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Conservative semisprays on Finsler manifolds II

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Abstract. In the first part we have presented a general theory of conservative torsion-free horizontal endomorphisms (non-linear connections) on a Finsler manifold (M, E). We investigated their existence problem giving a process to construct such kind of geometrical objects. As an application of these results here we describe a canonical one provided a Finslerian structure on the underlying manifold M.

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1. Preliminaries

M is an n-dimensional smooth, connected and paracompact manifold, $n \geq 2$. $\pi : TM \to M$ is the tangent bundle of M. $\pi_0 : \mathcal{T}M \to M$ is the bundle of nonzero tangent vectors. $\mathfrak{X}(M)$ is the module of vector fields on M, ι_X $(X \in \mathfrak{X}(M))$ and d are the *insertion operator* and the *exterior derivative*, respectively. $\mathfrak{X}^v(TM)$ is the module of vertical vector fields on the tangent manifold TM.

The vector 1-form J and $C \in \mathfrak{X}^v(TM)$ are the *vertical endomorphism*, (the canonical almost tangent structure) and the *Liouville vector field*, respectively:

$$\text{Im } J = \text{Ker } J = \mathfrak{X}^v(TM), \quad J^2 = 0.$$

Definition 1 ([1,2]). A mapping $S : v \in TM \to S(v) \in T_vTM$ is said to be a *semispray* on M if the following conditions are satisfied:

(i) S is smooth on $\mathcal{T}M$,

(ii) $JS = C$.

A semispray is called a *spray* if it has the homogeneity property $[C, S] = S$.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Definition 2 ([1, 2]). A vector 1-form $h : \mathfrak{X}(TM) \to \mathfrak{X}(TM)$ is said to be a *horizontal endomorphism* on M if it satisfies the following conditions:

(i) h is smooth on $\mathcal{T}M$,
(ii) h is a projector, i.e., $h^2 = h$,
(iii) Ker $h = \mathfrak{X}^v(TM)$.

A horizontal endomorphism h is *torsion-free* if the *Frölicher–Nijenhuis bracket* of the vertical endomorphism J and h vanishes, i.e.,

$$t := [J, h] = 0.$$

h is homogeneous if its *tension* $H := [h, C]$ vanishes. The vector 1-form

$$T := \iota_S t + H,$$

where S is a semispray on M, is said to be the *strong torsion* of h.

Remark 3. By Grifone's theory of non-linear connections any horizontal endomorphism is uniquely determined by the vector forms t, H, T and the so-called *associated semispray* $S_h := h(S)$.

Theorem 4 (The fundamental lemma of Finsler geometry, [1]). *Let (M, E) be a Finsler manifold with fundamental form $\omega := dd_J E$. There exists a unique horizontal endomorphism h such that*

(i) $d_h E = 0$*, i.e., h is conservative,*
(ii) *h is torsion-free,*
(iii) *h is homogeneous.*

Explicitly,

$$h = \frac{1}{2}(1 + [J, S]),$$

where S is the canonical spray given by the formula

$$\iota_S \omega = -dE;$$

h is called the Barthel endomorphism *of* (M, E).

Definition 5 ([5]). Let (M, E) be a Finsler manifold with canonical spray S. A horizontal endomorphism $\hat{h}$ on M is said to be *conservative* (with respect to the energy function E) if

$$d_{\hat{h}} E = 0.$$

A *semispray* $\hat{S}$ is *conservative* if the induced horizontal endomorphism

$$\hat{h} = \frac{1}{2}(1 + [J, \hat{S}])$$

is conservative. A *vector field* $V \in \mathfrak{X}^v(\mathcal{T}M)$ is *conservative* if the semispray

$$\hat{S} := S + V,$$

is conservative.

Theorem 6 ([5]). *A vector field $V \in \mathfrak{X}^v(TM)$ is conservative if and only if*

$$\iota_V \omega = d_J(VE).$$

Theorem 7 ([5]). *Let (M, E) be a Finsler manifold and suppose that the function $\varphi \in C^\infty(\mathcal{T}M)$ is homogeneous of degree 1, i.e., $C\varphi = \varphi$. Then the vector field $V \in \mathfrak{X}^v(\mathcal{T}M)$ defined by the formula*

$$\iota_V \omega = d_J \varphi$$

is conservative.

2. The main result

Theorem 8. *Let (M, E) be a Finsler manifold. There exists a unique horizontal endomorphism $\hat{h}$ such that*

1. *$d_{\hat{h}} E = 0$, i.e., $\hat{h}$ is conservative,*
2. *$\hat{h}$ is torsion-free,*
3. *$\widehat{H} = (1/L)(J - (1/L)\, d_J L \otimes C)$, where L is the fundamental function of the Finsler manifold:*

$$E = \tfrac{1}{2} L^2, \quad L \geq 0.$$

Proof. As it is well-known the energy function of a Finsler manifold is homogeneous of degree 2, i.e., $CE = 2E$. This homogeneity property implies that the fundamental function L must be homogeneous of degree 1. Using Theorem 7 we have a canonical conservative vector field $V \in \mathfrak{X}^v(\mathcal{T}M)$ defined by the formula

$$\iota_V \omega = d_J L.$$

Since $\iota_C \omega = d_J E$, a routine calculation shows that

$$V = \frac{1}{L} C,$$

i.e., V is just the normalized Liouville vector field. Consider the horizontal endomorphism

$$\hat{h} := \tfrac{1}{2}(1 + [J, \hat{S}]),$$

where $\hat{S} := S + V$.Then $\hat{h}$ is obviously conservative.

In order to prove (2.) observe that $\hat{h}$ and the Barthel endomorphism are related as follows:

$$\hat{h} = h + \tfrac{1}{2}[J, V].$$

Since J is integrable, i.e., $[J, J] = 0$ we get by the graded Jacobi identity that

$$[J, [J, V]] = 0 \Rightarrow [J, \hat{h}] = 0,$$

using the property (ii) of the Barthel endomorphism. This means that $\hat{h}$ is torsion--free.

Omitting the troublesome details (see [5, Corollary 3]) we note that the tension of $\hat{h}$ is just the Frölicher–Nijenhuis bracket $[J, V]$, where as it was proved above, $V = (1/L)C$. By the homogeneity property $[J, C] = J$ we have that

$$\widehat{H} = [J, (1/L)C] = \frac{1}{L}[J, C] + \left(d_J \frac{1}{L}\right) \otimes C = \frac{1}{L}\left(J - \frac{1}{L} d_J L \otimes C\right).$$

It remains only to show the uniqueness statement.To see this suppose that $\hat{h}$ satisfies the conditions (1.–3.) and let us define a horizontal endomorphism

$$h := \hat{h} - \tfrac{1}{2}\widehat{H}.$$

An easy but a little lengthy calculation shows that h satisfies the properties (i)–(iii) of the Theorem 4, i.e., it must be the Barthel endomorphism of the Finsler manifold. □

3. An application

In ([4]) the authors constructed special Finsler connections on a Finsler manifold starting out from conservative torsion-free horizontal endomorphisms. Among others it was proved that for any conservative torsion-free horizontal endomorphism h there exists a unique Finsler connection (D, h) such that

- D is metrical;
- The $(v)v$-torsion of D vanishes;
- The $(h)h$-torsion of D vanishes.

Using our main result we can introduce further *canonical* Finsler connection on a Finsler manifold such as

(i) A *Berwald-type Finsler connection* (see [4]; 4.3. Theorem, p. 47),
(ii) A *Cartan-type Finsler connection* (see [4]; 4.5. Theorem, p. 47),
(iii) A *Chern–Rund-type Finsler connection* (see [4]; 4.9. Theorem, p. 50),
(iv) A *Hashiguchi-type Finsler connection* (see [3]; Corollary 5, p. 31)

associated with $\hat{h}$.

It seems to be an important application of our results to the theory of Finsler connections.

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