E(f \mid \mathcal{A})$ is the conditional expectation of f with respect to $\mathcal{A}$), such that $\|P^n - K\|_{\text{BV}} < 1$. Now the Uniform Ergodic Theorem (see Chapter 8.8 in [2]) gives the following result.

Theorem 6. *Let* $T : [0,1] \to [0,1]$ *be an expanding piecewise monotonic map with respect to the finite partition* $\mathcal{Z}$ *of* $[0,1]$, *and suppose that* $1/|T'|$ *is a function of bounded variation. Then the spectrum* $\text{Sp}_{\text{BV}}(P)$ *of the operator* $P : \text{BV} \to \text{BV}$ *can* *be written as $D \cup G$, where D is contained in a disc of radius $r < 1$, and G is a nonempty finite union of finite subgroups of the unit circle. Every $\lambda \in G$ is a simple pole of the resolvent of P, and the eigenspace of λ is finite dimensional. For $\lambda \in G$ there exists a projection Q_λ to the eigenspace of λ, and there exists a continuous linear operator $R : \mathrm{BV} \to \mathrm{BV}$, such that $Q_{\lambda_1} Q_{\lambda_2} = 0$ for $\lambda_1 \neq \lambda_2 \in G$, $Q_\lambda R = R Q_\lambda = 0$ for $\lambda \in G$, the spectral radius $r_{\mathrm{BV}}(R)$ of R satisfies $r_{\mathrm{BV}}(R) \le r$, and*

$$P = \sum_{\lambda \in G} \lambda Q_\lambda + R.$$

These operators can be extended to continuous linear operators $Q_\lambda : L^1([0, 1]) \to L^1([0, 1])$ and $R : L^1([0, 1]) \to L^1([0, 1])$. Furthermore the eigenspace of 1 contains a basis $\{\varphi_1, \varphi_2, \dots, \varphi_s\}$ with $\varphi_j \ge 0$ and $\int_0^1 \varphi_j(x)\, dx = 1$ for $j \in \{1, 2, \dots, s\}$, such that $\min(\varphi_{j_1}, \varphi_{j_2}) = 0$ Lebesgue almost everywhere, whenever $j_1 \neq j_2 \in \{1, 2, \dots, s\}$.

We can use this result to characterize the T-invariant Borel probability measures on [0, 1], which are absolutely continuous with respect to the Lebesgue measure. The notation is as in Theorem 6.

Corollary 6.1. *Let $T : [0, 1] \to [0, 1]$ be an expanding piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of [0, 1], and suppose that $1/|T'|$ is a function of bounded variation.*

(1) *A Borel probability measure μ on [0, 1] is T-invariant and absolutely continuous with respect to the Lebesgue measure, if and only if there exist nonnegative numbers $c_1, c_2, \dots, c_s$ with $\sum_{j=1}^{s} c_j = 1$, such that $\mu(B) = \sum_{j=1}^{s} c_j \int_B \varphi_j(x)\, dx$ for every Borel set B.*

(2) *A T-invariant Borel probability measure μ on [0, 1], which is absolutely continuous with respect to the Lebesgue measure, is ergodic, if and only if there is a $j \in \{1, 2, \dots, s\}$ with $\mu(B) = \int_B \varphi_j(x)\, dx$ for every Borel set B.*

(3) *A T-invariant Borel probability measure μ on [0, 1], which is absolutely continuous with respect to the Lebesgue measure, is weak mixing, if and only if there is a $j \in \{1, 2, \dots, s\}$ with $\mu(B) = \int_B \varphi_j(x)\, dx$ for every Borel set B and $\int_0^1 |Q_\lambda(f)|\, d\mu = 0$ for every $\lambda \in G \setminus \{1\}$ and every $f \in \mathrm{BV}$.*

Using Theorem 6 we get that there exists an $n \in \mathbb{N}$, such that $\mathrm{Sp}_{\mathrm{BV}}(P^n) \cap \{z \in \mathbb{C} : |z| = 1\} = \{1\}$. Hence Corollary 6.1 gives that every ergodic T^n-invariant Borel probability measure μ on [0, 1], which is absolutely continuous with respect to the Lebesgue measure, is weak mixing. It can be proved that μ is also weak Bernoulli, and hence exact and strong mixing (Theorem 3 and Theorem 4 in [6] and Theorem 5 in [17]). We call μ *exact*, if $B \in \mathcal{B}_\infty := \bigcap_{n=0}^{\infty} T^{-n}\mathcal{B}$ implies $\mu(B) = 0$ or $\mu(B) = 1$, where $\mathcal{B}$ denotes the σ-algebra of Borel sets on [0, 1]. Now we shall define weak Bernoullicity. For $n \in \mathbb{N}$ let $\mathcal{F}^n$ be the σ-algebra generated by $\{T^{-j}Z : Z \in \mathcal{Z},\ j \in \mathbb{N}_0,\ 0 \le j \le n\}$, and let $\mathcal{F}_n^\infty$ be the σ-algebra generated by $\{T^{-j}Z : Z \in \mathcal{Z},\ j \in \mathbb{N}_0,\ j \geq n\}$. Now define for $n \in \mathbb{N}$

$$b_\mu(n) := \sup_{t\in\mathbb{N}} \int_0^1 \sup_{A\in\mathcal{F}_{n+t}^\infty} \left|E_\mu(1_A \mid \mathcal{F}^t) - \mu(A)\right| d\mu. \tag{2.8}$$

Then μ is called *weak Bernoulli*, if $\lim_{n\to\infty} b_\mu(n) = 0$.

Theorem 7. *Let $T : [0,1] \to [0,1]$ be an expanding piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of $[0,1]$, and suppose that $1/|T'|$ is a function of bounded variation. Let μ be an ergodic T-invariant Borel probability measure on $[0,1]$, which is absolutely continuous with respect to the Lebesgue measure on $[0,1]$. Then the following properties are equivalent.*

(1) *The measure μ is weak mixing.*
(2) *The measure μ is strong mixing.*
(3) *The measure μ is exact.*
(4) *The measure μ is weak Bernoulli.*
(5) *There exist $C, r \in \mathbb{R}$ with $C \geq 0$ and $0 \leq r < 1$, such that $b_\mu(n) \leq Cr^n$.*

Now set $\mu(B) := \int_B Q_1 1\, dx$ for every Borel set $B \subseteq [0,1]$, and we assume that μ is weak mixing. Set

$$\tau_0 := \sup\{z \in \mathrm{Sp}_{\mathrm{BV}}(P) : z \neq 1\}. \tag{2.9}$$

Using Theorem 6 and Corollary 6.1 we get $\tau_0 < 1$ and $\tau_0 = r_{\mathrm{BV}}(R)$. Now we define the rate of decay of correlations of $([0,1], T, \mu)$ for functions in BV, which measures how mixing μ is.

Definition 10. The infimum r_0 of all nonnegative real numbers r with the property that for every $f, g \in \mathrm{BV}$ there exists a nonnegative $C \in \mathbb{R}$ (C depends only on r, $\|f\|_{\mathrm{BV}}$ and $\|g\|_{\mathrm{BV}}$) such that

$$\left|\int_0^1 (f\circ T^n) g\, d\mu - \int_0^1 f\, d\mu \int_0^1 g\, d\mu\right| \leq Cr^n$$

for all $n \in \mathbb{N}$, is called the *rate of decay of correlations* of $([0,1], T, \mu)$ for functions in BV.

We want to prove that $\tau_0 = r_0$. Set $h := Q_1 1$ and observe that

$$Q_1 f = \Big(\int_0^1 f(x)\, dx\Big) h$$

for every $f \in \mathrm{BV}$. Let $f, g \in \mathrm{BV}$ (note that $gh \in \mathrm{BV}$, $f \in L^\infty([0,1])$ and $\|f\|_\infty \leq \|f\|_{\mathrm{BV}}$). Then Theorem 6 gives

$$\begin{aligned}\int_0^1 (f \circ T^n) g \, d\mu &= \int_0^1 (f \circ T^n)(x) g(x) h(x)\, dx \\ &= \int_0^1 P^n\big((f \circ T^n) gh\big)\, dx \\ &= \int_0^1 f(x) P^n(gh)(x)\, dx \\ &= \int_0^1 f(x) Q_1(gh)(x)\, dx + \int_0^1 f(x) R^n(gh)(x)\, dx \\ &= \int_0^1 f(x) \left(\int_0^1 g(y) h(y)\, dy \right) h(x)\, dx + \int_0^1 f(x) R^n(gh)(x)\, dx \\ &= \int_0^1 f \, d\mu \int_0^1 g\, d\mu + \int_0^1 f(x) R^n(gh)(x)\, dx.\end{aligned}$$

Therefore $|\int_0^1 f(x) R^n(gh)(x)\, dx| \leq \|f\|_\infty \|R^n(gh)\|_1 \leq \|f\|_{\mathrm{BV}} \|R^n(gh)\|_{\mathrm{BV}} \leq \|f\|_{\mathrm{BV}} \|R^n\|_{\mathrm{BV}} \|gh\|_{\mathrm{BV}}$ implies the desired result. Hence we have proved the following theorem.

Theorem 8. *Let* $T : [0,1] \to [0,1]$ *be an expanding piecewise monotonic map with respect to the finite partition* $\mathcal{Z}$ *of* $[0,1]$, *and suppose that* $1/|T'|$ *is a function of bounded variation. Suppose that* $\mathrm{Sp}_{\mathrm{BV}}(P) \cap \{z \in \mathbb{C} : |z| = 1\} = \{1\}$, *and that* $\dim\{f \in \mathrm{BV} : Pf = f\} = 1$. *Then there exists a unique* T*-invariant Borel probability measure* μ *on* $[0,1]$, *which is absolutely continuous with respect to the Lebesgue measure on* $[0,1]$. *This measure* μ *is weak mixing, and* τ_0 *is the rate of decay of correlations of* $([0,1], T, \mu)$ *for functions in* BV.

2.3. The perturbed Perron–Frobenius operator

In Section 2.1 and 2.2 we have considered the dynamical system $([0,1], T)$ with a random perturbation of the initial value. Now we shall investigate small perturbations at each time n.

Throughout this section we assume that $T : [0,1] \to [0,1]$ is a continuous expanding piecewise monotonic map with respect to the partition $\mathcal{Z}$ of $[0,1]$, that $\inf_{x \in [0,1]} |T'(x)| > 1$, and that $1/|T'|$ is a function of bounded variation. For simplicity we assume that there exists an $\varepsilon_0 > 0$ such that $T[0,1] \subseteq [\varepsilon_0, 1 - \varepsilon_0]$. Furthermore we assume that $\mathrm{Sp}_{\mathrm{BV}}(P) \cap \{z \in \mathbb{C} : |z| = 1\} = \{1\}$, and that $\dim\{f \in \mathrm{BV} : Pf = f\} = 1$. By Corollary 6.1 this assertion is equivalent to the property that there exists a unique T-invariant Borel probability measure μ on $[0,1]$, and that μ is weak mixing. Let $h \in L^1([0,1])$ be the unique function with $h \geq 0$, $\int_0^1 h(x)\,dx = 1$ and $Ph = h$.

Let $\varepsilon > 0$ and assume $\varepsilon \leq \varepsilon_0$. Now let $\theta_\varepsilon : \mathbb{R} \to \mathbb{R}$ be an integrable function with $\theta_\varepsilon \geq 0$, $\{x \in \mathbb{R} : \theta_\varepsilon(x) > 0\} \subseteq [-\varepsilon, \varepsilon]$, and $\int_{-\infty}^{\infty} \theta_\varepsilon(x)\,dx = 1$. We consider a sequence $(X_n)_{n \in \mathbb{N}_0}$ of random variables with $P(X_n \in A \mid X_{n-1} = x) = \int_A \theta_\varepsilon(y - Tx)\,dy$ for every $n \in \mathbb{N}$ and every Borel set $A \subseteq [0,1]$ (this means: if μ is a Borel probability measure on $[0,1]$ with $P(X_{n-1} \in A) = \mu(A)$ for every Borel set $A \subseteq [0,1]$ then $P(X_n \in B \mid X_{n-1} \in A) = \int_A \int_B \theta_\varepsilon(y - Tx)\,dy\,d\mu(x)$ for all Borel sets $A, B \subseteq [0,1]$, and in particular $P(X_n \in A) = \int_0^1 \int_A \theta_\varepsilon(y - Tx)\,dy\,d\mu(x)$ for every Borel set $A \subseteq [0,1]$). Roughly spoken, if we know that $X_{n-1} = x$ then we get X_n by applying the random perturbation given by θ_ε to Tx. Instead of thinking of a random perturbation of Tx, we can also imagine to choose randomly a t according to the density θ_ε and then applying the map T_t on x (this means calculating $T_t x$), where $T_t x := Tx + t$. Hence we can also think of a randomly chosen deterministic perturbation. As in the preceeding sections we assume that there exists a nonnegative, integrable function f of bounded variation, such that $P(X_0 \in A) = \int_A f(x)\,dx$ for every Borel set $A \subseteq [0,1]$. In order to calculate $P(X_n \in A)$ we define the perturbed Perron–Frobenius operator.

Definition 11. Define for $f \in L^1([0,1])$

$$P_\varepsilon f(x) := \int_0^1 \theta_\varepsilon(x - Ty) f(y)\,dy. \tag{2.10}$$

Then P_ε is called the *perturbed Perron–Frobenius operator*.

Let $A \subseteq [0,1]$ be a Borel set, and assume that $f \in L^1([0,1])$ satisfies $P(X_{n-1} \in B) = \int_B f(x)\,dx$ for every Borel set $B \subseteq [0,1]$. Then Fubini's Theorem gives

$$\begin{aligned} P(X_n \in A) &= \int_0^1 \left(\int_A \theta_\varepsilon(x - Ty)\,dx \right) f(y)\,dy \\ &= \int_A \left(\int_0^1 \theta_\varepsilon(x - Ty) f(y)\,dy \right) dx = \int_A P_\varepsilon f(x)\,dx. \end{aligned}$$

Therefore the distribution at time n is ${P_\varepsilon}^n f$, if the distribution at time 0 was f. Hence we are now interested in the behaviour of the sequence $f, P_\varepsilon f, {P_\varepsilon}^2 f, {P_\varepsilon}^3 f, \ldots$ for a suitable initial density f.

The following result is a consequence of Theorem 8, Theorem 9, Corollary 17 and Lemma 19 in [8].

Theorem 9. *Let* $T : [0,1] \to [0,1]$ *be a continuous piecewise monotonic map with respect to the partition* $\mathcal{Z}$ *of* $[0,1]$, *such that* $\inf_{x\in[0,1]} |T'(x)| > 1$ *and that* $1/|T'|$ *is a function of bounded variation. Assume that there exists an* $\varepsilon_0 > 0$ *such that* $T[0,1] \subseteq [\varepsilon_0, 1-\varepsilon_0]$. *Suppose that* $\mathrm{Sp}_{\mathrm{BV}}(P) \cap \{z \in \mathbb{C} : |z| = 1\} = \{1\}$, *and that* $\dim\{f \in \mathrm{BV} : Pf = f\} = 1$. *Then there exists an* $\varepsilon_1 > 0$, *such that for every* $\varepsilon \in (0, \varepsilon_1)$ *the following properties hold*:

(1) *The map* $P_\varepsilon : \mathrm{BV} \to \mathrm{BV}$ *is a nonnegative continuous linear operator with respect to the norm* $\|\cdot\|_{\mathrm{BV}}$.

(2) *There exists a unique* $h_\varepsilon \in \mathrm{BV}$ *with* $h_\varepsilon \geq 0$, $\int_0^1 h_\varepsilon(x)\,dx = 1$ *and* $P_\varepsilon h_\varepsilon = h_\varepsilon$.

Now denote by τ_ε the rate of decay of correlations of $([0,1], T, \mu_\varepsilon)$ for functions in BV, where $\mu_\varepsilon(A) = \int_A h_\varepsilon(x)\,dx$ for every Borel set $A \subseteq [0,1]$. Set

$$\sigma := \lim_{n\to\infty} \frac{1}{\sqrt[n]{\inf_{x\in[0,1]} |(T^n)'(x)|}}. \tag{2.11}$$

We do not know that the difference between h_ε and h is small. In fact, it is shown in Section 6 of [8], that this kind of stability is not true in general. Now we present a stability result, which is proved in [1] (Theorem 3 in [1]). Observe that the set of inner endpoints is exactly $E(T) \setminus \{0,1\}$.

Theorem 10. *Let* $T : [0,1] \to [0,1]$ *be a continuous piecewise monotonic map with respect to the partition* $\mathcal{Z}$ *of* $[0,1]$, *such that* $\inf_{x\in[0,1]} |T'(x)| > 1$ *and that* $1/|T'|$ *is a function of bounded variation. Assume that there exists an* $\varepsilon_0 > 0$ *such that* $T[0,1] \subseteq [\varepsilon_0, 1-\varepsilon_0]$. *Suppose that* $\mathrm{Sp}_{\mathrm{BV}}(P) \cap \{z \in \mathbb{C} : |z| = 1\} = \{1\}$, *and that* $\dim\{f \in \mathrm{BV} : Pf = f\} = 1$. *If no inner endpoint of* T *is periodic then* $\lim_{\varepsilon\to 0} \|h_\varepsilon - h\|_1 = 0$. *Furthermore, if additionally* $\tau_0 > \sqrt{\sigma}$, *then* $\lim_{\varepsilon\to 0} \tau_\varepsilon = \tau_0$.

In [1] there are also obtained stability results in the case of periodic inner endpoints. We call θ_ε *symmetric*, if $\theta_\varepsilon(-x) = \theta_\varepsilon(x)$ for all $x \in \mathbb{R}$.

Theorem 11. *Let* $T : [0,1] \to [0,1]$ *be a continuous piecewise monotonic map with respect to the partition* $\mathcal{Z}$ *of* $[0,1]$, *such that* $\inf_{x\in[0,1]} |T'(x)| > 1$ *and that* $1/|T'|$ *is a function of bounded variation. Assume that there exists an* $\varepsilon_0 > 0$ *such that* $T[0,1] \subseteq [\varepsilon_0, 1-\varepsilon_0]$. *Suppose that* $\mathrm{Sp}_{\mathrm{BV}}(P) \cap \{z \in \mathbb{C} : |z| = 1\} = \{1\}$, *and that* $\dim\{f \in \mathrm{BV} : Pf = f\} = 1$.

(1) *If* $\sigma < \frac{1}{2}$ *then* $\lim_{\varepsilon\to 0} \|h_\varepsilon - h\|_1 = 0$. *Furthermore, if additionally* $\tau_0 > \sqrt{2\sigma}$ *then* $\lim_{\varepsilon\to 0} \tau_\varepsilon = \tau_0$.

(2) *Suppose that* θ_ε *is symmetric. If* $\sigma < \frac{2}{3}$ *then* $\lim_{\varepsilon\to 0} \|h_\varepsilon - h\|_1 = 0$. *Furthermore, if additionally* $\tau_0 > \sqrt{3\sigma/2}$ *then* $\lim_{\varepsilon\to 0} \tau_\varepsilon = \tau_0$.

3. Deterministic perturbations of a piecewise monotonic map

In this section we discuss deterministic perturbations of a piecewise monotonic map $T : X \to \mathbb{R}$. We study the influence of small perturbations of T on the topological entropy $h_{\mathrm{top}}(R(T), T)$, the pressure $p(R(T), T, f)$, the Hausdorff dimension $\mathrm{HD}\big(R(T)\big)$, and the structure of the nonwandering set (see Theorem 4).

At first we have to describe topologies on piecewise monotonic maps. Then we introduce a graph, which will be usefull in order to give upper bounds of the jumps up of entropy, pressure and Hausdorff dimension. The results of this section give that entropy, pressure and Hausdorff dimension are lower semi-continuous, but in general they are not upper semi-continuous. In the case of a continuous piecewise monotonic map T we get continuity of entropy, pressure and Hausdorff dimension, if no inner endpoint of T is periodic. This stability condition has already occurred in Theorem 10.

3.1. Topologies on piecewise monotonic maps

In order to consider small deterministic perturbations of a piecewise monotonic map we define topologies for piecewise monotonic maps and piecewise continuous functions as in [11] and [15].

Let $\varepsilon > 0$. We say that two continuous functions $f : (a, b) \to \mathbb{R}$ and $\tilde{f} : (\tilde{a}, \tilde{b}) \to \mathbb{R}$ are *ε-close*, if

(1) $|a - \tilde{a}| < \varepsilon$ and $|b - \tilde{b}| < \varepsilon$,
(2) $|f(x) - \tilde{f}(x)| < \varepsilon$ for all $x \in (a, b) \cap (\tilde{a}, \tilde{b})$,
(3) $\sup_{x\in(a,\tilde{a})} |f(x) - \tilde{f}(\tilde{a}^+)| < \varepsilon$, if $a < \tilde{a}$, or
$\sup_{x\in(\tilde{a},a)} |\tilde{f}(x) - f(a^+)| < \varepsilon$, if otherwise $\tilde{a} \le a$,
(4) $\sup_{x\in(\tilde{b},b)} |f(x) - \tilde{f}(\tilde{b}^-)| < \varepsilon$, if $\tilde{b} < b$, or
$\sup_{x\in(b,\tilde{b})} |\tilde{f}(x) - f(b^-)| < \varepsilon$, if otherwise $b \le \tilde{b}$.

Observe that if ε is small enough then (1) gives that $(a, b) \cap (\tilde{a}, \tilde{b}) \neq \emptyset$.

Suppose that X and $\tilde{X}$ are finite unions of closed intervals. Let $f : X \to \mathbb{R}$ be piecewise continuous with respect to the finite partition $\mathcal{Z}$ of X, and let $\tilde{f} : \tilde{X} \to \mathbb{R}$ be piecewise continuous with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$. Suppose that $\mathcal{Z} = \{Z_1, Z_2, \ldots, Z_K\}$ with $Z_1 < Z_2 < \cdots < Z_K$ and $\tilde{\mathcal{Z}} = \{\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_{\tilde{K}}\}$ with $\tilde{Z}_1 < \tilde{Z}_2 < \cdots < \tilde{Z}_{\tilde{K}}$. Then $(f, \mathcal{Z})$ and $(\tilde{f}, \tilde{\mathcal{Z}})$ are said to be *ε-close in the R^0-topology*, if

(1) $\operatorname{card} \mathcal{Z} = \operatorname{card} \tilde{\mathcal{Z}}$, and
(2) $f|_{Z_j}$ and $\tilde{f}|_{\tilde{Z}_j}$ are ε-close in the sense defined above for $j = 1, 2, \ldots, K$.

Definition 12. Let $n \in \mathbb{N} \cup \{0, \infty\}$, let $f : X \to \mathbb{R}$ be a piecewise continuous function of class R^n with respect to the finite partition $\mathcal{Z}$ of X, and let $\tilde{f} : \tilde{X} \to \mathbb{R}$

be a piecewise continuous function of class R^n with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$. Then $(f, \mathcal{Z})$ and $(\tilde{f}, \tilde{\mathcal{Z}})$ are said to be *ε-close in the R^n-topology*, if $(f^{(j)}, \mathcal{Z})$ and $(\tilde{f}^{(j)}, \tilde{\mathcal{Z}})$ are ε-close in the R^0-topology for every $j \in \mathbb{N}_0$ with $j \le n$.

The R^0-topology is equivalent to the following topology. Two piecewise continuous functions $(f, \mathcal{Z})$ and $(\tilde{f}, \tilde{\mathcal{Z}})$ are said to be close at level ε, if $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ have the same number of elements (in the case of piecewise monotonic maps this means that they have the same number of intervals of monotonicity), and the graph of $\tilde{f}|_{\tilde{Z}_j}$ is contained in an ε-neighbourhood of the graph of $f|_{Z_j}$ considered as subsets of $\mathbb{R}^2$ for $j = 1, 2, \ldots, K$.

In general the entropy, the pressure and the Hausdorff dimension are not continuous. Now we give an example, which shows that the entropy is not continuous with respect to the R^0-topology (similar examples can be found for the pressure and the Hausdorff dimension). Set $X := [0, 1]$, $\mathcal{Z} := \left\{(0, \frac{1}{9}), (\frac{1}{9}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, 1)\right\}$ and define $T : [0, 1] \to [0, 1]$ by

$$Tx := \begin{cases} \frac{1}{6} + \frac{3}{2}x & \text{for } x \in [0, \frac{1}{9}], \\ \frac{1}{2} - \frac{3}{2}x & \text{for } x \in [\frac{1}{9}, \frac{1}{3}], \\ 2x - \frac{2}{3} & \text{for } x \in [\frac{1}{3}, \frac{2}{3}], \\ 2 - 2x & \text{for } x \in [\frac{2}{3}, 1]. \end{cases} \tag{3.1}$$

Then T is an expanding piecewise monotonic map of class R^∞ with respect to $\mathcal{Z}$. Furthermore we have $\Omega([0, 1], T) \subseteq [0, \frac{1}{3}] \cup \{\frac{2}{3}\}$ and $h_{\text{top}}([0, 1], T) = \log \frac{3}{2}$. For $s \in (0, \frac{1}{9})$ set

$$T_s x := \begin{cases} Tx & \text{for } x \in [0, \frac{1}{9}], \\ \frac{1}{2} - \frac{s}{2} - \left(\frac{3}{2} - \frac{9}{2}s\right) x & \text{for } x \in [\frac{1}{9}, \frac{1}{3}], \\ Tx + s & \text{for } x \in [\frac{1}{3}, 1]. \end{cases} \tag{3.2}$$

Then $T_s : [0, 1] \to [0, 1]$ is an expanding piecewise monotonic map of class R^∞ with respect to $\mathcal{Z}$. For a given $\varepsilon > 0$ we have that $(T_s, \mathcal{Z})$ is ε-close to $(T, \mathcal{Z})$ in the R^∞-topology, if $s < \frac{2}{9}\varepsilon$. The nonwandering set of $([0, 1], T_s)$ satisfies $[\frac{2}{3} - s, \frac{2}{3} + s] \subseteq \Omega([0, 1], T_s) \subseteq [s, \frac{1}{3}] \cup [\frac{2}{3} - s, \frac{2}{3} + s]$, and we have $h_{\text{top}}([0, 1], T_s) = \log 2 > \log \frac{3}{2}$.

3.2. Other oriented graphs associated to T

Although entropy, pressure and Hausdorff dimension are not upper semi-continuous (see Section 3.1), upper bounds for the jumps up can be given. To this end we introduce an oriented graph $(\mathcal{G}, \to)$ as in [15] (this graph is similar to the graph considered in [11]).

Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X. We assume that $\mathcal{Z} = \{Z_1, Z_2, \ldots, Z_K\}$ with $Z_1 < Z_2 < \cdots < Z_N$. Then there exist $a_1, a_2, \dots, a_{2K} \in \mathbb{R}_{\mathcal{Z}}$ with $a_1 < a_2 < \dots < a_{2K}$, such that $\hat{Z}_j = \{x \in \mathbb{R}_{\mathcal{Z}} : a_{2j-1} \le x \le a_{2j}\}$. For $i \in \{1, 2, \dots, 2K\}$ define

$$j(i) := \min\{j \in \mathbb{N} : T_{\mathcal{Z}}{}^j a_i \notin X_{\mathcal{Z}}\}, \tag{3.3}$$

where we set $j(i) := \infty$, if $T_{\mathcal{Z}}{}^j a_i \in X_{\mathcal{Z}}$ for all $j \in \mathbb{N}$. Now set

$$a_{i,j} := T_{\mathcal{Z}}{}^j a_i \quad \text{for } i \in \{1, 2, \dots, 2K\} \text{ and } j \in \mathbb{N}_0, 0 \le j < j(i). \tag{3.4}$$

Set $\mathcal{G} := \{a_{i,j} : i \in \{1, 2, \dots, 2K\},\ j \in \mathbb{N}_0,\ 0 \le j < j(i)\}$. For $a, b \in \mathcal{G}$ we introduce an arrow $a \to b$, if and only if $T_{\mathcal{Z}} a = b$ or $\pi_{\mathcal{Z}}(b) \in E_1(T)$ ($E_1(T)$ is the set of inner endpoints) and $\pi_{\mathcal{Z}}(T_{\mathcal{Z}} a) = \pi_{\mathcal{Z}}(b)$. Observe that this graph does not change, if we replace $T_{\mathcal{Z}}{}^j a_i$ in the definition of $a_{i,j}$ by $T_{\mathcal{Y}}{}^j a_i$ for a finite partition $\mathcal{Y}$ of X refining $\mathcal{Z}$.

Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, let $f : X \to \mathbb{R}$ be a piecewise continuous function with respect to the finite partition $\mathcal{Z}(f)$ of X, and let $\mathcal{Y}$ be a finite partition of X refining both $\mathcal{Z}$ and $\mathcal{Z}(f)$. For $a, b \in \mathcal{G}$ define

$$G_{a,b}(f) := \begin{cases} e^{f_{\mathcal{Y}}(a)} & \text{if } a \to b, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Set $G(f) := \big(G_{a,b}(f)\big)_{a,b \in \mathcal{G}}$. Simple calculations show the following result (cf. [15]).

Lemma 6. *The map $u \mapsto uG(f)$ is continuous and linear $\ell^1(\mathcal{G}) \to \ell^1(\mathcal{G})$, and the map $v \mapsto G(f)v$ is continuous and linear $\ell^\infty(\mathcal{G}) \to \ell^\infty(\mathcal{G})$. These two operators have the same norm $\|G(f)\|$ and the same spectral radius $r\big(G(f)\big)$. Furthermore we have*

$$\|G(f)\| = \sup_{a \in \mathcal{G}} \sum_{b \in \mathcal{G}} G_{a,b}(f), \tag{3.6}$$

$$\|G(f)^n\| = \sup_{a \in \mathcal{G}} \sum_{b_0 = a \to b_1 \to \dots \to b_n} \prod_{j=0}^{n-1} e^{f_{\mathcal{Y}}(b_j)} \quad \text{for every } n \in \mathbb{N}, \tag{3.7}$$

where the sum is taken over all paths $b_0 \to b_1 \to \dots \to b_n$ of length n in $\mathcal{G}$ with $b_0 = a$, and

$$r\big(G(f)\big) = \lim_{n \to \infty} \|G(f)^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|G(f)^n\|^{\frac{1}{n}}. \tag{3.8}$$

The next result gives an upper bound for $r\big(G(f)\big)$, if a certain condition is satisfied (cf. Lemma 2 in [15]). This condition will imply that entropy, pressure and Hausdorff dimension are continuous at T.

Lemma 7. *Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, let $f : X \to \mathbb{R}$ be a piecewise continuous function with respect to the finite partition $\mathcal{Z}(f)$ of X, and let $\mathcal{Y}$ be a finite partition of X refining both $\mathcal{Z}$ and $\mathcal{Z}(f)$.*

(1) *If $\mathcal{G}$ contains no closed paths $b_0 \to b_1 \to \cdots \to b_n$ with $b_0 = b_n$ and $\pi_{\mathcal{Y}}(b_0) \in E_1(T)$ then*

$$\log r\big(G(f)\big) \leq \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f) \leq p(R(T), T, f).$$

(2) *If T is continuous and no inner endpoint of T is periodic then*

$$\log r\big(G(f)\big) \leq \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f) \leq p(R(T), T, f).$$

The proof of the continuity results for entropy, pressure and Hausdorff dimension relies on a result, which describes how the Markov diagram reacts on small perturbations of T. In order to describe this result we need the notion of variants of the Markov diagram of T with respect to $\mathcal{Y}$ as introduced in [15]. For the definition of this concept see Section 2 of [15]. We describe shortly its most important properties. If $(\mathcal{A}, \to)$ is a variant of the Markov diagram of T with respect to $\mathcal{Y}$ then there exists a function $A : \mathcal{A} \to \mathcal{D}$ with $c \to d$ in $\mathcal{A}$ implies $A(c) \to A(d)$ in $\mathcal{D}$. Furthermore, if $c \in \mathcal{A}$, $D \in \mathcal{D}$, and $A(c) \to D$ in $\mathcal{D}$ then there exists a $d \in \mathcal{A}$ with $c \to d$ in $\mathcal{A}$ and $A(d) = D$. We can write $\mathcal{A} = \bigcup_{r=0}^{\infty} \mathcal{A}_r$ with $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ and $A(\mathcal{A}_r) = \mathcal{D}_r$.

A stability result for the Markov diagram is given in Lemma 6 of [15]. Roughly spoken it says, that for every piecewise monotonic map $T : X \to \mathbb{R}$ with respect to a finite partition $\mathcal{Z}$ of X, and for every $r \in \mathbb{N}$ there exists a $\delta > 0$, such that for every piecewise monotonic map $\tilde{T} : \tilde{X} \to \mathbb{R}$ with respect to a finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$, which satisfies that $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology, there exists a variant $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Z}$ and a variant $(\tilde{\mathcal{A}}, \to)$ of the Markov diagram of $\tilde{T}$ with respect to $\tilde{\mathcal{Z}}$, there exists an injective map $\varphi : \mathcal{A}_r \to \tilde{\mathcal{A}}_r$, such that $c \to d$ in $\mathcal{A}$ is equivalent to $\varphi(c) \to \varphi(d)$ in $\tilde{\mathcal{A}}$ for $c, d \in \mathcal{A}_r$, and there exists a map $\psi : \tilde{\mathcal{A}}_r \setminus \varphi(\mathcal{A}_r) \to \mathcal{G}$, such that $c \to d$ in $\tilde{\mathcal{A}}$ implies $\psi(c) \to \psi(d)$ in $\mathcal{G}$ for $c, d \in \tilde{\mathcal{A}}_r \setminus \varphi(\mathcal{A}_r)$. In fact this result is more complicated.

3.3. Continuity results for the pressure and the Hausdorff dimension

It is shown in Theorem 1 of [15] that the pressure is lower semi-continuous (cf. also Theorem 9 of [18]) and upper bounds for the jumps up are given in Theorem 2 of [15]. According to [15] the main ideas in the proof of the continuity results of the pressure are the following. Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, and suppose that $f : X \to \mathbb{R}$ is piecewise continuous with respect to $\mathcal{Z}$. Then for every $\varepsilon > 0$ there exists a finite partition $\mathcal{Y}$ of X refining $\mathcal{Z}$ and a function $f_\varepsilon : X \to \mathbb{R}$, which is piecewise constant with respect to $\mathcal{Y}$, such that

$|p(R(T), T, f) - p(R(T), T, f_\varepsilon)| < \varepsilon$. By Theorem 1 $p(R(T), T, f_\varepsilon)$ is the spectral radius of the matrix $F(f_\varepsilon)$ defined in (1.2). Now using Lemma 6 of [15] the following result can be proved.

Theorem 12. *Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, and suppose that $f : X \to \mathbb{R}$ is a piecewise continuous function with respect to $\mathcal{Z}$.*

(1) *For every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every piecewise monotonic map $\tilde{T} : \tilde{X} \to \mathbb{R}$ with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$ and for every piecewise continuous function $\tilde{f} : \tilde{X} \to \mathbb{R}$ with respect to $\tilde{\mathcal{Z}}$ the property "$(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology and $(\tilde{f}, \tilde{\mathcal{Z}})$ is δ-close to $(f, \mathcal{Z})$ in the R^0-topology" implies*

$$p(R(\tilde{T}), \tilde{T}, \tilde{f}) < \max\left\{p(R(T), T, f), \log r\big(G(f)\big)\right\} + \varepsilon.$$

(2) *If*

$$p(R(T), T, f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f)$$

then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every piecewise monotonic map $\tilde{T} : \tilde{X} \to \mathbb{R}$ with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$ and for every piecewise continuous function $\tilde{f} : \tilde{X} \to \mathbb{R}$ with respect to $\tilde{\mathcal{Z}}$ the property "$(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology and $(\tilde{f}, \tilde{\mathcal{Z}})$ is δ-close to $(f, \mathcal{Z})$ in the R^0-topology" implies

$$\begin{aligned} p(R(T), T, f) - \varepsilon &< p(R(\tilde{T}), \tilde{T}, \tilde{f}) \\ &< \max\left\{p(R(T), T, f), \log r\big(G(f)\big)\right\} + \varepsilon. \end{aligned}$$

(3) *If $\mathcal{G}$ contains no closed paths $b_0 \to b_1 \to \cdots \to b_n$ with $b_0 = b_n$ and $\pi_{\mathcal{Z}}(b_0) \in E_1(T)$, and if*

$$p(R(T), T, f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f)$$

then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every piecewise monotonic map $\tilde{T} : \tilde{X} \to \mathbb{R}$ with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$ and for every piecewise continuous function $\tilde{f} : \tilde{X} \to \mathbb{R}$ with respect to $\tilde{\mathcal{Z}}$ the property "$(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology and $(\tilde{f}, \tilde{\mathcal{Z}})$ is δ-close to $(f, \mathcal{Z})$ in the R^0-topology" implies

$$|p(R(\tilde{T}), \tilde{T}, \tilde{f}) - p(R(T), T, f)| < \varepsilon.$$

If both T and f are continuous we can give a description of this upper bound, which is easier to calculate (Corollary 2.2 in [15]). Define

$$G_T(f) := \max\left\{\frac{\operatorname{card}\big\{j \in \{0, 1, \dots, n-1\} : T^j x \in E_1(T)\big\}}{n} \log 2 + \frac{1}{n}\sum_{j=0}^{n-1} f(T^j x) : x \in E_1(T) \text{ is a point of period } n\right\}, \tag{3.9}$$

where we set $G_T(f) := -\infty$, if $E_1(T)$ contains no periodic points.

Corollary 12.1. *Let $T : X \to \mathbb{R}$ be a continuous piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, and suppose that $f : X \to \mathbb{R}$ is a continuous function.*

(1) *For every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every piecewise monotonic map $\tilde{T} : \tilde{X} \to \mathbb{R}$ with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$ and for every piecewise continuous function $\tilde{f} : \tilde{X} \to \mathbb{R}$ with respect to $\tilde{\mathcal{Z}}$ the property "$(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology and $(\tilde{f}, \tilde{\mathcal{Z}})$ is δ-close to $(f, \mathcal{Z})$ in the R^0-topology" implies*

$$p(R(\tilde{T}), \tilde{T}, \tilde{f}) < \max\left\{p(R(T), T, f), G_T(f)\right\} + \varepsilon.$$

(2) *If*

$$p(R(T), T, f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f)$$

then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every piecewise monotonic map $\tilde{T} : \tilde{X} \to \mathbb{R}$ with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$ and for every piecewise continuous function $\tilde{f} : \tilde{X} \to \mathbb{R}$ with respect to $\tilde{\mathcal{Z}}$ the property "$(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology and $(\tilde{f}, \tilde{\mathcal{Z}})$ is δ-close to $(f, \mathcal{Z})$ in the R^0-topology" implies

$$\begin{aligned} p(R(T), T, f) - \varepsilon &< p(R(\tilde{T}), \tilde{T}, \tilde{f}) \\ &< \max\left\{p(R(T), T, f), G_T(f)\right\} + \varepsilon. \end{aligned}$$

(3) *If no inner endpoint of T is periodic, and if*

$$p(R(T), T, f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f),$$

then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every piecewise monotonic map $\tilde{T} : \tilde{X} \to \mathbb{R}$ with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$ and for every piecewise continuous function $\tilde{f} : \tilde{X} \to \mathbb{R}$ with respect to $\tilde{\mathcal{Z}}$ the property "$(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology and $(\tilde{f}, \tilde{\mathcal{Z}})$ is δ-close to $(f, \mathcal{Z})$ in the R^0-topology" implies

$$|p(R(\tilde{T}), \tilde{T}, \tilde{f}) - p(R(T), T, f)| < \varepsilon.$$

In Section 4 of [15] an example is given, where the pressure is not lower semi-continuous (cf. also [18]).

If we set $f = 0$ we get that the entropy is lower semi-continuous (Theorem 5 of [12], note that the lower semi-continuity of the entropy is trivial, if $h_{\text{top}}(R(T), T) = 0$), and we get upper bounds for the jumps up of the entropy (Theorem 1 of [10] and Theorem 2 of [11]).

Corollary 12.2. *Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X.*

(1) *For every* $\varepsilon > 0$ *there exists a* $\delta > 0$, *such that for every piecewise monotonic map* $\tilde{T} : \tilde{X} \to \mathbb{R}$ *with respect to the finite partition* $\tilde{\mathcal{Z}}$ *of* $\tilde{X}$ *the property* "$(\tilde{T}, \tilde{\mathcal{Z}})$ *is* δ*-close to* $(T, \mathcal{Z})$ *in the* R^0*-topology" implies*

$$h_{\mathrm{top}}(R(T), T) - \varepsilon < h_{\mathrm{top}}(R(\tilde{T}), \tilde{T}) < \max\left\{h_{\mathrm{top}}(R(T), T), \log r\big(G(0)\big)\right\} + \varepsilon.$$

If T *is continuous then we get*

$$h_{\mathrm{top}}(R(T), T) - \varepsilon < h_{\mathrm{top}}(R(\tilde{T}), \tilde{T}) < \max\left\{h_{\mathrm{top}}(R(T), T), G_T(0)\right\} + \varepsilon.$$

(2) *If* $\mathcal{G}$ *contains no closed paths* $b_0 \to b_1 \to \cdots \to b_n$ *with* $b_0 = b_n$ *and* $\pi_{\mathcal{Z}}(b_0) \in E_1(T)$ *then for every* $\varepsilon > 0$ *there exists a* $\delta > 0$, *such that for every piecewise monotonic map* $\tilde{T} : \tilde{X} \to \mathbb{R}$ *with respect to the finite partition* $\tilde{\mathcal{Z}}$ *of* $\tilde{X}$ *the property* "$(\tilde{T}, \tilde{\mathcal{Z}})$ *is* δ*-close to* $(T, \mathcal{Z})$ *in the* R^0*-topology" implies*

$$|h_{\mathrm{top}}(R(\tilde{T}), \tilde{T}) - h_{\mathrm{top}}(R(T), T)| < \varepsilon.$$

(3) *If* T *is continuous and no inner endpoint of* T *is periodic then for every* $\varepsilon > 0$ *there exists a* $\delta > 0$, *such that for every piecewise monotonic map* $\tilde{T} : \tilde{X} \to \mathbb{R}$ *with respect to the finite partition* $\tilde{\mathcal{Z}}$ *of* $\tilde{X}$ *the property* "$(\tilde{T}, \tilde{\mathcal{Z}})$ *is* δ*-close to* $(T, \mathcal{Z})$ *in the* R^0*-topology" implies*

$$|h_{\mathrm{top}}(R(\tilde{T}), \tilde{T}) - h_{\mathrm{top}}(R(T), T)| < \varepsilon.$$

In the case of an expanding piecewise monotonic map we can obtain a similar continuity result for the Hausdorff dimension. In order to give the upper bound for the jumps up we need the following result, which is proved in Section 5 of [15].

Lemma 8. *Let* $T : X \to \mathbb{R}$ *be an expanding piecewise monotonic map with respect to the finite partition* $\mathcal{Z}$ *of* X. *Then either there exists a* $d_T \in [0, 1]$ *with*

$$\log r\big(G(-t \log |T'|)\big) > 0 \quad \text{for every } t \geq 0 \text{ with } t < d_T, \text{ and}$$
$$\log r\big(G(-t \log |T'|)\big) < 0 \quad \text{for every } t > d_T,$$

or there exists an $s > 1$ *with* $\log r\big(G(-t \log |T'|)\big) \geq 0$ *for every* $t \in [0, s]$.

Set $d_T := 1$, if there exists an $s > 1$ with $\log r\big(G(-t \log |T'|)\big) \geq 0$ for every $t \in [0, s]$. If T is continuous, and $|T'|$ can be extended to a continuous function on X then define

$$\hat{D}_T := \max\left\{\frac{\mathrm{card}\big\{j \in \{0, 1, \ldots, n-1\} : T^j x \in E_1(T)\big\}}{\log |(T^n)'|(x)} \log 2 : \right.$$
$$\left. x \in E_1(T) \text{ is a point of period } n\right\}, \tag{3.10}$$

and set

$$D_T := \begin{cases} 0 & \text{if } E_1(T) \text{ contains no periodic points,} \\ \hat{D}_T & \text{if } \hat{D}_T \in [0,1], \\ 1 & \text{if } \hat{D}_T > 1. \end{cases} \tag{3.11}$$

Using Theorem 3, Theorem 12 and Lemma 8 one can prove the following result (Theorem 3 and Corollary 3.2 in [15]).

Theorem 13. *Let $T : X \to \mathbb{R}$ be an expanding piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X.*

(1) *For every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every expanding piecewise monotonic map $\tilde{T} : \tilde{X} \to \mathbb{R}$ with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$ the property "$(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^1-topology" implies*

$$\mathrm{HD}\big(R(T)\big) - \varepsilon < \mathrm{HD}\big(R(\tilde{T})\big) < \max\big\{\mathrm{HD}\big(R(T)\big), d_T\big\} + \varepsilon.$$

(2) *If T is continuous, and $|T'|$ can be extended to a continuous function on X then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every expanding piecewise monotonic map $\tilde{T} : \tilde{X} \to \mathbb{R}$ with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$ the property "$(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^1-topology" implies*

$$\mathrm{HD}\big(R(T)\big) - \varepsilon < \mathrm{HD}\big(R(\tilde{T})\big) < \max\big\{\mathrm{HD}\big(R(T)\big), D_T\big\} + \varepsilon.$$

(3) *If $\mathcal{G}$ contains no closed paths $b_0 \to b_1 \to \cdots \to b_n$ with $b_0 = b_n$ and $\pi_{\mathcal{Z}}(b_0) \in E_1(T)$ then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every expanding piecewise monotonic map $\tilde{T} : \tilde{X} \to \mathbb{R}$ with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$ the property "$(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^1-topology" implies*

$$\big|\mathrm{HD}\big(R(\tilde{T})\big) - \mathrm{HD}\big(R(T)\big)\big| < \varepsilon.$$

(4) *If T is continuous and no inner endpoint of T is periodic then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every expanding piecewise monotonic map $\tilde{T} : \tilde{X} \to \mathbb{R}$ with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$ the property "$(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^1-topology" implies*

$$\big|\mathrm{HD}\big(R(\tilde{T})\big) - \mathrm{HD}\big(R(T)\big)\big| < \varepsilon.$$

3.4. Stability of the structure of the nonwandering set

Finally we investigate the influence of small perturbations of T on the structure of the nonwandering set, which is described in Theorem 4. In Section 1.4 we learned that the most interesting part of the dynamics takes place on the at most countable union of maximal topologically transitive subsets with positive entropy. Therefore we investigate, how the set $\mathcal{M}(R(T), T)$ defined in (1.20) changes under small perturbations of T. Unfortunately $\mathcal{M}(R(T), T)$ is not stable. It is shown in Section 3 of [16] that $\operatorname{card}\mathcal{M}(R(T), T)$ is not continuous (there is given an example of a piecewise monotonic map $T : [0, 1] \to [0, 1]$, where the entropy is continuous in T, but $\operatorname{card}\mathcal{M}(R(T), T)$ is not continuous).

In [16] the following stability result for $\mathcal{M}(R(T), T)$ is shown (Theorem 2 and Theorem 3 in [16]).

Theorem 14. *Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, let $L \in \mathcal{M}(R(T), T)$, and let $f_1, f_2, \ldots, f_n : X \to \mathbb{R}$ be finitely many functions, which are piecewise continuous with respect to $\mathcal{Z}$, and suppose that*

$$p(L, T, f_j) > \lim_{n\to\infty} \frac{1}{n} S_n(L, f_j) \quad \text{for } j \in \{1, 2, \ldots, n\}.$$

(1) *For every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every piecewise monotonic map $\tilde{T} : \tilde{X} \to \mathbb{R}$ with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$ and for all piecewise continuous functions $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n : \tilde{X} \to \mathbb{R}$ with respect to $\tilde{\mathcal{Z}}$, the property "$(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology and $(\tilde{f}_j, \tilde{\mathcal{Z}})$ is δ-close to $(f_j, \mathcal{Z})$ in the R^0-topology for $j \in \{1, 2, \ldots, n\}$" implies that there exists a topologically transitive subset $\tilde{L}$ of $R(\tilde{T})$, which satisfies*

(a) *$\tilde{L}$ and L are ε-close in the Hausdorff metric,*
(b) *$|h_{\text{top}}(\tilde{L}, \tilde{T}) - h_{\text{top}}(L, T)| < \varepsilon$ and*
(c) *$|p(\tilde{L}, \tilde{T}, \tilde{f}_j) - p(L, T, f_j)| < \varepsilon$ for $j \in \{1, 2, \ldots, n\}$.*

(2) *If T is expanding then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every expanding piecewise monotonic map $\tilde{T} : \tilde{X} \to \mathbb{R}$ with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$ and for all piecewise continuous functions $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n : \tilde{X} \to \mathbb{R}$ with respect to $\tilde{\mathcal{Z}}$, the property "$(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^1-topology and $(\tilde{f}_j, \tilde{\mathcal{Z}})$ is δ-close to $(f_j, \mathcal{Z})$ in the R^0-topology for $j \in \{1, 2, \ldots, n\}$" implies that there exists a topologically transitive subset $\tilde{L}$ of $R(\tilde{T})$, which satisfies*

(a) *$\tilde{L}$ and L are ε-close in the Hausdorff metric,*
(b) *$|h_{\text{top}}(\tilde{L}, \tilde{T}) - h_{\text{top}}(L, T)| < \varepsilon$,*
(c) *$|p(\tilde{L}, \tilde{T}, \tilde{f}_j) - p(L, T, f_j)| < \varepsilon$ for $j \in \{1, 2, \ldots, n\}$, and*
(d) *$|\operatorname{HD}(\tilde{L}) - \operatorname{HD}(L)| < \varepsilon$.*

Using Theorem 5 the lower semi-continuity results for the pressure, the entropy and the Hausdorff dimension can be deduced from Theorem 14. In [16] there are also obtained results in the other direction (Theorem 4, Corollary 4.1 and Theorem 5 in [16]), which imply the results on the upper bounds for the jumps up.

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