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Variational problems defined by local data[1]

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Abstract. We study, in the framework of the global inverse problem of the calculus of variations, the first order variational problems defined by a family of local Lagrangian densities. We will show that those local variational problems for which the differential of the Poincaré–Cartan form is globally defined, admit a geometrical formulation which apart from some cohomological obstructions is closely related to the one given in the ordinary case. In particular, we will give a criterion for deciding when a local variational problem is indeed a global one. We also give in this setup the proper definition of an infinitesimal symmetry and its associated Noether invariants. We will show that although every infinitesimal symmetry has a virtual Noether invariant, there is a cohomological obstruction to the existence of a global Noether invariant for a given infinitesimal symmetry. We end with a discussion of the associated Poisson algebras.

Keywords. Calculus of variations, global inverse problem, Poincaré–Cartan forms, infinitesimal symmetries, Noether invariants.

MS classification. 70S05, 70S10, 58A20, 58A12, 55N30.

1. Introduction

The variational problems associated with a global Lagrangian density admit a well-known geometric description, see for example [5, 6, 7, 8]. The aim of this paper is to adapt this geometric description to first order variational problems defined by a family of local Lagrangian densities. In what follows we shall refer to them, for simplicity, as local variational problems, and as we shall see they can be described, essentially, as a "recollement" of ordinary variational problems.

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An example of this situation is provided by a particle on a Riemannian manifold with nontrivial topology subject to the action of an electromagnetic field. Another instance is given by the Chern–Simons theory on a three-dimensional manifold and its generalization for any secondary characteristic class [12].

In this paper we will concentrate, for brevity, on the definition of the local variational symmetries, their extremals, infinitesimal symmetries and their Noether invariants. We will also study the corresponding Poisson algebras. The rest of the geometric theory for variational problems, such as the regularity conditions, the definition of Jacobi vector fields, the formal pre-symplectic structure, etc., could also be developed along the same guidelines and will be carried out elsewhere.

2. First order variational problems defined by local data

The local variational problems appear naturally in the study of the inverse problem of the Calculus of Variations, which deals with the question of deciding when a system of differential equations arises as the Euler–Lagrange equations of some Lagrangian. This problem has been studied and solved by several authors using different techniques, among them we may cite [1, 2, 4, 11, 13]. Recently new ways of attacking these problems have appeared, see [9, 14, 15].

In this section we give an equivalent formulation of the inverse problem in terms of Lagrangian densities, but we start first by recalling some well-known concepts.

Let $\pi : Y \to X$ be a surjective submersion, with $n = \dim X$. Let us denote by $\pi^k_l : J^kY \to J^lY$ the natural projection between the jet bundles for any $l < k$, and $\pi^k : J^kY \to X$ the composition $\pi^k = \pi \circ \pi^k_0$. We shall denote by $\Omega^{p,q}_{J^kY} \subset \Omega^{p+q}_{J^kY}$ the subsheaf of the sheaf of differential forms which are p-contact and q-horizontal.

Let us recall [10] that starting from the exterior differential one constructs two $\mathbb{R}$-derivations of degree 1, the vertical differential d_v, and the horizontal differential d_h, which are sheaf morphisms

$$d_v,\ d_h : \Omega^p_{J^kY} \to (\pi^{k+1}_k)_*\Omega^{p+1}_{J^{k+1}Y}$$

such that

$$d_v : \Omega^{p,q}_{J^kY} \to (\pi^{k+1}_k)_*\Omega^{p+1,q}_{J^{k+1}Y},$$

$$d_h : \Omega^{p,q}_{J^kY} \to (\pi^{k+1}_k)_*\Omega^{p,q+1}_{J^{k+1}Y}.$$

Then we have

Definition 1. The sheaf $\mathcal{L}ag_Y$ of first order Lagrangians on Y is the direct image under π^1_0 of the sheaf $\Omega^{0,n}_{J^1Y}$. That is

$$\mathcal{L}ag_Y = (\pi^1_0)_*\Omega^{0,n}_{J^1Y}.$$

The following proposition is well known [1, 10].

Proposition 1. *The Euler–Lagrange sheaf morphism*

$$E : \mathcal{L}ag_Y \to (\pi^2_0)_* \Omega^{1,n}_{J^2Y}$$

is an $\mathbb{R}$*-linear sheaf morphism which associates to every Lagrangian its Euler–Lagrange form on* J^2Y*. Moreover, if* L *is a local section of* $\mathcal{L}ag_Y$*, one has*

$$E(L) = (\pi^2_1)^* d\Theta_L + d_v(\theta \wedge \Omega_L) \tag{1}$$

where θ *is the structure form on* J^1Y *and* Θ_L*,* Ω_L *are the Poincaré–Cartan and the Legendre forms associated with* L*, respectively.*

We can now define the variational problems given by local data

Definition 2. Let $\mathcal{L} = \{L_\alpha \in \mathcal{L}ag_Y(U_\alpha)\}_{\alpha \in I}$ be a family of local sections of $\mathcal{L}ag_Y$ subordinate to an open covering $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ of Y. We shall say that $\{\mathfrak{U}, \mathcal{L}\}$ are the data of a local variational problem if the Euler–Lagrange morphism applied to $\mathcal{L}$, $E_{\mathcal{L}} = \{E(L_\alpha)\}_{\alpha \in I}$, defines a global section in $H^0(Y, (\pi^2_0)_* \Omega^{1,n}_{J^2Y}) = \Omega^{1,n}(J^2Y)$; that is

$$E(L_\alpha)|_{U_\alpha \cap U_\beta} = E(L_\beta)|_{U_\alpha \cap U_\beta} \quad \forall \alpha, \ \beta \in I.$$

We shall say that two data $\{\mathfrak{U}, \mathcal{L}\}$, $\{\mathfrak{U}', \mathcal{L}'\}$ are equivalent if $E_{\mathcal{L}} = E_{\mathcal{L}'}$.

A local variational problem is given by an equivalence class $[\{\mathfrak{U}, \mathcal{L}\}]$ of local variational data.

A local variational problem $[\{\mathfrak{U}, \mathcal{L}\}]$ is termed global if there exists a global Lagrangian density $L \in H^0(Y, \mathcal{L}ag_Y)$ such that $E(L) = E_{\mathcal{L}}$.

Remark 1. This definition implies that the extremals of $L_\alpha|_{U_\alpha \cap U_\beta}$ and $L_\beta|_{U_\alpha \cap U_\beta}$ coincide, allowing us to give a coherent definition of the global extremals. Moreover, the extremality condition only depends on the equivalence class $[\{\mathfrak{U}, \mathcal{L}\}]$.

Therefore, we have

Definition 3. We shall say that a local section s of $\pi : Y \to X$ is critical for the local variational problem $[\{\mathfrak{U}, \mathcal{L}\}]$ if $s|_{U_\alpha}$ is critical for the variational problem associated with L_α for every $\alpha \in I$.

The following proposition is an easy consequence of the definitions and the usual characterizations of the critical sections [1, 6]

Proposition 2. *A local section* s *is critical for a local variational problem* $[\{\mathfrak{U}, \mathcal{L}\}]$ *if and only if*

$$(i_D E_{\mathcal{L}})|_{j^2 s} = 0 \qquad \forall\, D \in \mathfrak{X}(J^2Y).$$

Equivalently, s *is critical if and only if*

$$(i_D d\Theta_\alpha)|_{j^1 s} = 0 \qquad \forall \alpha \in I, \ \forall\, D \in \mathfrak{X}(J^1Y).$$

As a consequence of the equality (1) in Proposition 1 one can prove

Proposition 3. *Let* $\mathcal{L} = \{L_\alpha \in \mathcal{L}ag_Y(U_\alpha)\}_{\alpha\in I}$ *be a family of sections of* $\mathcal{L}ag_Y$ *subordinate to an open covering* $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ *of* Y*, and such that* $\{d\Theta_{L_\alpha}\}_{\alpha\in I}$ *defines an element* $\Sigma_{\mathcal{L}} \in H^0(Y, (\pi^1_0)_*\Omega^{n+1}_{J^1Y}) = \Omega^{n+1}(J^1Y)$*; that is*

$$d\Theta_{L_\alpha}|_{U_\alpha\cap U_\beta} = d\Theta_{L_\beta}|_{U_\alpha\cap U_\beta} \quad \forall\alpha,\ \beta\in I.$$

Under these conditions, $\{\mathfrak{U}, \mathcal{L}\}$ *are the data of a local variational problem.*

Therefore we can give the following

Definition 4. We shall say that $\{\mathfrak{U}, \mathcal{L}\}$ are the data of a local variational problem of restricted class if

$$\Sigma_{\mathcal{L}} = \{d\Theta_{L_\alpha}\}_{\alpha\in I} \in H^0\big(Y, (\pi^1_0)_*\Omega^{n+1}_{J^1Y}\big) = \Omega^{n+1}(J^1Y).$$

We shall say that two data of restricted class $\{\mathfrak{U}, \mathcal{L}\}$, $\{\mathfrak{U}', \mathcal{L}'\}$ are equivalent if $\Sigma_{\mathcal{L}} = \Sigma_{\mathcal{L}'}$. A variational problem of restricted class is given by an equivalence class of variational data of restricted class.

Remark 2. From the relationship between the Euler–Lagrange operator and the Poincaré–Cartan form it follows that two data of restricted class are equivalent if and only if they are equivalent as the data of a local variational problem.

In the case of variational problems of restricted class the global inverse problem allow us to prove the following

Theorem 1. *Let* $\{\mathfrak{U}, \mathcal{L}\}$ *be the data of a local variational problem of restricted class. Then*

1. $\Sigma_{\mathcal{L}} \in H^0(Y, (\pi^1_0)_*Z^{n+1}_{J^1Y}) = Z^{n+1}_{DR}(J^1Y)$*, where* $Z^{n+1}_{DR}(J^1Y)$ *denotes the De Rham* $(n+1)$*-cocycles.*
2. *The variational problem* $[\{\mathfrak{U}, \mathcal{L}\}]$ *is global if and only if the cohomology class* $[\Sigma_{\mathcal{L}}] \in H^{n+1}(J^1Y, \mathbb{R})$ *vanishes.*
3. *If* $[\Sigma_{\mathcal{L}}] = 0$ *then there exists* $L \in H^0(Y, \mathcal{L}ag_Y)$ *such that* $E(L) = E_{\mathcal{L}}$ *and* $\Sigma_{\mathcal{L}} = d\Theta_L$*.*

Proof. The first part is evident from the definition of $\Sigma_{\mathcal{L}}$. The second part is a consequence of Theorem 4.2 in [1].

In fact, using the notation of [1], there exists a unique morphism χ_2 which renders commutative the following diagram

$$\begin{array}{ccccc}
(\pi^1_0)_*\Omega^n_{J^1Y} & \xrightarrow{\ d\ } & (\pi^1_0)_*Z^{n+1}_{J^1Y} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \chi_2} & & \\
\mathcal{L}ag_Y & \xrightarrow{\ E\ } & \mathcal{E}_{J^2Y} & \longrightarrow & 0.
\end{array}$$

where ψ_2 is the horizontalization morphism and $\mathcal{E}_{J^2Y} = \operatorname{Im} E$ is the image sheaf under the sheaf morphism given by the Euler–Lagrange operator.

Under these conditions, the obstruction class $\delta(E_{\mathcal{L}})$ for the existence of a global Lagrangian is determined by the cohomology class of any $\omega \in H^0(Y, (\pi_0^1)_* Z^{n+1}_{J^1Y})$ such that $\chi_2(\omega) = E_{\mathcal{L}}$. This is equivalent to finding an open covering $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ of Y and differential forms $\Theta_\alpha \in \Omega^n_{J^1Y}((\pi_0^1)^{-1}(U_\alpha))$ such that

$$\{d\Theta_\alpha\}_{\alpha\in I} \in Z^{n+1}_{DR}(J^1Y),$$
$$E\big(\psi_2(\Theta_\alpha)\big) = E_{\mathcal{L}}|_{(\pi_0^1)^{-1}(U_\alpha)}.$$

It is easy to check that, for a local variational problem of restricted class, we can take $\Theta_\alpha = \Theta_{L_\alpha}$, which finishes the proof. The last part of the theorem follows from the previous ones. $\Box$

Remark 3. Let us note that the obstruction to the existence of a global Lagrangian lives in $H^{n+1}(Y, \mathbb{R})$ since J^1Y is a deformation retract of Y.

We can characterize the critical sections in terms of $\Sigma_{\mathcal{L}}$

Proposition 4. *A local section s is critical for a local variational problem* $[\{\mathfrak{U}, \mathcal{L}\}]$ *of restricted class if and only if*

$$(i_D\Sigma_{\mathcal{L}})|_{j^1s} = 0 \quad \forall\, D \in \mathfrak{X}(J^1Y).$$

3. Infinitesimal symmetries and Noether invariants

In what follows we are going to consider only local variational problems of restricted class.

Given a vector field $D \in \mathfrak{X}(Y)$ we will denote by $\bar{D} \in \mathfrak{X}(J^1Y)$ its 1-jet prolongation.

Definition 5. Let $\{\mathfrak{U}, \mathcal{L}\}$ be the data of a local variational problem. We shall say that $D \in \mathfrak{X}(Y)$ is an infinitesimal symmetry of the variational problem $[\{\mathfrak{U}, \mathcal{L}\}]$ if, for every $\alpha \in I$, $D|_{U_\alpha}$ is an infinitesimal symmetry of the variational problem associated with L_α. That is, one has

$$L_{\bar{D}}\Theta_\alpha = -d\eta^\alpha_D$$

with $\Theta_\alpha = \Theta_{L_\alpha}$ and $\eta^\alpha_D \in (\pi_0^1)_*\Omega^{n-1}_{J^1Y}(U_\alpha)$.

The local Noether invariant on U_α associated with the infinitesimal isometry D on U_α is the class

$$\omega^\alpha_D = i_{\bar{D}}\Theta_\alpha + \eta^\alpha_D + (\pi_0^1)_* Z^{n-1}_{J^1Y}(U_\alpha),$$

where $Z^{n-1}_{J^1Y}$ is the sheaf of De Rham $(n-1)$-cocycles on J^1Y.

Remark 4. Let us note that in this definition of infinitesimal symmetry it is not essential that the vector field $\bar{D} \in \mathfrak{X}(J^1Y)$ be the 1-jet prolongation of a vector field $D \in \mathfrak{X}(Y)$. This definition can be extended to every vector field $\widehat{D} \in \mathfrak{X}(J^1Y)$.

One can now prove the following

Proposition 5. *If $D \in \mathfrak{X}(Y)$ is an infinitesimal symmetry of the local variational problem associated with $\{\mathfrak{U}, \mathcal{L}\}$, then $\{\omega_D^\alpha\}_{\alpha\in I}$ defines an element*

$$[\omega_D] \in H^0(J^1Y, \Omega^{n-1}_{J^1Y}/Z^{n-1}_{J^1Y}).$$

Definition 6. Let $D \in \mathfrak{X}(Y)$ be an infinitesimal symmetry of the local variational problem $[\{\mathfrak{U}, \mathcal{L}\}]$. The element

$$[\omega_D] \in H^0(J^1Y, \Omega^{n-1}_{J^1Y}/Z^{n-1}_{J^1Y})$$

is called the virtual Noether invariant associated with D.

We shall say that D admits a global Noether invariant $\omega_D \in H^0(J^1Y, \Omega^{n-1}_{J^1Y})$ if its image in $H^0(J^1Y, \Omega^{n-1}_{J^1Y}/Z^{n-1}_{J^1Y})$ is $[\omega_D]$.

For symmetries with a global Noether invariant we have the Noether theorem

Theorem 2. *If D is an infinitesimal symmetry with a global Noether invariant ω_D, then for every critical section s one has*

$$d(\omega_D)|_{j^1s} = 0,$$

thus $(j^1s)^*\omega_D \in Z^{n-1}(X)$.

As a consequence of the abstract De Rham theorem we can prove

Proposition 6. *Let $D \in \mathfrak{X}(Y)$ be an infinitesimal symmetry of the local variational problem* $[\{\mathfrak{U}, \mathcal{L}\}]$

1. *The virtual Noether invariant $[\omega_D]$ associated with D gets identified with*

$$\xi_D = -i_{\bar{D}}\Sigma_{\mathcal{L}} \in H^0(J^1Y, Z^n_{J^1Y}) = Z^n_{DR}(J^1Y).$$

2. *D admits global Noether invariants if and only if the cohomology class*

$$[i_{\bar{D}}\Sigma_{\mathcal{L}}] \in H^n(J^1Y, \mathbb{R})$$

vanishes. That is, if and only if there exists $\omega_D \in \Omega^{n-1}(J^1Y)$ such that

$$i_{\bar{D}}\Sigma_{\mathcal{L}} = -d\omega_D.$$

In that case, the set of global Noether invariants associated with D is the class

$$\zeta_D = \omega_D + Z^{n-1}_{DR}(J^1Y)$$

that is,

$$\zeta_D \in \Omega^{n-1}(J^1Y)/Z^{n-1}_{DR}(J^1Y) \simeq B^n_{DR}(J^1Y).$$

Remark 5. Therefore the obstruction to the existence of global Noether invariants lives in $H^n(Y, \mathbb{R})$.

We can give a cohomological characterization of the infinitesimal symmetries and of those that admit global Noether invariants

Theorem 3. *Given a vector field $D \in \mathfrak{X}(Y)$ one has*

1. *D is an infinitesimal symmetry of the local variational problem $[\{\mathfrak{U}, \mathcal{L}\}]$ if and only if*

$$i_{\bar{D}}\Sigma_{\mathcal{L}} \in H^0(J^1Y, Z^n_{J^1Y}) = Z^n_{DR}(J^1Y).$$

Equivalently, D is an infinitesimal symmetry if and only if

$$L_{\bar{D}}\Sigma_{\mathcal{L}} = 0.$$

2. *D is an infinitesimal symmetry that admits global Noether invariants if and only if the cohomology class $[i_{\bar{D}}\Sigma_{\mathcal{L}}] \in H^n(J^1Y, \mathbb{R})$ vanishes.*

Proof. The direct implication of the first section is a consequence of the preceding proposition. Therefore, we have to see that $di_{\bar{D}}\Sigma_{\mathcal{L}} = 0$ implies that D is an infinitesimal symmetry. Without changing the cohomology class $[\{\mathfrak{U}, \mathcal{L}\}]$ we can take a good covering $\mathfrak{U}$ of Y, in a way such that U_α be contractible. This is not a real restriction since on every (paracompact) manifold we can construct a good covering starting from the geodesically convex neighborhoods associated with a Riemannian metric, see [3]. It is clear now that, for every $\alpha \in I$, there exists ω^α_D such that $-d\omega^\alpha_D = (i_{\bar{D}}\Sigma_{\mathcal{L}})|_{U_\alpha}$ which finishes the proof of the first part.

The second part of the first section is immediate from the first just by taking into account that $\Sigma_{\mathcal{L}}$ is closed. The second section is a consequence of the first and the preceding proposition. □

Theorem 4. *One has that*

1. *The set $\bar{\mathcal{D}}$ of infinitesimal symmetries of a local variational problem is a Lie subalgebra of $\mathfrak{X}(Y)$.*
2. *The set $\bar{\mathcal{D}}_0$ of infinitesimal symmetries which admit a global Noether invariant, is a Lie subalgebra of $\bar{\mathcal{D}}$ and*

$$[\bar{\mathcal{D}}, \bar{\mathcal{D}}] \subset \mathcal{D}_0.$$

Hence $\bar{\mathcal{D}}_0$ is an ideal of $\bar{\mathcal{D}}$. Moreover, there is a natural inclusion of the Lie algebra

$$\bar{\mathcal{D}}/\bar{\mathcal{D}}_0 \subset H^n(J^1Y, \mathbb{R}) \simeq H^n(Y, \mathbb{R}).$$

Corollary 1. *Let D_1 and D_2 be two infinitesimal symmetries belonging to $\bar{\mathcal{D}}_0$, then*

$$\omega_{[D_1, D_2]} = i_{\bar{D}_2} i_{\bar{D}_1}\Sigma_{\mathcal{L}}$$

is a global Noether invariant for $[D_1, D_2]$.

4. Poisson algebras

We now begin the study of the Poisson algebras that are naturally associated with a local variational problem of restricted class.

Proposition 7. *Let $\overline{\mathcal{J}_v} \subset Z^n_{DR}(J^1Y)$ be the set of virtual Noether invariants of a local variational problem. The $\mathbb{R}$-linear map*

$$\xi : \bar{\mathcal{D}} \to \overline{\mathcal{J}_v},$$

which associates to every infinitesimal symmetry D its virtual Noether invariant $\xi_D = -i_{\bar{D}}\Sigma_{\mathcal{L}}$, has $\overline{\mathrm{rad}_0 \Sigma_{\mathcal{L}}}$ as kernel; that is, the vector fields $D \in \mathfrak{X}(Y)$ such that $\bar{D}$ belongs to the nullity of $\Sigma_{\mathcal{L}}$.

Moreover, $\overline{\mathrm{rad}_0 \Sigma_{\mathcal{L}}}$ is an ideal of $\bar{\mathcal{D}}$.

Proof. The first statement is obvious.

For the second it is enough to take into account that $\xi_{[D_1,D_2]} = d(i_{\bar{D}_2} i_{\bar{D}_1} \Sigma_{\mathcal{L}})$ and thus, if $\bar{D}_1 \in \overline{\mathrm{rad}_0 \Sigma_{\mathcal{L}}}$ then $\xi_{[D_1,D_2]} = 0$. □

Therefore, the Lie algebra structure of $\bar{\mathcal{D}}$ passes to the quotient $\bar{\mathcal{D}}/\overline{\mathrm{rad}_0 \Sigma_{\mathcal{L}}} \simeq \overline{\mathcal{J}_v}$. Hence, we can give the following

Definition 7. We define on $\overline{\mathcal{J}_v}$ the following Lie algebra structure

$$\{\xi_{D_1}, \xi_{D_2}\}_v \equiv \xi_{[D_1,D_2]} = -i_{[\bar{D}_1,\bar{D}_2]}\Sigma_{\mathcal{L}} = d(i_{\bar{D}_2} i_{\bar{D}_1} \Sigma_{\mathcal{L}}).$$

The Lie algebra $(\overline{\mathcal{J}_v}, \{\cdot,\cdot\}_v)$ is called the virtual Poisson algebra of the local variational problem under consideration.

We can proceed in a similar way with the global Noether invariants

Proposition 8. *Let $\overline{\mathcal{J}} \subset \Omega^{n-1}(J^1Y)/Z^{n-1}_{DR}(J^1Y)$ be the set of global Noether invariants of a local variational problem. The $\mathbb{R}$-linear map*

$$\zeta : \bar{\mathcal{D}}_0 \to \overline{\mathcal{J}},$$

which associates to every infinitesimal symmetry $D \in \bar{\mathcal{D}}_0$ its global Noether invariant ζ_D, has $\overline{\mathrm{rad}_0 \Sigma_{\mathcal{L}}}$ as kernel.

Moreover, $\overline{\mathrm{rad}_0 \Sigma_{\mathcal{L}}}$ is an ideal of $\bar{\mathcal{D}}_0$.

Hence, in this case the Lie algebra structure also goes to the quotient and we have

Definition 8. We define on $\overline{\mathcal{J}}$ the following Lie algebra structure

$$\{\zeta_{D_1}, \zeta_{D_2}\} \equiv \zeta_{[D_1,D_2]} = [i_{\bar{D}_2} i_{\bar{D}_1} \Sigma_{\mathcal{L}}].$$

The Lie algebra $(\overline{\mathcal{J}}, \{\cdot,\cdot\})$ is called the Poisson algebra of the local variational problem under consideration.

There exists the following relationship between these Poisson algebras:

Proposition 9. *The differential $d : \overline{\mathcal{J}} \to \overline{\mathcal{J}}_v$ establishes an injective Lie algebras morphism, whose image is an ideal. Thus, for every $D_1,\ D_2 \in \bar{\mathcal{D}}_0$, one has*

$$d\{\zeta_{D_1}, \zeta_{D_2}\} = \{d\zeta_{D_1}, d\zeta_{D_2}\}_v.$$

Proof. Taking into account the exact De Rham cohomology sequence, one has that the differential gives us an isomorphism

$$\begin{aligned} d : \Omega^{n-1}(J^1Y)/Z^{n-1}_{DR}(J^1Y) \xrightarrow{\sim} & \, d\Omega^{n-1}(J^1Y) \\ & = B^n_{DR}(J^1Y) \subset Z^n_{DR}(J^1Y) \end{aligned}$$

and this proves the injectivity.

As a consequence of Corollary 1, d is a Lie algebra morphism. Finally, the image is an ideal due to the second section of Theorem 4. □

Next we introduce the Lie algebra obtained as the quotient of the previously defined Poisson algebras. This Lie algebra measures the deficiency that exists between the global Noether invariants and the virtual ones. We will see that this deficiency coincides with the one existing between the corresponding Lie algebras of infinitesimal symmetries. This fact explains the name that we shall give to the quotient of the Poisson algebras. It is remarkable that this algebra gets identified with a subspace of the cohomology and thus, it is always finite-dimensional.

Definition 9. We shall denote by $(\overline{\mathcal{J}}_{ev}, \{\cdot\,,\cdot\}_{ev})$ the quotient Lie algebra $\mathcal{J}_v/\mathcal{J}$ and we shall call it the strictly virtual Poisson algebra.

Proposition 10. *One has*

1. *The strictly virtual Poisson algebra gets identified in a natural way with a subspace* $\overline{\mathcal{J}}_{ev} \subset H^n(J^1Y, \mathbb{R}) \simeq H^n(Y, \mathbb{R})$.
2. *The strictly virtual Poisson algebra is isomorphic, as a Lie algebra, with the quotient Lie algebra* $\overline{\mathcal{D}}_{ev} = \overline{\mathcal{D}}/\overline{\mathcal{D}}_0$.

Proof. For the first part it is enough to take into account the following commutative diagram of Lie algebras, whose rows and columns are exact

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \overline{\mathcal{J}} & \longrightarrow & B^n_{DR}(J^1Y) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \overline{\mathcal{J}}_v & \longrightarrow & Z^n_{DR}(J^1Y) \\ & & \downarrow & & \downarrow \\ & & \overline{\mathcal{J}}_{ev} & & H^n(J^1Y, \mathbb{R}) \\ & & \downarrow & & \downarrow \\ & & 0 & & 0. \end{array}$$

We now deduce the natural inclusion $\overline{\mathcal{J}}_{ev} \subset H^n(J^1Y, \mathbb{R})$.

For the second part we have the commutative diagram, with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{\mathrm{rad}_0 \Sigma_{\mathcal{L}}} & \longrightarrow & \overline{\mathcal{D}}_0 & \longrightarrow & \overline{\mathcal{J}} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{\mathrm{rad}_0 \Sigma_{\mathcal{L}}} & \longrightarrow & \overline{\mathcal{D}} & \longrightarrow & \overline{\mathcal{J}}_v & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \overline{\mathcal{D}}/\overline{\mathcal{D}}_0 & & \overline{\mathcal{J}}_{ev} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Hence, we have the isomorphism $\overline{\mathcal{D}}/\overline{\mathcal{D}}_0 \simeq \overline{\mathcal{J}}_{ev}$. □

As a consequence of this proposition, we have the following:

Corollary 2. *Let us suppose that* $H^n(Y, \mathbb{R}) = 0$ *then*

1. *The strictly virtual Poisson algebra vanishes*

$$\overline{\mathcal{J}}_{ev} = 0.$$

2. *Every infinitesimal symmetry admits global Noether invariants; that is*

$$\overline{\mathcal{D}} = \overline{\mathcal{D}}_0.$$

Among all the infinitesimal symmetries a special role is played by the ones that are π-projectable, we thus have the set $\mathcal{D} = \{D \in \overline{\mathcal{D}} : D \text{ is } \pi \text{ projectable}\}$.

One can easily prove the following:

Proposition 11. *$\mathcal{D}$ is a Lie algebra. The subset $\mathcal{D}^\vee$ of vertical vector fields is an ideal of $\mathcal{D}$ and the sequence of Lie algebras*

$$0 \longrightarrow \mathcal{D}^\vee \longrightarrow \mathcal{D} \xrightarrow{\pi} \pi\mathcal{D} \longrightarrow 0$$

is exact.

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