SOME ASPECTS OF TOPOLOGICAL TRANSITIVITY — A SURVEY

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ABSTRACT

This is intended as a survey article on topological transitivity of a dynamical system given by a continuous selfmap of a compact metric space. On one hand it introduces beginners to the study of topological transitivity and gives an overview of results on the topic, but, on the other hand, it covers some of the recent developments. The paper is purely 'topological' and only occasionally reflects differentiable dynamics or ergodic theory aspects of the topic.

1. Introduction

Topological transitivity is a global characteristic of a dynamical system. Though the local structure of a topologically transitive dynamical system fulfills certain conditions such as, for example, the absence of attracting invariant sets, there is a variety of such

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systems. Say, some of them have dense periodic points while some of them may be minimal and so without any periodic point.

The concept of topological transitivity goes back to G.D. Birkhoff. According to [45], he used it in 1920, cf. [18], vol. 2, p. 108 and p. 221 (see also [19]).

We will consider a discrete (semi)dynamical system (X, f) given by a metric space (phase space) X and a continuous map $f : X \to X$ (see below for further restrictions). A point $x \in X$ "moves," its trajectory being the sequence $x, f(x), f^2(x), f^3(x), \dots,$ where f^n is the nth iteration of f. The point $f^n(x)$ is the position of x after n units of time. The set of points of the trajectory of x under f is called the orbit of x, denoted by $\mathrm{orb}_f(x)$.

As a motivation for the notion of topological transitivity of (X, f) one may think of a real physical system, where a state is never given or measured exactly, but always up to a certain error. So instead of points one should study (small) open subsets of the phase space and describe how they move in that space. If for instance the minimality of (X, f) is defined by requiring that every point $x \in X$ visit every open set V in X (i.e., $f^n(x) \in V$ for some $n \in \mathbb{N}$) then, instead, one may wish to study the following concept: every nonempty open subset U of X visits every nonempty open subset V of X in the following sense: $f^n(U) \cap V \neq \emptyset$ for some $n \in \mathbb{N}$. If the system (X, f) has this property, then it is called *topologically transitive*. We also say that f itself is topologically transitive if no misunderstanding can arise concerning the underlying phase space.

Intuitively, a topologically transitive map f has points which eventually move under iteration from one arbitrarily small neighbourhood to any other. Consequently, the dynamical system cannot be broken down or decomposed into two subsystems (disjoint sets with nonempty interiors) which do not interact under f, i.e., are invariant under the map ($A \subset X$ is invariant if $f(A) \subset A$).

The terminology is not unified—instead of 'topologically transitive' some authors say 'regionally transitive' [45], [79], 'topologically ergodic' [106], cf. [79], 'topologically indecomposable (or irreducible)' [90], 'nomadic' [31].

On the other hand, some authors working with the notion of topological transitivity are using definitions of this notion which are different from or even, generally speaking, non-equivalent with our definition. For instance f is relatively often called topologically transitive if there is a point $x \in X$ whose orbit is dense in X, see, e.g., [7], [71], [107]. More on this see below in Section 2 where, in particular, we show that if X is compact (even less) then both mentioned definitions are equivalent.

We also want to bring the attention of the reader to the fact that the topological transitivity (even topological dynamics as a whole) may be studied in different settings than it is in the present paper. (Recall that, if not stated otherwise, X is a metric space (later we will restrict ourselves more) and f is a continuous selfmap of X, in general non-invertible.) Without going into details note at least that

– instead of discrete dynamical systems one can consider continuous ones,
– considering a discrete system, one can be less or more restrictive concerning the phase space than we are (say, X = topological space, continuum, differentiable manifold,...),
– one can be less or more restrictive concerning the map than we are. Though taking a general non-continuous map f can hardly give anything reasonable and there are only rare attempts to study the dynamics of, say, bijections, Baire α functions, etc., it is important to study the dynamics of, say, interval exchange transformations or general piecewise monotone piecewise continuous maps on the interval. On the other hand, one can consider differentiable maps or invertible maps—homeomorphisms, diffeomorphisms (especially in higher dimensions). For invertible maps both transitivity and one-sided transitivity are considered (the terminology is not unified). They are not equivalent ([107], pp. 127–129). Finally, let us also mention that instead of a single map f one can be concerned with general transformation groups (this is the case of, e.g., the classical book [45]).

Convention. Throughout the paper 'transitive' will always mean 'topologically transitive'.

Note that there is also 'metrical transitivity' (defined for systems with invariant probability measure) which in fact is 'ergodicity' in the sense of ergodic theory [77], [48], [37], [79], [107], [62], [58].

2. Definitions of topological transitivity

2.1. Two most frequently used definitions and their non-equivalence. Our choice of the definition. Let X be a metric space and $f : X \to X$ continuous. Consider the following two conditions:

(TT) for every pair of nonempty open sets U and V in X, there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$,

(DO) there is a point $x_0 \in X$ such that the orbit of x_0 is dense in X.

As usual, *we adopt the condition* (*TT*) *as the definition of topological transitivity*, but note that some authors take (DO) instead. Any point with dense orbit is called a *transitive point*. A point which is not transitive is called *intransitive*. The set of transitive or intransitive points of (X, f) will be denoted by $\mathrm{tr}(f)$ or $\mathrm{intr}(f)$ respectively, provided no misunderstanding can arise concerning the phase space. In such a case we will also speak on transitive or intransitive points of f.

The two conditions are independent in general. In fact, take $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ endowed with the usual metric and $f : X \to X$ defined by $f(0) = 0$ and $f(1/n) = 1/(n+1)$, $n = 1, 2, \ldots$. Clearly, f is continuous. The point $x_0 = 1$ is (the only) transitive point for (X, f) but the system is not topologically transitive (take, say, $U = \{\frac{1}{2}\}$, $V = \{1\}$). So, (DO) does not imply (TT).

We show that neither (TT) implies (DO). To this end take $I = [0, 1]$ and the standard tent map $g(x) = 1 - |2x - 1|$ from I to itself. Let X be the set of all periodic points of g and $f = g|_X$ (a point x is *periodic* for g if $g^n(x) = x$ for some positive integer n; the least such n is called the period of x). Then the system (X, f) does not satisfy the condition (DO), since X is infinite (dense in I) while the orbit of any periodic point is finite. But the condition (TT) is fulfilled. This follows from the fact that for any nondegenerate subinterval J of I there is a positive integer k with $g^k(J) = I$ (see Example 3.1.3). Hence, whenever J_1 and J_2 are nonempty open subintervals of I, there is a periodic orbit of g which intersects both J_1 and J_2. This gives (TT) for (X, f).

2.2. On the equivalent formulations of the definition. Standard dynamical systems. Nevertheless, under some additional assumptions on the phase space (or on the map) the two definitions (TT) and (DO) are equivalent. In fact, we have

Theorem 2.2.1 ([97]). *If X has no isolated point then* (*DO*) *implies* (*TT*)*. If X is separable and second category then* (*TT*) *implies* (*DO*)*.*

Topological dynamics (in discrete setting) traditionally studies qualitative properties of homeomorphisms of a compact metric space or at least a topological Hausdorff space (see [106], cf. [107], [37], [20]). As it was said above, in the present paper the map will be more general, namely an arbitrary continuous selfmap of a metric space. If one wishes to consider the classical situation when X is compact metric, then (TT) implies (DO) but not conversely and they are equivalent provided X has no isolated point, see Theorem 2.2.1. Therefore we will sometimes restrict ourselves to compact metric spaces with no isolated points.

Let us also remark that in general compact metric spaces, (TT) and (DO) are equivalent for onto maps. If f is transitive (i.e., it satisfies (TT)) then f is onto. If a compact metric space admits a transitive map (i.e., if there exists a continuous selfmap f of X satisfying (TT)) then X has no isolated point if and only if it is infinite.

Convention. Throughout the paper, a couple (X, f) where X is a compact metric space and $f : X \to X$ is continuous is said to be a *dynamical system*. To express that (X, f) is a dynamical system we will equivalently write $f \in C(X)$. If, additionally, X has no isolated point then (X, f) is said to be a *standard dynamical system*, an SDS for short.

Given a dynamical system (X, f), the *ω-limit set* of a point $x \in X$ under f, $\omega_f(x)$, is the set of all limit points of the trajectory of x, i.e., $y \in \omega_f(x)$ if and only if $f^{n_k}(x) \to y$ for some sequence of integers $n_k \to \infty$.

A point $x \in X$ is called *nonwandering* if for every neighbourhood U of x there is a positive integer n such that $f^n(U) \cap U \neq \emptyset$. The set of all nonwandering points of f will be denoted by $\Omega(f)$.

The next theorem follows easily from the definitions and parts of it can be found in any book dealing at least partially with topological dynamics (cf. [20], [37], [106], [107]).

Theorem 2.2.2. *Let (X, f) be a dynamical system. Then the following are equivalent*:

(1) *f is topologically transitive* (*i.e.*, (*TT*) *is fulfilled*),
(2) *for every pair of nonempty open sets U and V in X, there is a nonnegative integer n such that $f^n(U) \cap V \neq \emptyset$,*
(3) *for every nonempty open set U in X, $\bigcup_{n=1}^{\infty} f^n(U)$ is dense in X,*
(4) *for every nonempty open set U in X, $\bigcup_{n=0}^{\infty} f^n(U)$ is dense in X,*
(5) *for every pair of nonempty open sets U and V in X, there is a positive integer n such that $f^{-n}(U) \cap V \neq \emptyset$,*
(6) *for every pair of nonempty open sets U and V in X, there is a nonnegative integer n such that $f^{-n}(U) \cap V \neq \emptyset$,*
(7) *for every nonempty open set U in X, $\bigcup_{n=1}^{\infty} f^{-n}(U)$ is dense in X,*
(8) *for every nonempty open set U in X, $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is dense in X,*
(9) *if $E \subset X$ is closed and $f(E) \subset E$ then $E = X$ or E is nowhere dense in X,*
(10) *if $U \subset X$ is open and $f^{-1}(U) \subset U$ then $U = \emptyset$ or U is dense in X,*
(11) *there exists a point $x \in X$ such that $\omega_f(x) = X$,*
(12) *there exists a G_δ-dense set $A \subset X$ such that $\omega_f(x) = X$ whenever $x \in A$,*
(13) *the set* $\mathrm{tr}(f)$ *is G_δ-dense,*
(14) *the map f is onto and the set* $\mathrm{tr}(f)$ *is nonempty,*
(15) $\Omega(f) = X$ *and* $\mathrm{tr}(f)$ *is nonempty,*
(16) *there is a point $x \in X$ such that the set $\{f^n(x) : n = 1, 2, \dots\}$ is dense in X.*

Further, the above conditions imply that

(17) *the set* $\mathrm{tr}(f)$ *is nonempty* (*i.e.*, (*DO*) *is fulfilled*).

If, additionally, (X, f) is a standard dynamical system then also (17) *is equivalent to the rest.*

For another condition in this line (too "ergodic-like" for this paper) see [59], cf. [79], p. 152 (notice that (at least in [79]) the map is a homeomorphism).

2.3. Topological transitivity and conjugacy. Note also that topological transitivity is preserved by *topological conjugacy*. More precisely, let (X, f) and (Y, g) be two dynamical systems and suppose they are topologically conjugate, i.e., there is a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. Then f is topologically transitive on X if and only if g is topologically transitive on Y.

If h is not a homeomorphism but only a continuous surjection (a *semiconjugacy*), then the transitivity of f implies the transitivity of g but not conversely. In this case (X, f) is an *extension* of (Y, g) and (Y, g) is a *factor* of (X, f).

If h is a continuous injection then neither of two implications between the transitivity of f and the transitivity of g holds. But if h is a (not necessarily continuous) open injection ('open' means that it sends open sets to open sets) then the transitivity of g implies the transitivity of f. The converse is still not true (even if h is continuous).

3. Examples of transitive maps

In no case we are able to give a complete overview of examples of transitive systems in the literature. But we wish to mention some of them.

3.1. Introductory examples. First recall that a dynamical system is called *minimal* if all points are transitive. Trivially, minimality of a dynamical system implies its transitivity but not conversely.

Perhaps the most popular and/or simplest examples of transitive systems are the following ones.

Example 3.1.1. Let (X, f) be any dynamical system and let $x_0 \in X$ be a periodic point of f. Denote the (finite) orbit of x_0 by Y and let $g = f|_Y$. Then the dynamical system (Y, g) is transitive. Though it is not an SDS, it satisfies both (TT) and (DO), hence all 17 conditions from Theorem 2.2.2. Notice also that (Y, g) is minimal.

Example 3.1.2. Let $\mathbf{S}$ be the unit circle and $f : \mathbf{S} \to \mathbf{S}$ be an irrational rotation. Then $(\mathbf{S}, f)$ is topologically transitive, in fact minimal.

Example 3.1.3. Let $I = [0, 1]$ and $f \in C(I)$ be the standard tent map defined by $f(x) = 1 - |2x - 1|$. If J is a closed subinterval of I which does not contain $\frac{1}{2}$ then $f(J)$ is twice as long as J. Therefore $f^k(J)$ contains $\frac{1}{2}$ for some k. Then $f^{k+2}(J)$ is a closed interval containing 0 and repeating the argument with doubling length we get that $f^n(J) = I$ for some n. This property easily implies transitivity (in general the property is stronger than transitivity, see also Theorem 6.1.2; compare with the definition of exactness [65], p. 66). The system is not minimal, since the set of periodic points is dense in I. (No minimal systems exist on I.)

Example 3.1.4. The map $g \in C(I)$ defined by $g(x) = 4x(1-x)$ is topologically transitive. This follows from the fact that the tent map f is topologically conjugate to g, the conjugating homeomorphism being $h(x) = \sin^2(\pi x/2)$.

Example 3.1.5. Let Σ be the set of all infinite sequences of 0's and 1's. Define the distance between two sequences $s = (s_0 s_1 s_2 \dots)$ and $t = (t_0 t_1 t_2 \dots)$ by $\varrho(s,t) = \sum_{i=0}^{\infty}(1/2^i)|s_i - t_i|$. The *shift map* $\sigma : \Sigma \to \Sigma$ is given by $\sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 s_3 \dots)$. Then (Σ, σ) is a standard dynamical system. It is transitive, since the point $s^* = (0100011011000001\dots)$ constructed by successively listing all blocks of 0's and 1's of length 1, then length 2, etc., has dense orbit. The system is not minimal since it has periodic points. The set of periodic points is dense in Σ. Indeed, taking a point $s = (s_0 s_1 s_2 \dots) \in \Sigma$, the point $t_n = (s_0 \dots s_n s_0 \dots s_n \dots)$ is periodic (with period dividing $n+1$) and $t_n \to s$ when $n \to \infty$.

Let us also remark that the system (Σ, σ) apears in a natural way when one investigates the dynamics of an interval map which has a 2-*horseshoe* (see Section 9 for the definition of a horseshoe), i.e., is turbulent in the terminology of [20]. See, e.g., [20], p. 35.

Example 3.1.6. Represent the Cantor set C as the set Σ of all infinite sequences of 0's and 1's with the metric defined in Ex. 3.1.5. Take the '*adding machine*' (see, e.g., [74], p. 14 or [20], p. 133 or [97]), i.e., the map $f : \Sigma \to \Sigma$ defined by $f(s_0 s_1 s_2 \dots) = 1 + (s_0 s_1 s_2 \dots)$. Instead of precise definition of the binary addition we give two examples: $1 + (0000\dots) = (1000\dots)$, $1 + (11001000\dots) = (00101000\dots)$. (More precisely, the abelian group $(\Sigma, +)$ rather than the map f itself is usually called the adding machine.) Then f is transitive, $s = (000\dots)$ being a transitive point (f is even minimal, see [20], p. 134).

Let us also remark that the dynamics on C induced by f is an example of so-called *solenoidal* dynamics.

The next example is more difficult.

Example 3.1.7. Consider the 2-torus $\mathbf{T} = \mathbf{S} \times \mathbf{S}$ and a 2×2 matrix A such that all entries of A are integers, $\det(A) = +1$ or -1 and A is hyperbolic, i.e., none of its eigenvalues has absolute value one. The map induced on $\mathbf{T}$ by A is called a hyperbolic toral automorphism (it is in fact a diffeomorphism). All hyperbolic toral automorphisms are topologically transitive (but not minimal, since they have dense periodic points). See, e.g., [38] for details, cf. [10].

Topological transitivity is a necessary but not sufficient condition for ergodicity of a dynamical system ([48], [91], [93]). Often the proof of topological transitivity of a system was followed by the proof of its ergodicity. It is out of scope of this paper to survey these aspects of the transitivity.

In the literature one can find a flood of sofisticated examples of transitive systems. Though we do not attempt to give a relatively complete survey of them, we wish to mention at least a few of them, in particular results due to Ye. A. Sidorov (though they are not the strongest ones today), since they do not seem to be widely known. For a moment we drop the assumption that the phase space is compact, since there are many interesting examples in the plane or in the cylinder $\mathbf{S} \times \mathbb{R}$ (note that, by Theorem 2.2.1, (TT) and (DO) are still equivalent).

3.2. Examples in the plane and in $\mathbb{R}^n$. It seems that Shnirelman [88] was the first who gave an example of a transitive map in the plane. Later Besicovitch [16], [17] constructed such maps in a simpler way (see also [75], [77] for this and also for higher dimensional and stronger results having ergodic theory aspects). In [16] he proved that for any map $f(\varphi)$ from a subfamily of the family of all continuous 2π-periodic functions there exists an irrational number Θ such that the map from the plane into itself defined by

$$F(\varrho e^{i\varphi}) = \varrho e^{f(\varphi)} \cdot e^{i(\varphi+2\pi\Theta)}$$

is transitive (maps of this form are sometimes called Shnirelman–Besicovitch maps and in the smooth case they could be called Shnirelman–Besicovitch–Anosov maps, cf. [92]). Transitive maps in [88] and [17] are constructed in an analogous way. Then Sidorov having been inspired by these results and methods obtained the following stronger and more general result.

Theorem 3.2.1 ([90]). *Let $\{k_i\}_{i=1}^{\infty}$ be an arbitrary increasing sequence of positive integers and let n be an integer ≥ 2. Then there exists a map $T_n : \mathbb{R}^n \to \mathbb{R}^n$ such that for some point $x \in \mathbb{R}^n$, the set $\{T_n^{k_i}(x) : i \in \mathbb{N}\}$ is dense in $\mathbb{R}^n$.*

3.3. Examples in regions in $\mathbb{R}^n$. Sidorov [92] proved that for any connected region $D \subset \mathbb{R}^n$ ($n \geq 3$) there exists a transitive C^∞ selfmap of D. (He proved this for flows and then used the result due to Oxtoby and Ulam (see [77], Theorem 6) saying that if T_λ, $-\infty < \lambda < +\infty$ is a topologically transitive continuous flow in a separable metric space such that it has no isolated streamline, then for all values of λ, except a set of first category on the line $-\infty < \lambda < +\infty$, the automorphisms T_λ are topologically transitive.)

In dimension 2, it is proved in [92] that for any region $D \subset \mathbb{R}^2$ diffeomorphic to the unit disk there exists a transitive C^∞ selfmap of D. (To prove this, Sidorov showed that an appropriately modified Shnirelman–Besicovitch–Anosov map in the unit disc satisfies all the requirements.)

For stronger results see, e.g., [9]. See also [76], [77] and, for flows, [8], [98], [99], [66].

3.4. Category method versus explicit examples. The category method (see [76]) has been succesfully used to show the existence of transitive maps in some spaces. To illustrate this recall again that an explicit example of a transitive automorphism of the plane was given by Besicovitch [16]. It seemed that it was (see [76], page 71) 'not easy to exhibit one for the closed unit square, let alone one that preserves area or leaves the boundary points fixed'. The existence of such transformations was first established by the category method [75]. Today explicit examples of transitive maps in the square based on Shnirelman–Besicovitch maps are known, see Subsection 3.3 and [109]. Such explicit examples exist even in the class of triangular maps of the square, i.e., continuous maps of the form $F(x, y) = (f(x), g(x, y))$, see Theorem 10.2.

The category method has also been used to establish the existence of metrically (hence also topologically) transitive automorphisms [77], [73].

3.5. An example on the torus. Finally, we wish to mention the following example of a minimal homeomorphism on the torus.

Consider a homeomorphism of the 2-torus, $S : \mathbf{T} \to \mathbf{T}$, of the form $S(x, y) = (x+\alpha, y+\beta)$, where $1, \alpha, \beta \in \mathbb{R}$ are rationally independent and $+ : \mathbb{R}/\mathbb{Z}\times\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is defined in the obvious way. Then S is minimal (and ergodic with respect to Lebesgue measure). M. Rees [83] found a minimal homeomorphism S_1 which is an extension of S (i.e., $\varphi \circ S_1 = S \circ \varphi$ for some continuous surjection φ of $\mathbf{T}$; see Section 2.3) such that S_1 has positive topological entropy. See [83] also for references to related results.

3.6. Pseudo-Anosov homeomorphisms. Let M be a compact connected oriented surface possibly with boundary and $f : M \to M$ be a homeomorphism. There are two basic types of homeomorphisms which appear in Nielsen-Thurston classification: the finite order homeomorphisms and pseudo-Anosov ones.

A homeomorphism f is said to be of *finite order* if $f^n = \mathrm{id}$ for some $n \in \mathbb{N}$. Finite order homeomorphisms have topological entropy zero.

A homeomorphism f is said to be *pseudo-Anosov* if there is a real number $\lambda > 1$ and a pair of transverse measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that $f(\mathcal{F}^s) = \lambda^{-1}\mathcal{F}^s$ and $f(\mathcal{F}^u) = \lambda\mathcal{F}^u$. Pseudo-Anosov homeomorphisms are topologically transitive, have positive topological entropy, and have Markov partitions [39], see also [104], [49].

3.7. On examples on cylinders and other examples. For examples of topologically transitive maps on cylinders ($\mathbf{S} \times \mathbb{R}$ or more general) see [88], [16], [17], [46], [45] Section 14, [94], [95], [63], see also [50].

For constructions of transitive flows see, e.g., [8], [98], [99] and [66]. For some other examples see, e.g., [106], [77], [45], as well as books on ergodic theory.

4. Transitive and intransitive points

4.1. Basic facts. First realize that in a dynamical system (X, f), $x \in \mathrm{intr}(f) \Rightarrow f(x) \in \mathrm{intr}(f)$ and $f(x) \in \mathrm{tr}(f) \Rightarrow x \in \mathrm{tr}(f)$. If the system is standard then we have equivalences instead of implications.

Further, if a dynamical system (X, f) is transitive then by Theorem 2.2.2((1) $\Rightarrow$ (13)) it has a G_δ-dense set of transitive points. We show how this can be proved (provided (1) $\Rightarrow$ (8) has been established). Realize that a point is transitive if and only if it visits all sets from a countable base $\{U_i\}_{i=1}^{\infty}$ of open sets (note that X is compact). Thus

$$\mathrm{tr}(f) = \bigcap_{k=1}^{\infty} \left(\bigcup_{n=0}^{\infty} f^{-n}(U_k) \right).$$

Now it is sufficient to use (1) $\Rightarrow$ (8) in Theorem 2.2.2.

Notice also that as an immediate consequence of the previous paragraph we get that if (X, f) is transitive and a point $x_0 \in X$ is isolated then it is a transitive point. But in fact transitivity of the system implies more: the point x_0 is periodic and X is just the orbit of x_0. Since we usually know whether the phase space under consideration has an isolated point, Theorem 2.2.2 has some value only for standard dynamical systems.

4.2. Finite and infinite systems. It is also instructive to distinguish two cases depending on whether the space X is finite or not. In considerations it is sometimes useful to distinguish whether there are isolated points in X or not.

First consider a dynamical system (X, f) with a finite phase space X. If the system consists just of a single periodic orbit then it is transitive and all points are transitive. Otherwise it is not transitive and at most one point is transitive.

Now suppose that X is infinite. The system may have no transitive points, of course. The example used in Section 2.1 to prove that (DO) does not imply (TT) shows that the system may have a unique transitive point. It is easy to show that if the system has two transitive points a, b then it is transitive (and so has a G_δ-dense set of transitive points). (*Proof*: In an SDS one transitive point is sufficient to get transitivity of (X, f), so let X have an isolated point x_0. If both a and b are isolated then, being transitive points, they belong to the same periodic orbit and X is just this orbit, hence the system is transitive. So suppose that a is not isolated. Since a is transitive and f is continuous, $f^n(U) = \{x_0\}$ for some $n > 0$ and some neighbourhood U of a. Since a is transitive and U is infinite there is $k > 0$ with $f^{n+k}(a) \in U$. Then $\mathrm{orb}(a)$ is finite whence $X = \mathrm{orb}(a)$. If a is periodic the system is transitive. If not, the point b being different from a and belonging to $\mathrm{orb}(a)$ cannot be transitive.)

4.3. Standard dynamical systems. Finally, assume that (X, f) is not an arbitrary infinite dynamical system but a *standard* one. Then transitive points behave as we wish. Namely, if x is a transitive point in an SDS then $f^n(x)$ is a transitive point for every n, all points from the trajectory of x are mutually different and the trajectory of x visits any ball in X infinitely many times. Recall also that if in an SDS there is a transitive point then the system is transitive and hence the set of transitive points is dense G_δ. So, in an SDS (X, f) there are the following possibilities:

(a) $\mathrm{tr}(f) = \emptyset$, $\mathrm{intr}(f) = X$,

(b) $\mathrm{tr}(f)$ is dense G_δ and

- (b1) $\mathrm{intr}(f) = \emptyset$ (minimality) or
- (b2) $\mathrm{intr}(f)$ is dense (Ex. 3.1.3).

The following statement shows that no other possibility exists.

Theorem 4.3.1 (cf. [3]). *Let (X, f) be an SDS. Then the set* $\mathrm{intr}(f)$ *is either empty or dense in X* (*equivalently*: *if* $\mathrm{tr}(f)$ *has nonempty interior then the system is minimal*).

The statement must be known but we are unable to give a reference except of [3], p. 77, exercise 30 which implicitly contains this result. Since Hint to the exercise assumes the knowledge of the theory explained in the book, we present here a straightforward proof based on ideas from [3]. For a result similar to Theorem 4.3.1 in case of a homeomorphism in a non-compact space see [96] (and other papers cited there).[1]

Proof. Suppose that $\mathrm{Int}(\mathrm{tr}(f)) \neq \emptyset$. This implies the transitivity of the system because it is standard. Since the preimage of a transitive point is a transitive point and the orbit of any transitive point intersects $\mathrm{Int}(\mathrm{tr}(f))$, we have

$$\mathrm{tr}(f) = \bigcup_{n=0}^{\infty} f^{-n}\left(\mathrm{Int}(\mathrm{tr}(f))\right).$$

Hence $\mathrm{tr}(f)$ is open and, the system being transitive, dense. Then the set $\mathrm{intr}(f)$ is closed and nowhere dense. Moreover, $f(\mathrm{tr}(f)) = \mathrm{tr}(f)$ and $f(\mathrm{intr}(f)) = \mathrm{intr}(f)$ (note that f is onto). We wish to prove that $\mathrm{intr}(f) = \emptyset$. Suppose on the contrary that this is not the case and take a closed neighbourhood $U \neq X$ of the set $\mathrm{intr}(f)$. Then $\bigcap_{n=0}^{\infty} f^{-n}(U) = \mathrm{intr}(f)$ since the orbit of any point from $U \setminus \mathrm{intr}(f)$ intersects the open set $X \setminus U$.

The set $f(X \setminus \mathrm{Int}\, U)$ is compact and disjoint with $\mathrm{intr}(f)$. So one can find in U a closed neighbourhood V of $\mathrm{intr}(f)$ with $f^{-1}(V) \subset U$. Consequently,

$$\mathrm{intr}(f) = \bigcap_{n=1}^{\infty} f^{-n}(V) \subset \bigcap_{n=0}^{\infty} f^{-n}(U) = \mathrm{intr}(f).$$

[1] After submitting this article the authors found that, according to [29], Theorem 4.3.1 is proved in [60] and a related result in [52].

Denote $V_n = \bigcap_{k=1}^{n} f^{-k}(V)$. Then $\{V_n\}_{n=1}^{\infty}$ is a decreasing sequence of closed sets with $\bigcap_{n=1}^{\infty} V_n = \mathrm{intr}(f) \subset \mathrm{Int}\, V$. So there exists m such that

$$V_m = f^{-1}(V) \cap f^{-2}(V) \cap \cdots \cap f^{-m}(V) \subset \mathrm{Int}\, V.$$

Now define

$$W = V \cap V_{m-1} = V \cap f^{-1}(V) \cap \cdots \cap f^{-(m-1)}(V).$$

Then $W \subset V$ and $f^{-1}(W) = f^{-1}(V) \cap f^{-2}(V) \cap \cdots \cap f^{-m}(V) = V_m = V \cap V_m \subset V \cap V_{m-1} = W$. Finally, realize that W is a closed neighbourhood of $\mathrm{intr}(f)$. But the existence of a set W such that $\mathrm{Int}(X \setminus W) \neq \emptyset$, $\mathrm{Int}(W) \neq \emptyset$ and $f^{-1}(W) \subset W$ contradicts the transitivity of f (the orbit of a point $x \in \mathrm{tr}(f) \cap (X \setminus W)$ does not meet the set W). □

4.4. On the measure of the set of transitive points. If (X, f) is a topologically transitive dynamical system and X is simultaneously a measure space (with, say, some ‘usual’ measure), one can ask what is the measure of the set $\mathrm{tr}(f)$. Since the problem involves both topology and measure, one can expect that usually the answer will not be easy and, on the other hand, that ‘wild’ examples should be possible.

Here we only say that it is easy to show that the tent map (Ex. 3.1.3) has full Lebesgue measure set of transitive points but it is not difficult to find an example of a transitive map $f : I \to I$ with $\lambda(\mathrm{tr}(f)) < 1$ (λ is the Lebesgue measure). One possibility is to take in I a Cantor-like set C with $\lambda(C) > 1 - \varepsilon$, $\min C = 0$, $\max C = 1$. All the points from C will be fixed points of f and in every contiguous interval of C the map f will consist of three linear pieces—increasing, decresing, increasing (just take care of having sufficiently big ‘peaks’ in every contiguous interval to ensure transitivity (even a stronger property: for every interval $J \subset I$ there is some n with $f^n(J) = I$), but not too big ‘peaks’ to ensure continuity of the map). Then $\lambda(\mathrm{tr}\, f) < \varepsilon$. In [93] there is an example of a topologically transitive map in any bounded connected region in $\mathbb{R}^3$ with $\lambda(\mathrm{tr}(f)) = 0$.

Examples of maps (homeomorphisms) with similar properties on the cylinder $\mathbf{S} \times \mathbb{R}$ can be found in [94].

Though it is out of scope of this ‘topological’ paper to survey the measure-theoretical aspects of the transitivity, we wish add also some recent papers dealing at least partly with the problem of the measure of the set of transitive points (in dimension one): [102], [70], [32], [51].

4.5. Independence of transitive points. Topological weak mixing. In connection with transitive points we wish to mention also results of A. Iwanik.

Two transitive points x, y in a standard dynamical system (X, f) (see [53], [54] for a more general situation) are called *independent* if (x, y) is transitive in the product system $(X, f)^2 = (X \times X, f \times f)$ (here $f \times f$ is defined by $(f \times f)(x, y) = (f(x), f(y))$). If two such points exist then (X, f) is called *topologically weakly mixing*. More generally, a subset E of X is called independent for the system (X, f) [53] if for any finite collection $x_1, x_2, \dots, x_n$ of distinct points of E the point $(x_1, x_2, \dots, x_n)$ is transitive in the product system $(X, f)^n$ (then each x_i is transitive in (X, f)). A subset E of X is called *totally independent* for (X, f) [54] if for any positive integer r the set E is independent for the system (X, f^r) (do not confuse with $(X, f)^r$).

In [53] it is shown that, among other results, in every topologically weakly mixing system there exists an uncountable independent set. Totally independent sets have been studied in [54].

For 'mixing' notions in topological sense (they are anologues of corresponding notions from ergodic theory) see also [42], [37], [79], cf. [107]. Here only recall that topological strong mixing (defined in Section 6 as 'topological mixing') implies topological weak mixing, which implies topological transitivity.

5. Transitivity of a map and its iterates

Let (X, f) be a dynamical system. If $n \in \mathbb{N}$ and f^n is transitive then trivially f is transitive. The converse is not true in general. As a simple counterexample one can take a map f defined on the union of, say four, disjoint compact intervals in such a way that it is transitive (hence cyclically permutes the intervals). Then f^n is not transitive if n is even. To see that this is possible also if X is connected, take the 3-star (looking as the letter Y, sometimes is called the Y-space) and a transitive map which cyclically permutes the three edges, the branching point being fixed. (E.g., f maps cyclically each edge to the next one, two of them linearly and the remaining one as in the tent map, piecewise linearly with two pieces.) Then take the third iterate. See also the example of a transitive but not bitransitive map in Section 6.

The following theorem is probably a folk result (cf. [35], see [3], p. 77, Exercise 31 and, for homeomorphisms, [96]).

Theorem 5.1. *Let (X, f) be a dynamical system and $n \geq 2$. If f is transitive but f^n is not, then there is a closed set $K \neq X$ with nonempty interior and a divisor $m > 1$ of n ($m = n$ is possible) such that*

(1) $f^m(K) = K$,
(2) $K \cup f(K) \cup \dots \cup f^{m-1}(K) = X$,
(3) $\mathrm{Int}(f^i(K) \cap f^j(K)) = \emptyset$ *for* $0 \leq i, j \leq m-1, i \neq j$.

Finally, note that if f^m and f^n are transitive then so is f^{mn}, cf. [35] (but, as we showed above, f^{m+n} may not be transitive).

6. Transitivity in dimension one

6.1. Transitivity on the interval. As usual, let I denote a real compact interval, say $I = [0, 1]$. When speaking on an interval we will always mean a nondegenerate interval (also in further sections).

There are two typical examples of transitive maps on I. One of them is the tent map (Ex. 3.1.3). It is easily seen that all iterates of the tent map are transitive. In particular, f^2 is transitive. A map $f : I \to I$ such that f^2 is transitive is sometimes called *bitransitive*. Notice that if f is bitransitive then it is also transitive.

The other example is the continuous piecewise linear map defined by $f(0) = \frac{1}{2}$, $f(\frac{1}{4}) = 1$, $f(\frac{1}{2}) = \frac{1}{2}$ and $f(1) = 0$. Then f is transitive but f^2 is not, the subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ being invariant for f^2. But notice that the restriction of f^2 to either of these intervals is bitransitive.(Nevertheless, if $f \in C(I)$ is transitive then f^n is transitive for all odd $n > 0$, see [35].)

The following theorem shows that there are no other possibilities.

Theorem 6.1.1 ([14], cf. [23] and [20], p. 156). *Let $f \in C(I)$ be transitive. Then either f is bitransitive or there is a point $c \in$ Int$[0, 1]$ such that $f([0, c]) = [c, 1]$, $f([c, 1]) = [0, c]$ and both $f^2|_{[0,c]}$ and $f^2|_{[c,1]}$ are bitransitive.* (*Clearly, the point c is the only fixed point of f.*)

Hence, if $f \in C(I)$ is transitive and has at least two fixed points then it is bitransitive. But there are bitransitive maps with only one fixed point.

A map $f \in C(I)$ is *piecewise monotone* if there are points $0 = a_0 < a_1 < \cdots < a_n = 1$ such that for every $k \in \{1, 2, \dots, n\}$, the restriction of f to the interval $[a_{k-1}, a_k]$ is (not necessarily strictly) monotone. Of course, if f is transitive then all pieces are strictly monotone. Note also that in this paper 'piecewise monotone' means 'piecewise monotone with finite number of pieces of monotonicity'.

Before going to next theorem recall that a dynamical system (X, f) or the map f itself is called *topologically mixing* if, for every pair of nonempty open sets U and V, there is a positive integer N such that $f^n(U) \cap V \neq \emptyset$ for all $n > N$. Clearly if f is topologically mixing then it is also transitive but not conversely (see Ex. 3.1.2). Below we will see that on the interval, topological mixing is equivalent to bitransitivity.

Theorem 6.1.2 ([14]+[15]+[35], cf. [20], pp. 157–159). *Let $f \in C(I)$. Then the following statements are equivalent*:

(1) *f is bitransitive* (*i.e., f^2 is transitive*),
(2) *f^n is transitive for every $n > 0$,*
(3) *f is transitive and has a periodic point of odd period > 1,*
(4) *f is topologically mixing,*

(5) *for every* $\varepsilon > 0$ *and every interval* $J \subset I$ *there is a positive integer* N *such that* $f^n(J) \supset [\varepsilon, 1-\varepsilon]$ *for all* $n > N$.

Furthermore, if f *is piecewise monotone, then the following statement is equivalent to the rest*:

(6) *for every interval* $J \subset I$ *there is some positive integer* n *with* $f^n(J) = I$.

There are examples showing that without the assumption of piecewise monotonicity (6) is strictly stronger than (1)–(5).

One can see that it is possible to add two other statements, each of them being equivalent to each of the statements (1)–(5):

(5′) f^n is topologically mixing for some $n > 0$,

(5″) f^n is topologically mixing for every $n > 0$.

6.2. Transitivity on the circle. We have mentioned that if f is a transitive interval map then f^n is transitive for all odd n. Realize also that f has a fixed point and compare this with

Theorem 6.2.1 ([35]). *If* $f \in C(\mathbf{S}, \mathbf{S})$ *is transitive and has a fixed point, then* f^n *is transitive for every odd* $n > 0$.

For the circle there is an analogue of Theorem 6.1.2.

Theorem 6.2.2 ([35]). *Let* $f \in C(\mathbf{S}, \mathbf{S})$. *Then the following statements are equivalent*:

(1) *there is an* m *such that* f^m *is transitive and has a fixed point and a point of odd period greater than one,*

(2) *there is an* m *such that* f^{2m} *is transitive and* f^m *has a fixed point,*

(3) f^n *is transitive for every* $n > 0$ *and* f *has periodic points,*

(4) f *is topologically mixing.*

Furthermore, if f *is piecewise monotone, then the following statement is equivalent to the rest*:

(5) *for every interval* $J \subset \mathbf{S}$ *there is some positive integer* n *with* $f^n(J) = \mathbf{S}$.

6.3. Transitivity on one-dimensional ramified manifolds (graphs). By a ‘one-dimensional ramified manifold’ Blokh [25] means any compact metric space whose *local* structure at arbitrary point x can be described as ‘a finite number of intervals having the point x as a common endpoint’ (notice that a circle is also covered by this definition and that connectedness is not required). In [26] (where, in contrast to [25], only statements without proofs are given) he calls them shortly ‘graphs.’ In the sequel the set of all periodic points of f will be denoted by $P(f)$.

We need also recall the definition of so-called specification property. (X, f) (the metric on X being denoted by ϱ) or the map f itself are said to satisfy the *specification property* if the following holds: for any $\varepsilon > 0$ there exists an integer $M(\varepsilon)$ such that for any $k \geq 2$, for any k points $x_1, x_2, \ldots, x_k \in X$, for any integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_i - b_{i-1} \geq M(\varepsilon)$ for $2 \leq i \leq k$, and for any integer p with $p \geq M(\varepsilon) + b_k - a_1$, there exists a point $x \in X$ with $f^p(x) = x$ such that $\varrho(f^n(x), f^n(x_i)) \leq \varepsilon$ for $a_i \leq n \leq b_i$, $1 \leq i \leq k$.

If f has the specification property, then f is transitive, $P(f)$ is dense and the topological entropy of f is positive, see [37].

Theorem 6.3.1 ([25], [26]). *Let K be a one-dimensional ramified manifold* (*a graph*) *and $f \in C(K, K)$ be transitive. Then for some n, $K = \bigcup_{i=0}^{n-1} K_i$, where all the K_i are connected compact sets, $K_i \cap K_j$ is finite for $i \neq j$, $f(K_i) = K_{i+1}$, $i = 0, 1, \ldots, n-2$, $f(K_{n-1}) = K_0$ and two cases are possible*:

(1) *$P(f) \neq \emptyset$, then $f^n|_{K_i}$ has the specification property, $i = 0, 1, \ldots, n-1$,*

(2) *$P(f) = \emptyset$, then all the K_i are homeomorphic to circles and the $f^n|_{K_i}$ are conjugate to irrational rotations, $i = 0, 1, \ldots, n-1$.*

7. Transitivity and dense periodicity

Recall that we denote the set of all periodic points of $f \in C(X)$ by $P(f)$. If $P(f)$ is dense in X, we sometimes say that f or (X, f) has dense periodicity.

Of course, dense periodicity cannot imply the transitivity if the phase space has more than one point (take the identity map). In some spaces neither the converse is true (an irrational rotation on the circle). But there are spaces where transitivity implies dense periodicity.

Sharkovskii ([86], cf. [14], [20], p. 156) proved that if $f \in C(I)$ is transitive then $P(f)$ is dense in I. This classical result can be substantially generalized. But first some terminology.

A connected space X has a *disconnecting interval* if there is an open subset J of X, homeomorphic to an open interval, such that $X \setminus J$ is not connected.

Theorem 7.1 ([5]). *If in the system (X, f) the space X is connected and has a disconnecting interval and f is transitive then $P(f)$ is dense in X.*

If X is the union of two disjoint circles then it has a disconnecting interval but X is not connected and there is a transitive map $f \in C(X)$ without periodic points (cf. Theorem 6.3.1, case (2)).

If X is just one circle then it is connected but has no disconnecting interval. Again, transitive map can have no periodic point.

On the circle it is sufficient to add an additional assumption:

Theorem 7.2 ([35]). *If* $f \in C(\mathbf{S}, \mathbf{S})$ *is transitive and* $P(f)$ *is nonempty then* $P(f)$ *is dense in* $\mathbf{S}$.

On the other hand, recall the following

Theorem 7.3 ([11]). *Let* $f \in C(\mathbf{S}, \mathbf{S})$. *Then* f *is transitive and* $P(f) = \emptyset$ *if and only if* f *is topologically conjugate to an irrational rotation.*

Concerning transitive maps on graphs, notice that *in Theorem* 6.3.1 *in case* (1) *we have dense periodicity*, since the specification property implies it (and $x \in P(f^n)$ if and only if $x \in P(f)$).

On the square I^2 *transitivity does not imply dense periodicity.* Even, there is a triangular continuous selfmap of the square which is transitive and has nowhere dense set of periodic points, see Theorem 10.2.

See also Ex. 3.1.6 and the map S in Subsection 3.5 to see that *on the Cantor set and on the torus transitivity does not imply dense periodicity.* More generally, if X is infinite and admits a minimal map f then $P(f) = \emptyset$ and, consequently, transitivity on X does not imply dense periodicity.

8. Transitivity and chaos

The term chaos in a connection with a map was first used by Li and Yorke [69], although without giving any formal definition. Today there are various definitions of what it means for a map to be chaotic, some of them working reasonably only in special phase spaces. Though one could say that 'as many authors, as many definitions of chaos,' behind them it is usually the idea of unpredictability of behaviour of all trajectories or 'many' trajectories or at least one trajectory when the position of the point whose trajectory is considered is given with an error (instability of points or sensitive dependence to initial conditions are terms usually used to describe this phenomenon).

Since topological transitivity is usually either a part of the definition of a chaos or transitivity implies or is implied by the chaos (at least in some spaces), we mention here some kinds of chaos and indicate their connection with transitivity. If for no other purposes, by this we wish to show that transitivity is worth of further investigating.

8.1. Chaos on the interval as a large Sharkovskii type of the map. Sometimes $f \in C(I)$ is called chaotic if it has a periodic orbit whose period is not a power of 2, cf. [20], p. 33 (on the interval this is in fact equivalent to the positivity of the topological entropy of f, see [7], p. 231; a connection between transitivity and entropy will be dealt with in Section 9). The above definition is connected with the Sharkovskii ordering of the set $\mathbb{N} \cup \{2^\infty\}$:

$$3 \succ 5 \succ 7 \succ \cdots \succ 2 \cdot 3 \succ 2 \cdot 5 \succ 2 \cdot 7 \succ \cdots \succ 4 \cdot 3 \succ 4 \cdot 5 \succ 4 \cdot 7 \succ \cdots \succ \cdots$$
$$\succ 2^n \cdot 3 \succ 2^n \cdot 5 \succ 2^n \cdot 7 \succ \cdots \succ \cdots \succ 2^\infty \succ \cdots \succ 2^n \succ \cdots \succ 4 \succ 2 \succ 1.$$

We will also use the symbol $\succeq$ in the natural way. For $t \in \mathbb{N} \cup \{2^\infty\}$ we denote by $S(t)$ the set $\{k \in \mathbb{N} : t \succeq k\}$ ($S(2^\infty)$ stands for the set $\{1, 2, 4, \dots, 2^k, \dots\}$). Let $f \in C(I)$ and $\mathrm{Per}(f)$ be the set of periods of its periodic points.

Sharkovskii Theorem ([85], [87]). *For every $f \in C(I)$ there exists a $t \in \mathbb{N} \cup \{2^\infty\}$ with* $\mathrm{Per}(f) = S(t)$. *On the other hand for every $t \in \mathbb{N} \cup \{2^\infty\}$ there exists an $f \in C(I)$ with* $\mathrm{Per}(f) = S(t)$.

If $\mathrm{Per}(f) = S(t)$, then f is called to be *of type t*. When speaking of types we consider them to be ordered by the Sharkovskii ordering. So the considered definition says that an interval map is chaotic if its type is greater than 2^∞.

If $f \in C(I)$ is bitransitive then its type is an odd number $2r + 1$, where $r > 0$ (see Theorem 6.1.2). This type may not be 3, i.e., it is possible that $r > 1$ (e.g., the map in [7], p. 37, Fig. 2.2.2, is bitransitive and its type is 7).

So, if f is transitive but not bitransitive then (see Theorem 6.1.1) the type of f^2 is an odd number $2s+1$, $s > 0$ whence the type of f is $2\cdot(2s+1)$, $s > 0$. It is surprising that we can claim here that always $s = 1$. This follows from the facts that if f is transitive then, by [23], f^2 has so called Sharkovskii's L-scheme, (i.e., 2-horseshoe in another terminology, cf. Section 9) and that if g has an L-scheme then g has a periodic point of period 3 (see [85]). Hence, if f is transitive then it has a periodic point of period 6 (the statement that any transitive map has a periodic point of period 6 is also proved in [21]).Now it is sufficient to use the fact that f, being transitive but not bitransitive, has only a fixed point and periodic points of even periods (see Theorem 6.1.1) and so it has type 6.

Hence all transitive maps on the interval are chaotic in the considered sense but not conversely.

8.2. Chaos in the sense of Li and Yorke. The definition of Li–Yorke chaos was fixed on the basis of [69] and later other equivalent definitions were found [56]. At last it turned out [64] that a map $f \in C(I)$ is chaotic in the sense of Li and Yorke if and only if there are two points $x, y \in I$ with $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$ and $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$ (then the set $\{x, y\}$ is called a two-point scrambled set).

If $f \in C(I)$ is transitive then it is chaotic in the sense of Li and Yorke (in the next subsection a stronger result will be presented; compare this with circle where the irrational rotations have no two-point scrambled sets). The converse is not true (since Li–Yorke chaos allows that f be, say, constant in a subinterval of I).

But in [31] it is proved that, under the Continuum Hypotheses, a map $f \in C(I)$ is bitransitive if and only if there is an uncountable extremally scrambled set for f, i.e., an uncountable set S such that for every $x, y \in S$ with $x \neq y$, $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$ and $\limsup_{n\to\infty} |f^n(x) - f^n(y)| = 1$. (Note that 1 is the length of I.)

8.3. Generic chaos on the interval. This kind of chaos, proposed by A. Lasota (see [80]), is stronger than those considered above. Though the interval has dimension one, there is a two-dimensional aspect in this chaos.

For a function $f \in C(I)$ define the following planar set:

$$C(f) = \left\{(x, y) \in I^2 \colon \liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0 \text{ and } \limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0\right\}.$$

A compact interval $T \subset I$ will be called an invariant transitive interval of f if it is f-invariant and the restriction of f to the interval T is topologically transitive. According to A. Lasota a function $f \in C(I)$ is called generically chaotic if the set $C(f)$ is residual in I^2. Similarly, we will say that f is densely chaotic if $C(f)$ is dense in I^2 (see also definitions of generic and dense ε-chaos in [100], which are in fact equivalent to generic chaos).

In [100] several conditions equivalent to generic chaos have been found. In [101] it is shown that in a class of maps which contains all piecewise monotone maps the notion of dense chaos and that of generic chaos coincide (for an example (belonging to I. Mizera) of a densely chaotic map which is not generically chaotic, see [100]).

Topological transitivity implies generic chaos. The converse is not true, since there are generically chaotic maps arbitrarily close to a constant map. Nevertheless, there is only a 'small' difference between these two notions:

Theorem 8.3.1 ([100]). *Let $f \in C(I)$. The following conditions are equivalent*:

(a) *f is generically chaotic,*

(b) *the following two conditions are fulfilled simultaneously*:

(b_1) *f has a unique invariant transitive interval or two invariant transitive intervals having one point in common,*

(b_2) *for every interval J there is an invariant transitive interval T of f such that* $\mathrm{Int}(T) \cap \bigcup_{n=0}^{\infty} f^n(J) \neq \emptyset$.

8.4 Chaos in the sense of Ruelle and Takens (Auslander and Yorke). Let (X, f) be a dynamical system. Denote the metric on X by ϱ. Let $\varepsilon > 0$. The map f is called

Lyapunov ε-unstable at a point $x \in X$ if for every neighbourhood U of x, there is $y \in U$ and $n \geq 0$ with $\varrho(f^n(x), f^n(y)) > \varepsilon$ (some authors write $\geq \varepsilon$ but this has no influence on what follows). The map f is called unstable at a point x (or the point x itself is called unstable) if there is $\varepsilon > 0$ such that f is Lyapunov ε-unstable at x. For more information see, e.g., [47], [12], [24], [81].

In [84] (according to [12]) a system (X, f) with surjective f is called chaotic if every point is unstable and X contains a dense orbit. The instability of all points in a system implies that the system has no isolated points. So, the chaos in a standard dynamical system is defined as 'topological transitivity plus pointwise instability'.

Notice also that if a standard dynamical system has the property of pointwise instability then it can happen that there is no $\varepsilon > 0$ such that all points are ε-unstable with this ε. But if, additionally, the system has a dense orbit (i.e., is transitive) then pointwise instability (in fact the instability of the point with dense orbit) implies uniform pointwise instability, i.e., the existence of such common $\varepsilon > 0$. (Cf. the definition of sensitive dependence to initial conditions in the next subsection.)

Of course, in general there is no connection between transitivity and pointwise instability. But on the interval transitivity implies pointwise instability (the converse is still not true even if the uniform pointwise instability is assumed). See also the next subsection.

8.5. Chaos in the sense of Devaney. Recently Devaney's definition of chaos became rather popular, partially due to a redundancy in the definition which was found later. By this definition (see [38]), the system (X, f) is chaotic if

(1) f is transitive,
(2) the periodic points of f are dense in X,
(3) f has sensitive dependence on initial conditions.

Here sensitive dependence means that there exists $\varepsilon > 0$ such that f is Lyapunov ε-unstable at every point in X (ε is the same for all points, i.e., sensitive dependence = uniform pointwise instability). Recall also that the terminology is not unified. In the original Guckenheimer's definition of sensitive dependence on initial conditions (for interval maps) the uniform pointwise instability was required for a set of positive Lebesgue measure [47].

It turned out that (1) and (2) together imply (3) provided X is not a finite set (when transitivity implies that the system is just a single periodic orbit).

Theorem 8.5.1 ([97], [13], [43]). *Let X be an infinite metric space and $f : X \to X$ be continuous. If f is transitive and has dense periodic points then f has sensitive dependence on initial conditions.*

In fact in [43] the space is additionally assumed to be compact but, on the other hand, many related results can be found there. Also in [97] there are related results. See also [105].

For more information on Devaney's chaos on the interval see [67]. Among others, it is proved there that f has positive topological entropy if and only if there is a closed invariant set $D \subset X$ such that $f|_D$ is chaotic in the sense of Devaney. So, if an interval map f has positive entropy then it is transitive on a closed invariant set which (due to the sensitive dependency on initial conditions on this set) has no isolated point, so it is a perfect set. For a stronger result see Theorem 9.4.

9. Transitivity and topological entropy

To be able to discuss the connection between transitivity and (topological) entropy, we start with some definitions.

The *topological entropy* of a map $f \in C(X)$, denoted by $h(f)$, was introduced in [1] as an invariant of topological conjugacy and an analogue to the notion of metric (measure-theoretical) entropy (see, e.g., [37], [107]). We give here one of the definitions which were introduced by Bowen [28] and are equivalent to the original one.

So let, as usual, (X, ϱ) be a compact metric space and let f be a continuous map from X into itself. A subset E of X is called (n, ε)-*separated* if for every two different points $x, y \in E$ there exists $0 \leq j < n$ with $\varrho(f^j(x), f^j(y)) > \varepsilon$. Let $s_n(f, \varepsilon)$ be the maximal possible cardinality of an (n, ε)-separated set. Then the (topological) entropy of f is defined by

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(f, \varepsilon).$$

Recall that $h(f^n) = n \cdot h(f)$ for any nonnegative integer n.

For interval maps the following notion is very useful to compute the entropy (see [7] for more details).

A family of subintervals of an interval J forms a partition of J if they are mutually disjoint and their union is J. An interval J_1 f-covers an interval J_2 if $f(J_1) \supset J_2$. If $f \in C(I)$ and $s \geq 2$, then an s-*horseshoe* for f is an interval $J \subset I$ and a partition $\mathcal{D}$ of J into s subintervals such that the closure of each element of $\mathcal{D}$ f-covers J. If f has an s-horseshoe then $h(f) \geq \log s$, see [7], p. 207, cf. [72].

The question whether the positivity of the entropy of f implies the transitivity of f does not have a very good sense. Indeed, transitivity is a global characteristics. A map having two invariant sets A, B with nonempty interiors cannot be transitive. But the positivity of the entropy may be caused by the fact that $f|_A$ has positive entropy ($h(f) \geq h(f|_A)$).

On the other hand, the question whether transitivity implies the positivity of the entropy is chalenging. Moreover, if the answer is affirmative, one can ask what is the best lower bound for the entropy of transitive maps in the space under consideration.

There are spaces, in which transitive maps can have zero topological entropy. Apart from such trivial examples as finite spaces, classical examples exist on the circle and on the torus (irrational rotations, see Ex. 3.1.2 and the map S from Section 3.5).

But in some spaces this is not the case. Here is a well known result of this kind:

Theorem 9.1 ([24], [21], cf. [7], p. 260). *Let $f \in C(I)$ be transitive. Then f^2 has a 2-horseshoe and therefore $h(f) \geq \frac{1}{2}\log 2$.*

The example of a transitive but not bitransitive map at the beginning of Subsection 6.1 shows that the equality is possible.

Theorem 9.2 ([21]). *Let $f \in C(I)$ be transitive and has at least two fixed points (this is the same as to say that f is a bitransitive map with at least two fixed points) then f has a 2-horseshoe and therefore $h(f) \geq \log 2$.*

Again, the equality is possible (the standard tent map).

A piecewise monotone map f is called piecewise linear (of the constant slope β) if all pieces of monotonicity are linear with the slope coefficient of absolute value β. The next theorem shows that a piecewise monotone transitive interval map is always conjugate to a 'nicer' map (with the same entropy, of course).

Theorem 9.3 ([78], cf. [7], p. 260 and [82], p. 57). *If $f \in C(I)$ is piecewise monotone and transitive then f is topologically conjugate to some piecewise linear map of the constant slope $\beta = \exp(h(f))$.*

Before stating next result recall that in Section 6 topological mixing has been defined as a notion stronger than topological transitivity.

Theorem 9.4 ([20], pp. 162 and 218; cf. [108]). *A map $f \in C(I)$ has positive topological entropy if and only if there exists a positive integer n and an infinite closed set X such that X is invariant under f^n and the restriction of f^n to X is topologically mixing.*

See also the end of Subsection 8.5 and [20], p. 229, Theorem 28 ((i)$\Leftrightarrow$ (vi)).

The situation on the circle is more complicated. The substantial part of the following result is implicitly contained in Theorem 6.3.1. (For the definition of the degree of a circle map we refer the reader to, say, [7].)

Theorem 9.5 ([7], p. 267). *Let $f \in C(\mathbf{S}, \mathbf{S})$ be transitive. Then either $h(f) > 0$ or f is conjugate to an irrational rotation (via a homeomorphism of degree 1).*

The best lower bound of the entropy of transitive circle maps of a given degree is as follows:

Theorem 9.6 ([22]+[26]+[11]+[5]). *Let $f \in C(\mathbf{S}, \mathbf{S})$ be a transitive circle map of degree d. Then the following statements hold.*

(a) *If $|d| > 1$, then $h(f) \geq \log |d|$.*
(b) *If $d = 0$, then $h(f) \geq \log 2$.*
(c) *If $d = -1$, then $h(f) \geq \frac{1}{2} \log 3$.*
(d) *If $d = 1$ and f has periodic points, then $h(f) > 0$.*
(e) *If $d = 1$ and f has no periodic points, then $h(f) = 0$.*

Moreover, there exist transitive circle maps f_0, f_{-1} and f_d with degree $0, -1$ and $d \in \mathbb{Z} \setminus \{-1, 0, 1\}$ respectively, such that $h(f_0) = \log 2$, $h(f_{-1}) = \frac{1}{2} \log 3$ and $h(f_d) = \log |d|$.

Parts (d) and (e) were proved in [26] and [11] respectively (see also [68]), and part (a) was proved in [22] (see also [7]). The rest was proved in [5].

A natural question is: Does there exist a positive lower bound for the topological entropy of a transitive circle map of degree one with periodic points? The negative answer is given by the following theorem.

Theorem 9.7 ([5]). *The infimum of the (positive) topological entropies for the transitive circle maps of degree 1 with periodic points is zero.*

We have stated results for the interval and the circle. Concerning general graphs, recall that from Theorem 6.3.1 we get that if a transitive map on a graph has periodic points then it has positive topological entropy (but this entropy can be arbitrarily small, see below).

In particular, if the graph is a tree (i.e., a connected space that is homeomorphic to the union of finitely many copies of the unit interval, but does not contain a subset homeomorphic to a circle), then any transitive f has positive entropy (since f has a fixed point) and again a positive lower bound does not exist if we do not fix the number of endpoints of the tree (see below).

A natural question arises what is the best lower bound for the topological entropy of transitive selfmaps of trees (so called tree maps), depending on the number of endpoints and on the number of edges of the tree. A general formula allowing to compute the infimum of topological entropies of all transitive selfmaps of a given tree is not known and it seems that it would be quite complicated.

Franks and Misiurewicz in [41] studied a special class of transitive tree maps that arises naturally in the study of the homeomorphisms of the disk. For such class of maps the bound $(\log 2)/n$ for the topological entropy has been obtained, where n is the number of endpoints of the tree. In [40] a similar problem for the tree maps obtained from pseudo-Anosov diffeomorphisms of a punctured disk has been considered. The bound of the topological entropy obtained in this case is $(\log(1+\sqrt{2}))/k$ where k is the number of punctures.

A (positive) lower bound for the entropies of transitive tree-maps is known, though in general it is not the best lower bound:

Theorem 9.8 ([4]). *Let f be a continuous transitive tree map. Then $h(f) \geq (\log 2)/n$ where n is the number of endpoints of the tree.*

In [5] the same was proved for n-star maps (n-star is the subspace of the complex plane which is most easily described as the set of all complex numbers z such that z^n is in the unit interval $[0, 1]$, i.e., a central point (the origin) with n copies of the interval $[0, 1]$ attached to it. Notice that the 1-star and the 2-star are homeomorphic. For $n > 2$, the n-star is a tree having a unique branching point (the origin) which will be denoted by b. In what follows we only consider the n-star with $n \geq 2$. Any continuous map from an n-star into itself is called an n-star map).

Theorem 9.9 ([5]). *Let f be a continuous transitive n-star map. Then the following statements hold.*

(1) *If $f(b) \neq b$, then $h(f) \geq (\log 2)/2$ (it is not known whether this is the best lower bound).*
(2) *If $f(b) = b$, then $h(f) \geq (\log 2)/n$ (equality is possible).*

Now it is natural to ask what is the connection between transitivity and topological entropy in higher dimensions. In the next Section we show (see Theorem 10.3) that in the square I^2 there is an transitive orientation preserving homeomorhism with zero entropy. Moreover, *in any cube I^n there is a transitive* (*triangular*) *map with zero entropy*.

Finally, we wish to mention the following aspect of the relation between transitivity and entropy.

Following [36] say that $f \in C(X)$ is *entropy-minimal* if the only nonempty, closed, f-invariant subset Y of X such that $h(f|_Y) = h(f)$ is $Y = X$. Clearly, every minimal map is entropy-minimal. Also a part of the following result holds in every dynamical system.

Theorem 9.10 ([36]). *Let $f \in C(X)$. If f is entropy-minimal then it is topologically transitive. If $X = I$ and f is piecewise monotone then also the converse is true.*

10. On some extensions of transitive maps

First, let us recall that the triangular maps are defined as follows. Let (X_i, ϱ_i) be metric spaces for $i = 1, 2, \dots, n$. We assume that the set $\prod_{i=1}^n X_i$ is endowed with the metric $\varrho(x, y) = \max_{1 \le i \le n} \varrho_i(x_i, y_i)$, where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. A map F from $\prod_{i=1}^n X_i$ into itself is called *triangular* if it is continuous and is of the form

$$F(x_1, x_2, \dots, x_n) = (F_1(x_1), F_2(x_1, x_2), \dots, F_n(x_1, x_2, \dots, x_n)) \qquad (*)$$

where $(x_1, x_2, \dots, x_n) \in \prod_{i=1}^n X_i$. So the triangularity means continuity and the dependence of the ith component map only on the first i variables for every i. The set of all triangular maps from $\prod_{i=1}^n X_i$ into itself will be denoted by $\mathcal{C}_\triangle(\prod_{i=1}^n X_i)$. When all the spaces X_i are the interval I, instead of $\mathcal{C}_\triangle(\prod_{i=1}^n X_i)$, we will simply write $\mathcal{C}_\triangle(I^n)$ and we will *always* understand that a map $F \in \mathcal{C}_\triangle(I^n)$ is triangular at *all* levels. That is, it satisfies (*). If $n \ge 2$, we can always think on the map F as a triangular map defined on the 'rectangle' $(\prod_{i=1}^{n-1} X_i) \times X_n$ by $F(x, y) = (f(x), g(x, y))$, where $x = (x_1, x_2, \dots, x_{n-1})$, $y = x_n$ and $f(x) = (F_1(x_1), F_2(x_1, x_2), \dots, F_{n-1}(x_1, x_2, \dots, x_{n-1}))$, $g(x, y) = F_n(x_1, x_2, \dots, x_n)$. Here f is also a triangular (and not an arbitrary continuous) map from $\prod_{i=1}^{n-1} X_i$ into itself and $g(x, y)$ is a map from $(\prod_{i=1}^{n-1} X_i) \times X_n$ to X_n.

We will denote by $\mathcal{C}_\triangle(X \times I)$ the set of all triangular maps from $X \times I$ into itself. Also, each map $F \in \mathcal{C}_\triangle(X \times I)$ will be written as $F(x, y) = (f(x), g(x, y))$ where $f \in C(X)$ and g is a continuous map from $X \times I$ to I. Instead of $g(x, y)$ we can also write $g_x(y)$, where $g_x : I \to I$ is a family of continuous maps depending continuously on $x \in X$. The map f is called the *basis map* of F and the maps g_x are called the *fibre maps*. The map F splits the rectangle $X \times I$ into one-dimensional fibres $I_x = \{x\} \times I$ for $x \in X$ such that each fibre is mapped by F into a fibre. Though, in this sense, triangular maps are similar to basis ones (more precisely, F is an extension of f, see semiconjugacy at the end of Subsection 2.3), there are essential differences in the dynamics between them for the case $X = I$ (see [61]).

Topological entropy of triangular maps has been studied in [61] and [6]. Here only recall that, by Bowen's formula [28],

$$\max\left\{h(f), \sup_{x \in X} h(F, I_x)\right\} \le h(F) \le h(f) + \sup_{x \in X} h(F, I_x),$$

where $h(F, I_x)$ is the entropy of F on the fibre I_x (consider only (n, ε)-separated sets lying in I_x).

Further, realize that if $F = (f, g_x)$ is transitive (in $X \times I$) then f is transitive (in I) but not conversely.

If $f \in C(X)$ is transitive, the question is, whether it can be extended to a transitive map $F \in \mathcal{C}_\triangle(X \times I)$ (this means, that the basis map of F will be f). It is easy to do this, it is sufficient to take $F(x, y) = (f(x), g_x(y))$ where, for every $x \in X$, $g_x(y) = g(y) = 1 - |2y - 1|$ (use the definition (TT) and the fact that for the standard tent map g it holds (6) from Theorem 6.1.2). But if $h(f)$ is finite then we get (see [44]) that $h(F) = h(f) + h(g) > h(f)$.

So, we modify the question. Given a transitive f, is it possible to extend it to a transitive F without increasing the entropy? (Cf. also [89] where the 'converse' question is considered: If (X, F) has positive entropy does it have a factor (Y, f) with strictly smaller entropy?)

Theorem 10.1 ([5]). *Let (X, ϱ) be a compact metric space and let $f \in C(X)$ be a transitive map which is not minimal. Then the map f can be extended to a map $F \in \mathcal{C}_\triangle(X \times I)$ (i.e., f is the basis map of F) in such a way that F is transitive and has the same entropy as f.*

The theorem, being proved by the category method, does not give explicit examples of transitive triangular maps whose entropy is not greater than the entropy of their basis maps. In the particular case when $X = I$, there are also explicit examples.

Theorem 10.2 ([5]). *The map*

$$F : (x, y) \mapsto (1 - |2x - 1|, y^{\exp(x-\beta)}),$$

where β is any irrational number from $(0, \frac{2}{3})$, is a transitive map from $\mathcal{C}_\triangle(I^2)$ with topological entropy $h(F) = \log 2$ (the same as the entropy of the basis map) such that the set of periodic points is contained in $I \times \{0, 1\}$ (and hence is nowhere dense in I^2).

From Theorem 10.1 we get the first three parts of the following

Theorem 10.3 ([5]). *The following statements hold.*

(a) *Let (X, ϱ) be a compact metric space such that there are no minimal maps in $C(X)$. Then*

$$\inf\{h(f) : f \in C(X) \text{ is transitive}\} = \inf\{h(F) : F \in \mathcal{C}_\triangle(X \times I) \text{ is transitive}\}$$

and if one of these infima is minimum then so is the other.

(b) *For every $n = 2, 3, \ldots,$*

$$\min\{h(F) : F \in \mathcal{C}_\triangle(I^n) \text{ is transitive}\} = \min\{h(f) : f \in C(I) \text{ is transitive}\} = \tfrac{1}{2}\log 2.$$

(c) *For every $n = 2, 3, \ldots,$ there are transitive maps from $C(I^n)$ with arbitrarily small positive topological entropies. Consequently,*

$$\inf\{h(f) : f \in C(I^n) \text{ is transitive and } h(f) > 0\} = 0.$$

(d) *There exists a transitive orientation preserving homeomorphism of I^2 with zero topological entropy. Consequently,*

$$\min\{h(f) : f \in C(I^2) \text{ is transitive}\} = 0.$$

(e) *For every* $n = 2, 3, \dots,$

$$\min\{h(f) : f \in C(I^n) \text{ is transitive }\} = 0.$$

The parts (c),(d) and (e) are probably known but we are unable to give other references. They could perhaps be deduced from [9] and [57] (even for C^∞ diffeomorphisms?). Nevertheless, in these two papers the notion of topological entropy is not mentioned at all.

11. Miscellaneous results

We wish to bring the attention of the reader to some other results concerning transitivity, mainly one-dimensional ones. These will by no means exhaust the topics not covered (or not sufficiently covered) by the previous sections or even the results known on these topics. Rather, we hope merely to indicate the diversity of results on transitivity in the literature.

(1) If in a dynamical system (X, f) a point x belongs to its ω-limit set $\omega_f(x)$ then the restriction of f to $\omega_f(x)$ is transitive, see, e.g., [3], p. 69.

(2) A piecewise monotone map $f \in C(I)$ is called *expanding* if there is a constant $\lambda > 1$ such that $|f(x) - f(y)| \geq \lambda|x - y|$ whenever both x and y belong to some interval on which f is monotone.

An expanding map need not be transitive and a transitive piecewise monotone map need not be expanding (Ex. 3.1.4). However, in [55] it is proved that if a piecewise monotone map $f \in C(I)$ is expanding then there exists $n \in \mathbb{N}$ and a closed interval $J \subset I$ such that J is invariant for f^n and $f^n|_J$ is transitive in J.

(3) Inverse limit spaces can be succesfully used to study transitivity. See, e.g., [15] (note that Lemma 0 was taken from [12] without the assumption that f is onto).

(4) In [27] all pairs (v, e) of numbers are found with the property that there is a transitive map $f \in C(I)$ whose variation is v and topological entropy is e.

(5) In [82] transitivity of piecewise monotone interval maps is thoroughly investigated.

(6) An effective procedure for determining whether a so-called linear Markov map on the interval is transitive can be found in [21].

(7) If $f \in C(I)$ and $P = \{p_1 < \cdots < p_n\}$ is a finite invariant set of f, let f_P be the map defined on $[p_1, p_n]$ which agrees with f on P and which is linear on $[p_i, p_{i+1}]$,

$i = 1, \ldots, n-1$. Then $h(f) = \sup h(f_P)$ where the supremum is taken over all finite invariant sets [103]. In [34] it is shown that for transitive maps, this supremum is attained at some finite invariant set if and only if the map is piecewise monotone and the set contains the endpoints of the interval and the turning points of the map (i.e., the points from Int I in which f is not monotone).

(8) Let $f_s \in C(I)$ be defined by $f_s(x) = sx$ if $0 \le x \le \frac{1}{2}$ and $f_s(x) = s(1-x)$ if $\frac{1}{2} \le x \le 1$ (cf. Ex. 3.1.3). If $s \in [\sqrt{2}, 2]$ then $I_s = [f_s^2(\frac{1}{2}), f_s(\frac{1}{2})]$ is an invariant interval for f. In [30] it is shown that the set of parameters for which the orbit of the critical point $\frac{1}{2}$ is dense in I_s, is G_δ-dense in $[\sqrt{2}, 2]$.

(9) In [2] it is proved that $F \subset I$ is the set of fixed points of some transitive map from $C(I)$ if and only if F is a nonempty closed nowhere dense set different from each of $\{0\}$, $\{1\}$ and $\{0, 1\}$.

(10) A compact set $A \subset X$ is called transitive under f if there is a point in A whose ω-limit set is the set A (equivalently, the system $(A, f|_A)$ is transitive). Let $x \sim y$ iff x, y belong to the same connected component of A. Consider the quotient space $K = A/\sim$ with the identification topology and the continuous map $\tilde{f} : K \to K$ induced by f. In [33] the system $(K, \tilde{f})$ is studied provided X is locally compact metric space (the influence of Liapunov stability or asymptotic stability of the set A is also investigated).

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