Differential Geometry and Its Applications
Proc. Conf., Opava (Czech Republic), August 27–31, 2001
Silesian University, Opava, 2001, 401–413

Gauge natural field theories and applications to conservation laws[1]

L. Fatibene, M. Francaviglia and M. Raiteri

Abstract. We review the notion of a gauge natural theory and we show in particular its role in implementing the very basic principles of any reasonable fundamental physical interaction. Conservation laws are investigated in this general framework by means of a skilful application of Nöther theorem together with the theory of superpotentials. Wess–Zumino model interacting with a gravitational fields is considered as an application of the gauge-natural framework.

Keywords. Gauge natural theories, conservation laws, Noether's theorem, supersymmetries.

MS classification. 70S15, 35L65, 58A32.

1. Introduction

The category of gauge natural bundles has been recently introduced (see [4, 20, 21]). Since then, gauge natural bundles have been recognized to be essential to describe the field theories that are relevant to the Physics of fundamental interactions (see [5, 7, 8, 9, 10]). We shall here review the subject and discuss how deep the link between classical field theory and gauge natural bundles goes. Our aim is to support the claim that the gauge natural framework is a very natural extension of the axioms of General Relativity; furthermore such an extension is really based on empirical evidences. Because of these we believe that the gauge natural framework for field theory will remain as a fundamental ingredient even in future developments, at least as long as field theory itself will remain as the cornerstone onto which our understanding of our physical world is based.

---

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

The contents of this paper are organized as follows. In Section 2 we shall review the basic definitions of natural and gauge natural field theories. A particular emphasis is devoted to clarify the differences regarding the fundamental transformations that are allowed in the two cases, since such differences will be essential for the physical interpretation.

In Section 3 we shall give a short though complete review of the results about conservation laws. The relation between Spencer cohomology and the standard manipulation of Bianchi's identities will be clarified leading to the proof of the theorem about existence of superpotentials in any gauge natural theory.

Section 4 will be devoted to develop a simple application to supersymmetry. The Wess–Zumino model presented here is one of the simplest supersymmetric models and it is currently under further investigation. As a first step we shall show that one can coherently investigate supersymmetries within the gauge natural framework; the problem of whether supersymmetries themselves can be regarded as "gauge natural transformations" is currently under investigation. The Wess–Zumino model has been chosen as an example for gauge natural theories also because it deals both with the frame formulation of General Relativity and with (anticommuting) spinor fields. The discussions of both these subsystems clarifies most of the issues introduced during our preceding discussion about gauge natural theories.

2. Natural and gauge natural field theories

Let us start by reviewing natural and gauge natural theories. Natural theories have been recognized to be the most appropriate framework to formulate the principle of general covariance which characterizes, in particular, General Relativity. A *natural field theory* is a triple $(\mathcal{B}, L, \Gamma)$ where:

(a) $\mathcal{B} = (B, M, \pi; F)$ is a natural bundle, called the *configuration bundle*, over the *base manifold* M (usually *spacetime*). Being $\mathcal{B}$ natural means that there exists a functorial representation of $\mathrm{Diff}(M)$ over the bundle $\mathcal{B}$; in other words, there is a way of functorially associating to any (local) diffeomorphism φ of M a (local) bundle morphism $\boldsymbol{\Phi} = (\phi, \varphi)$ of $\mathcal{B}$.

(b) L is a k-order Lagrangian covariant with respect to any diffeomorphism of M represented on $\mathcal{B}$ by means of the natural action of (a) above (see below for details). For now regard the Lagrangian either as a horizontal form over the k-order jet bundle $J^k\mathcal{B}$ or as a bundle morphism $L : J^k\mathcal{B} \to A_m(M)$ with values into the bundle $A_m(M)$ of m-forms over the base manifold M (m being its dimension); other global frameworks for variational calculus would also be allowed (e.g., sheaf theoretic or others);

(c) Γ, which is called *dynamical connection*, is a bundle morphism between (some jet prolongation of) the configuration bundle and the bundle $\mathrm{Con}(M)$ of connections over M. Sometimes this last item is not mentioned explicitly in the definition of a natural field theory (see [17]); we choose to add it in order to make canonical a number of relevant constructions (among others, conservation laws, higher order Poincaré–Cartan forms, superpotentials). The morphism Γ provides a way for assigning to each section σ of the configuration bundle $\mathcal{B}$ a connection $\Gamma(\sigma)$ over M. In other words, we assume that the configuration bundle is such that one can suitably construct a connection out of the fields. Notice that this axiom holds for all the theories considered in fundamental physics.

Sections of $\mathcal{B}$ are called *configurations* or *fields*. Different sections describe different configurations of the physical system under consideration. Examples are tensor fields, including metrics (possibly of a fixed signature) and connections over M. Base diffeomorphisms act on configurations by pull-back (recall that tensor fields as well as connections over M can be pulled-back along base diffeomorphisms).

Axiom (b) simply says that diffeomorphisms are symmetries of the Lagrangian; this is the active version of the *principle of general covariance* simply because diffeomorphisms are the active version of coordinate changes. See next section for a discussion about the physical content of this principle.

Since natural bundles are built functorially out of the base manifold M, one could say that they actually make explicit some information which is already encoded in M; in fact, sections of a natural bundle $\mathcal{B}$ are also called *natural objects over M*. As an example sections of natural bundle, we mention the tangent bundle $T(M)$, the points of which, i.e., tangent vectors, are equivalence classes of curves over M.

Hence a natural theory actually describes the dynamics of some structure over the base *spacetime* manifold.

On the contrary, in gauge natural theories one starts from a principal bundle

$$\mathcal{P} = (P, M, p; G),$$

called the *structure bundle*, a prototype of which is any principal G-bundle of a pure G-gauge theory (G being some suitable Lie group of gauge symmetries). Once this change of starting point has been accepted then a *gauge natural theory* is defined to be a quadruple $(\mathcal{P}, \mathcal{B}, L, \omega)$ where:

(i) $\mathcal{P} = (P, M, p; G)$ is the structure bundle. The Lie group G is called the *gauge group* of the theory;

(ii) $\mathcal{B}$ is a gauge natural bundle (called the *configuration bundle*) associated to the structure bundle, which means that one has a canonical functorial action of the structure bundle automorphisms on the configuration bundle itself. Hence a bundle morphism $\Phi_{\mathcal{B}} \in \mathrm{Aut}(\mathcal{B})$ is associated to each automorphism $\Phi \in \mathrm{Aut}(\mathcal{P})$ on the structure bundle. Automorphisms of $\mathcal{P}$ are called *gauge transformations* and vertical gauge transformations are called *pure gauge transformations*. In gauge natural theories gauge transformations act on fields.

(iii) L is a k-order Lagrangian covariant with respect to all gauge transformations represented on the configuration bundle $\mathcal{B}$.

(iv) ω, which is called *(fibered) dynamical connection*, is a bundle morphism between (some jet prolongation of) the configuration bundle and the fibered product $\mathrm{Con}(M) \times_M \mathrm{Con}(\mathcal{P})$ of "fibered connections" over $\mathcal{P}$. The morphism ω associates to each configuration σ a connection $\Gamma(\sigma)$ over M and a principal connection $\omega(\sigma)$ over $\mathcal{P}$ (see [16]).

The main technical difference between natural and gauge natural theories is that diffeomorphisms are completely replaced by gauge transformations. In gauge natural theories spacetime diffeomorphisms do not act at all on fields, since the only action one can define in general is that of gauge transformations. This is due to the fact that although pure gauge transformations are canonically embedded into the group of generalized gauge transformations, there is no canonical "horizontal" complement to be identified with $\mathrm{Diff}(M)$. Consequently, in natural theories the Lie derivative of a configuration σ with respect to any spacetime vector field ξ is defined by the *fully covariant* prescription:

$$\pounds_\xi \sigma = T\sigma(\xi) - \hat{\xi} \circ \sigma \tag{1}$$

where $\hat{\xi}$ is the vector field naturally and uniquely induced on the configuration bundle by the functorial lifting action. On the contrary, in gauge natural theories the Lie derivative of a configuration σ with respect to a right-invariant vector field Ξ of the structure bundle $\mathcal{P}$ is defined as follows

$$\pounds_\Xi \sigma = T\sigma(\xi) - \widehat{\Xi} \circ \sigma \tag{2}$$

where $\widehat{\Xi}$ is the vector field on the configuration bundle gauge-naturally induced by Ξ and ξ is its projection onto the spacetime M.

Let us stress that in gauge natural theories Lie derivatives with respect to generic spacetime vector fields are undefined. This is a further source of difference between natural and gauge natural theories. "Horizontal" symmetries, in fact, are generally associated to physically relevant conservation laws, such as energy, momentum and angular momentum. The definition of such quantities is almost trivial in natural theories; on the contrary, in gauge natural theories pure gauge transformations are easily associated to gauge charges (e.g., the electric charge in electromagnetism), while the absence of "horizontal" gauge transformations is a problem to be solved to appropriately define energy, momentum and angular momentum. For this reason, in gauge natural theories the dynamical connection plays an extra role in determining horizontal infinitesimal symmetries as the gauge generators Ξ which are horizontal with respect to the principal connection ω. Of course such a concept depends on the configuration σ which is necessary to define $\omega(\sigma)$.

3. Conservation laws

It is now time to be more specific about the language to be used when one wants to talk about variational principles and conservation laws. There are many options available; Lagrangian forms (see [17]), Lepagean forms (see [22]), Poincaré–Cartan forms (see [15, 18, 24]), bundle morphisms (see [8]), variational sequences (see [23, 26]) and even others. These languages have been investigated and compared for a long time and, except for matters of convenience for some particular issue, they can be considered as being practically equivalent (any choice among them being a matter of taste).

We define here the Lagrangian to be a bundle morphism of the type $L : J^k\mathcal{B} \to A_m(M)$ which, once evaluated onto the prolongation $j^k\sigma$ of a configuration, produces an m-form $L(j^k\sigma)$ over the m-dimensional spacetime M. This a special case of a particular class of bundle morphisms, called *variational morphisms*, which are defined as follows:

Definition 3.1. Let $\mathcal{E} = (E, M, \tau; V)$ be a vector bundle. A *variational morphism* relative to $\mathcal{E}$ is a bundle morphism of the form

$$\mathbb{R} : J^k\mathcal{B} \to (J^h\mathcal{E})^* \otimes A_{m-n}(M). \tag{3}$$

The integers (k, h, n) are called the *order*, the *rank* and the *codegree*, respectively.

The Lagrangian is a k-order variational morphisms relative to the trivial bundle $\mathcal{E} = (M \times \{0\}, M, \tau; \{0\})$ of rank 0. Of course, any variational morphism $\mathbb{R}$ can be evaluated on a suitable jet prolongation of a section X of $\mathcal{E}$; such an evaluation will be denoted as

$$\langle\mathbb{R}|j^h X\rangle : J^k\mathcal{B} \to A_{m-n}(M). \tag{4}$$

Hence any variational morphism $\mathbb{R}$ can be expanded as a linear combination of a section of $\mathcal{E}$ together with its symmetrized covariant derivatives up to order h, the covariant derivative being defined with respect to any fixed fibered connection (see [16]). We choose the basis of covariant derivatives so that each term of the linear combination

$$\begin{aligned}\langle\mathbb{R}|j^h X\rangle = [&p_a^{\lambda_1\cdots\lambda_n} X^a + p_a^{\lambda_1\cdots\lambda_n\mu}\nabla_\mu X^a + \cdots \\ &+ p_a^{\lambda_1\cdots\lambda_n\mu_1\cdots\mu_h}\nabla_{\mu_1\cdots\mu_h} X^a]\,\mathbf{ds}_{\lambda_1\cdots\lambda_n}\end{aligned} \tag{5}$$

has a global meaning ($\mathbf{ds}_{\lambda_1\cdots\lambda_n}$ denotes a (local) natural basis of $(m-n)$-forms). The coefficients

$$p_a^{\lambda_1\cdots\lambda_n\,\mu_1\cdots\mu_h}$$

are called *momenta* (relative to the fibered connection). They are antisymmetric with respect to the indices $[\lambda_1\cdots\lambda_n]$ and symmetric with respect to $(\mu_1\cdots\mu_h)$. Hence the variational morphisms can be identified with chains in Spencer cohomology (see [19]).

A variational morphism is *reduced* (with respect to the fixed fibered connection) if its coefficients satisfy the following condition

$$p_a^{[\lambda_1\cdots\lambda_n\mu_1]\,\mu_2\cdots\mu_l} = 0, \qquad \forall 0 < l < h. \tag{6}$$

Variational morphisms of codegree $n = 0$ are automatically reduced. One can prove that the notion of reduction depends on the fibered connection for high enough rank ($h \geq 2$).

We can prove general lemmas about splitting of variational morphisms. Let us denote by Div the *divergence operator* which is the "transcription" of exterior differential on forms in terms of the variational morphisms; it is defined by

$$\mathrm{Div}\big(\langle\mathbb{R}|j^h X\rangle\big) \circ j^{k+1}\sigma = \mathrm{d}\big(\langle\mathbb{R}|j^h X\rangle \circ j^k\sigma\big). \tag{7}$$

Then we have the following

Lemma 3.1. *For any fixed fibered connection each variational morphism* $\mathbb{R} : J^k\mathcal{B} \to J^h\mathcal{E}^* \otimes A_{m-n}(M)$ *can be canonically and algorithmically splitted as*

$$\langle \mathbb{R} | j^h X\rangle = \langle \mathbb{T} | j^{h_1} X\rangle + \mathrm{Div}\big(\langle \mathbb{S} | j^{h_2} X\rangle\big) \tag{8}$$

where $\mathbb{T} \equiv \mathbb{T}(\mathbb{R})$ *and* $\mathbb{S} \equiv \mathbb{S}(\mathbb{R})$ *are reduced variational morphisms of the type as below. If* $\mathbb{R}$ *is* 0*-codegree, then one has*

$$\begin{aligned} &\mathbb{T} : J^{k+h}\mathcal{B} \to \mathcal{E}^* \otimes A_m(M) \\ &\mathbb{S} : J^{k+h-1}\mathcal{B} \to (J^{h-1}\mathcal{E})^* \otimes A_{m-1}(M) \end{aligned} \tag{9}$$

i.e., $h_1 = 0$ *and* $h_2 = h - 1$*. If* $\mathbb{R}$ *is* n*-codegree* $n > 0$*, then one has*

$$\begin{aligned} &\mathbb{T} : J^{k+h}\mathcal{B} \to (J^h\mathcal{E})^* \otimes A_{m-n}(M) \\ &\mathbb{S} : J^{k+h-1}\mathcal{B} \to (J^{h-1}\mathcal{E})^* \otimes A_{m-n-1}(M) \end{aligned} \tag{10}$$

i.e., $h_1 = h$ *and* $h_2 = h - 1$*.*

The proof (see [12] and [13]) is completely algebraic and based on covariant integration by parts. For each connection the splitting (8) is unique. Once the fibered connection is fixed then $\mathbb{T}$ and $\mathbb{S}$ are uniquely determined by the requirement of being both reduced. Of course, depending on rank, when the fibered connection is changed then the splitting may change. In higher-order natural and gauge natural theories the canonical splitting is usually done with respect to the dynamical connection.

The variation of the Lagrangian can be directly identified with a variational splitting

$$\delta L : J^k\mathcal{B} \to V^*(J^k\mathcal{B}) \otimes A_m(M) \tag{11}$$

defined by

$$\langle \delta L | j^k X\rangle = \frac{\mathrm{d}}{\mathrm{ds}}(L \circ J^k\Phi_s)\bigg|_{s=0}. \tag{12}$$

Here $V^*(J^k\mathcal{B})$ is the dual of the vertical bundle $V(J^k\mathcal{B})$, the canonical isomorphism $V(J^k\mathcal{B}) \sim J^kV(\mathcal{B})$ has been used and $\Phi_s : \mathcal{B} \to \mathcal{B}$ is the flow of the vertical vector field X.

The splitting lemma defines two *global* variational morphisms

$$\begin{cases} \mathbb{E} \equiv \mathbb{T}(\delta L) : J^{2k}\mathcal{B} \to V^*(\mathcal{B}) \otimes A_m(M) \\ \mathbb{F} \equiv \mathbb{S}(\delta L) : J^{2k-1}\mathcal{B} \to V^*(J^{h-1}\mathcal{B}) \otimes A_{m-1}(M) \end{cases} \tag{13}$$

which are called *Euler–Lagrange morphism* and *Poincaré–Cartan morphism* of L, respectively. They are such that the so-called *first variation formula* holds:

$$\langle \delta L | j^k X\rangle = \langle \mathbb{E} | X\rangle + \mathrm{Div}\big(\langle \mathbb{F} | j^{k-1} X\rangle\big). \tag{14}$$

In any gauge natural theory, each right-invariant vector field Ξ over the structure bundle $\mathcal{P}$ projecting over a spacetime vector field ξ is required to be an infinitesimal symmetry of the Lagrangian, i.e., the following is assumed to hold:

(15) $$\langle \delta L \,|\, j^k \pounds_\Xi \sigma \rangle = \pounds_\xi(L) \equiv \mathrm{Div}(i_\xi L).$$

Notice that here the Lie derivative of configurations σ with respect to Ξ is involved, so that the gauge natural structure is essential to the following construction.

Nöther theorem is almost trivial using the notation introduced above. We can define the *Nöther current* and the *work form* as

(16) $$\begin{cases} \mathcal{E}(\Xi) = \langle \mathbb{F} \,|\, j^{h-1} \pounds_\Xi \sigma \rangle - i_\xi L \\ \mathcal{W}(\Xi) = -\langle \mathbb{E} | \pounds_\Xi \sigma \rangle \end{cases}$$

and they produce a *weak conservation law*

(17) $$\mathrm{Div}\mathcal{E}(\Xi) = \mathcal{W}(\Xi).$$

As a consequence the Nöther current is closed along critical sections.

Finally, we are ready to the last step. Both the Nöther current and the work form may be regarded as variational morphisms. In fact, Ξ are general sections of a gauge natural vector bundle $\mathcal{C}$ associated to $\mathcal{P}$ and Lie derivatives are linear in Ξ and its symmetrized covariant derivatives up to some finite order h. Thence we can define

(18) $$\begin{cases} \mathcal{E} = J^{2k-1}\mathcal{B} \to (J^{k+h-1}\mathcal{C})^* \otimes A_{m-1}(M) \\ \mathcal{W} = J^{2k}\mathcal{B} \to (J^h \mathcal{C})^* \otimes A_m(M) \end{cases}$$

such that

(19) $$\begin{cases} \langle \mathcal{E} \,|\, j^{k+h-1}\Xi \rangle = \mathcal{E}(\Xi) \\ \langle \mathcal{W} \,|\, j^h \Xi \rangle = \mathcal{W}(\Xi). \end{cases}$$

Once again we urge the reader to notice that the gauge natural structure is here essential; the set of infinitesimal symmetries has to be identified with *general* sections of a suitable vector bundle.

By splitting the work form one obtains a unique decomposition

(20) $$\langle \mathcal{W} \,|\, j^h \Xi \rangle = \langle B \,|\, \Xi \rangle + \mathrm{Div}\langle \tilde{\mathcal{E}} \,|\, j^{h-1} \Xi \rangle.$$

Since $\mathcal{W}$ is a pure divergence then B is identically zero and it is called *generalized Bianchi's identities*. The boundary part $\tilde{\mathcal{E}}$ is called *reduced current* and it vanishes along critical sections. By further splitting the Nöther current $\mathcal{E}$ we obtain

(21) $$\langle \mathcal{E} \,|\, j^{k+h-1}\Xi \rangle = \langle \hat{\mathcal{E}} \,|\, j^{k+h-1}\Xi \rangle + \mathrm{Div}\langle \mathcal{U} \,|\, j^{k+h-2}\Xi \rangle.$$

Once again we use weak conservation and obtain that $\hat{\mathcal{E}} = \tilde{\mathcal{E}}$. The (essentially unique up to divergences) morphism $\mathcal{U}$ is called the *superpotential*. Equation (21) proves that the Nöther current is not only closed along critical sections but even exact. We stress that this is an extremely general result, holding in *any* gauge natural theory and for *any* Nöther current arising from a flow of generalized gauge transformations (see [12]).

4. Wess–Zumino model

Wess–Zumino model (see [3, 14, 25, 27]) is one of the simplest examples with a supersymmetry in field theory. We here present some basic constructions to introduce supersymmetries in a fashion which is compatible with classical field theories. Of course we stress that this is not the end of the story. Supersymmetries are the starting point for quantum field theories (often involving gravitation). Here we want simply to present the classical roots of supersymmetries. We believe that understanding these classical roots is important to clarify the origin of supersymmetries (which often have developed in a completely different framework that was defined along the way). On the other hand we are aware that lots of important issues remain still open or even simply untouched here.

We first need a number of generalizations of the standard structures defined above.

4.1. Generalized vector fields

Generalized (vertical) vector fields of order k are sections X of the following pull-back bundle

(22)
$$\begin{array}{ccc} (\pi^k_0)^* V(\mathcal{B}) & \overset{i}{\dashrightarrow} & V(\mathcal{B}) \\ {\scriptstyle X}\uparrow\ \downarrow & & \downarrow \\ J^k\mathcal{B} & \xrightarrow{\ \pi^k_0\ } & \mathcal{B}. \end{array}$$

Locally, a generalized vector field is expressed as

(23) $$X = X^i(x^\mu, y^i, y^i_\mu, \dots, y^i_{\mu_1\cdots\mu_k})\partial_i$$

The composition $X_\sigma = i \circ X \circ j^k\sigma$ is a section of the fibration $V(\mathcal{B}) \to M$, i.e., a vector field over the section σ. Accordingly, we can define the Lie dragging and hence the Lie derivative of sections of $\mathcal{B}$ along generalized vector fields. The notion of symmetries and Nöther theorem easily extends to generalized vector fields (see [1, 14]).

4.2. Spin frames

Let us denote by $\mathrm{Spin}(r, s)$ the spin group of signature (r, s), where $r + s = m = \dim(M)$, and by $\mathrm{GL}(m)$ the general linear group. We denote by $\ell : \mathrm{Spin}(r, s) \to \mathrm{SO}(r, s)$ the covering map exhibiting the spin group as a double covering of the corresponding orthogonal group. Moreover, let us denote by $L(M)$ the principal bundle of linear frames on M, which has $\mathrm{GL}(m)$ as a structure group.

Spinors can be defined on a general spin-manifold M (see [2]) which shall be here assumed to admit global metrics of the given signature. The first step is to regard spin structures on M as dynamical variables by introducing as follows the so-called *spin frames* (see [7, 10, 11]). We fix a principal bundle Σ with $\mathrm{Spin}(r, s)$ as structure group; spin frames are principal morphisms $e : \Sigma \to L(M)$. We consider the action on $\mathrm{GL}(m)$

(24) $$\lambda : \mathrm{GL}(m) \times \mathrm{Spin}(r,s) \times \mathrm{GL}(m) \to \mathrm{GL}(m) : (J, S, e) \mapsto \ell(S) \cdot e \cdot J^{-1}$$

and we define the associated bundle $\Sigma_\lambda = (L(M) \times_M \Sigma) \times_\lambda \mathrm{GL}(m)$; it has fibered coordinates (x^μ, e^a_μ) and it is thence locally analogue to the frame bundle. For this reason spin frames are often identified in literature with standard frames (*vielbein*). We stress however that these similarities have only a local nature since spin frames and ordinary frames have completely different behaviours with respect to Lie-dragging and covariant derivative (see [7]). The bundle Σ_λ is by construction a gauge natural bundle and its sections are in one-to-one correspondence with spin frames.

Spin frames can be defined on each spin manifold (even on non-parallelizable ones) and each spin frame induces a metric by $g_{\mu\nu} = e^a_\mu \, \eta_{ab} \, e^b_\nu$, where η_{ab} is the standard diagonal matrix with signature (r, s).

4.3. Anticommuting spinors

Supersymmetries strongly rely on anticommuting spinors. Naively speaking, if ψ^i denote the components of a spinor ψ then quadratic forms such as $\psi^i \varphi^j$ are assumed to be antisymmetric in the exchange of the spinors, i.e., $\psi^i \varphi^j = -\varphi^j \psi^i$, in view of a path integral quantization of the field theory (according to Fermi–Dirac quantization rules for Fermions). Mathematically, this behaviour is implemented by assuming that each component ψ^i takes its value in the odd part of the (complexified) exterior algebra $\Lambda_-(W) \otimes \mathbb{C}$ of some suitable vector space W. Let us denote by $V = [\Lambda_-(W) \otimes \mathbb{C}]^n$ the space so generated; since $\Lambda_-(W) \otimes \mathbb{C}$ is a complex vector space of dimension $2^{\dim(W)-1}$, V is a complex vector space too and we are entitled to use anticommuting coordinates ψ^i in V. Each ψ^i is actually a block of $2^{\dim(W)-1}$ (complex) ordinary coordinates.

The spin group $\mathrm{Spin}(r, s)$ acts as follows on spinors by block matrix multiplication on anticommuting coordinates:

(25) $$\rho : \mathrm{Spin}(r, s) \times V \to V : (S, \psi) \mapsto S^i_{\ j} \, \psi^j.$$

Thence we can define the associated bundle $\Sigma_\rho : \Sigma \times_\rho V$ the sections of which are called *anticommuting spinors*.

Choosing the matrix representation of the spin group is equivalent to fix a set of Dirac matrices γ_a $(a = 1, \ldots, m)$ such that $\gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab}$. The charge conjugation operator is defined as the element C in the appropriate Clifford algebra such that

$$C \gamma_a C^{-1} = -{}^t\gamma_a,$$

where ${}^t\gamma_a$ is the transpose of γ_a. The *charge conjugated spinor* is defined as $\phi^c =$

$C({}^t\bar{\psi})$, where $\bar{\psi} = \psi^\dagger\gamma_0$ is the *adjoint spinor* and $\psi^\dagger$ is the transpose conjugate spinor. Spinors which are self charge conjugated ($\psi^c = \psi$) are called *Majorana spinors,* provided they exist (for example, this happens in 4-dimension Lorentzian spacetimes). In this case the spinor space V splits as $V = V_+ \oplus V_-$. The action of the spin group on V preserves this splitting and thence also the bundle Σ_ρ splits as $\Sigma_\rho = \Sigma_\rho^+ \oplus \Sigma_\rho^-$. Sections of Σ_ρ^+ are in one-to-one correspondence with Majorana spinors.

4.4. The model

We shall consider it on a general spin manifold M with a fixed spin bundle Σ. Fields are: a spin frame e^a_μ, an anticommuting Majorana spinor ψ and four scalar densities (A, B, C, D) of arbitrary weights $(\alpha, \beta, \gamma, \delta)$. Each scalar density, say A, is a section of a natural bundle $\mathcal{A}^\alpha = L(M) \times_{\lambda_\alpha} \mathbb{R}$ associated to $L(M)$ by means of the representation

$$\lambda_\alpha : \mathrm{GL}(m) \times \mathbb{R} \to \mathbb{R} : (J, A) \mapsto (\det J)^{-\alpha} A. \tag{26}$$

Fibered coordinates on $\mathcal{A}^\alpha$ are thence (x^μ, A).

The configuration bundle of the Wess–Zumino model is thence

$$\mathcal{C}_{WZ} = \Sigma_\lambda \times_M \Sigma_\rho^+ \times_M \mathcal{A}^\alpha \times_M \mathcal{A}^\beta \times_M \mathcal{A}^\gamma \times_M \mathcal{A}^\delta \tag{27}$$

which is a gauge natural bundle associated to the structure bundle Σ and it has fibered "coordinates" $(x^\mu, e^a_\mu, \psi^i, A, B, C, D)$. We recall that strictly speaking these are not coordinates since the ψ^i anticommute.

We consider the Lagrangian

$$\begin{aligned} L_{WZ} = {} & \frac{1}{2}\left[(\nabla_a A)(\nabla^a A)\, e^{2\alpha} + D^2\, e^{2\delta} + 2m AD\, e^{\alpha+\delta}\right] e\, \mathbf{ds} \\ & + \frac{1}{2}\left[(\nabla_a B)(\nabla^a B)\, e^{2\beta} + C^2\, e^{2\gamma} - 2m BC\, e^{\beta+\gamma}\right] e\, \mathbf{ds} \\ & - \bar{\psi}(i\gamma^a \nabla_a \psi + m\psi) e\, \mathbf{ds} \end{aligned} \tag{28}$$

where $\mathbf{ds}$ is the local volume element, e denotes the determinant of the spin frame e^a_μ, e^μ_a is the inverse of e^a_μ, the covariant derivatives of the scalar densities are defined as usual, e.g., $\nabla_a A = e^\mu_a(\mathrm{d}_\mu A - \alpha \Gamma^\lambda_{\lambda\mu} A)$; $\Gamma^\alpha_{\beta\mu}$ (Γ^{ab}_μ, respectively) is the Levi-Civita connection of the metric g (resp. the spin connection) uniquely induced by the spin frame e^a_μ and the covariant derivative of the spinor is

$$\nabla_a \psi = e^\mu_a\big(\mathrm{d}_\mu \psi + \tfrac{1}{8}\Gamma^{ab}_\mu[\gamma_a, \gamma_b]\psi\big). \tag{29}$$

Since $\mathcal{C}_{WZ}$ is gauge natural all automorphisms of Σ act canonically on $\mathcal{C}_{WZ}$. The Lagrangian (28) is covariant with respect to generalized gauge transformations and consequently the theory is gauge natural in the sense of [8]. One can thence canonically define conservation laws which admit superpotentials, as proved in general for any gauge natural theory (see [11]).

4.5. Supersymmetries

Supersymmetries can be now introduced as generalized vector fields. Let ϵ be a covariantly constant ($\nabla_\mu \epsilon = 0$) Majorana spinor. Let us consider the generalized vector field

$$X = \delta e^a_\mu \frac{\partial}{\partial e^a_\mu} + \delta\psi \frac{\partial}{\partial \phi} + \delta A \frac{\partial}{\partial A} + \delta B \frac{\partial}{\partial B} + \delta C \frac{\partial}{\partial C} + \delta D \frac{\partial}{\partial D} \tag{30}$$

where we set $\delta e^a_\mu = 0$ and

$$\begin{cases} \delta A = \frac{a}{2}(\bar\epsilon\psi)\, e^{-\alpha}, \qquad \delta C = -\frac{a}{2}(\bar\epsilon\gamma^5\gamma^a\nabla_a\psi)\, e^{-\gamma}, \\ \delta B = -i\frac{a}{2}(\bar\epsilon\gamma^5\psi)\, e^{-\beta}, \qquad \delta D = i\frac{a}{2}(\bar\epsilon\gamma^a\nabla_a\psi)\, e^{-\delta}, \\ \delta\psi = \frac{a}{2}\big[i(\gamma^a\epsilon)\,\nabla_a A\, e^{\alpha} + (\gamma^a\gamma^5\epsilon)\,\nabla_a B\, e^{\beta} + i(\gamma^5\epsilon)\, C\, e^{\gamma} + (\epsilon)\, D\, e^{\delta}\big]. \end{cases} \tag{31}$$

Notice that (30) is a generalized vector field since it depends on the derivatives of fields.

One can easily prove that these infinitesimal transformations are symmetries for the Lagrangian (28), modulo the divergence term

$$\begin{aligned} \alpha = \frac{a}{4}\Big[&\big(2imA(\bar\epsilon\gamma^\mu\psi) + 2\nabla^\mu A(\bar\epsilon\psi) - \nabla_\nu A(\bar\epsilon\gamma^\nu\gamma^\mu\psi)\big)\, e^{\alpha+1} \\ &+ \big(2mB(\bar\epsilon\gamma^5\gamma^\mu\psi) - 2i\nabla^\mu B(\bar\epsilon\gamma^5\psi) + i\nabla_\nu B(\bar\epsilon\gamma^5\gamma^\nu\gamma^\mu\psi)\big)\, e^{\beta+1} \\ &+ iD(\bar\epsilon\gamma^\mu\psi)\, e^{\delta+1} + C(\bar\epsilon\gamma^\mu\gamma^5\psi)\, e^{\gamma+1}\Big]\, \mathbf{ds}_\mu \end{aligned} \tag{32}$$

where $\mathbf{ds}_\mu = i_{\partial_\mu}\mathbf{ds}$ is the $(m-1)$-surface local volume.

5. Supersymmetric algebra and perspectives

We stress that the supersymmetry generators are not closed with respect to commutator. One can in fact check that, given two supersymmetries generated by ϵ_1 and ϵ_2, their commutator is the Lie derivative with respect to the vector field defined on the structure bundle Σ by

$$\hat\xi = \xi^\mu(\partial_\mu - \Gamma^{ab}_\mu\sigma_{ab}) \oplus (e^a_\mu\nabla_\nu\xi^\mu e^{b\nu})\sigma_{ab}, \qquad \xi^\mu = i\frac{a^2}{2}(\bar\epsilon_2\gamma^\mu\epsilon_1)\,. \tag{33}$$

We remark that the vector field (33) is the so-called *Kosmann lift* of the spacetime vector field $\xi = \xi^\mu\partial_\mu$; see [6].

The vector fields (33) and the supersymmetry generators form an algebra with the following commutation rules

$$[\delta_1, \delta_2] = \pounds_{\hat\xi}\,, \qquad [\delta_1, \pounds_{\hat\xi}] = 0\,, \qquad [\pounds_{\hat\zeta}, \pounds_{\hat\xi}] = 0\,. \tag{34}$$

Of course these vector fields do not span an ordinary Lie algebra, since some of the parameters are actually anticommuting, while the ordinary Lie algebra parameters are scalars. In fact, one should regard them as generators of a graded Lie algebra (also called *superalgebra*).

References

[1] I.M. Anderson, N. Kamran and P.J. Olver, Internal, external, and generalized symmetries, *Advances In Mathematics* 100 (1993) (1) 53–100.

[2] H. Blaine Lawson Jr. and M.-L. Michelsohn, *Spin Geometry* (Princeton University Press, New Jersey, 1989).

[3] L. Castellani, R. D'Auria and P. Fré, *Supergravity and Superstrings. A Geometric Perspective* (World Scientific, Singapore, 1991).

[4] D.J. Eck, Gauge-natural bundles and generalized gauge theories, *Mem. Amer. Math. Soc.* 33 (1981) (247).

[5] L. Fatibene, M. Ferraris and M. Francaviglia, Nöther formalism for conserved quantities in classical gauge field theories. II. The arbitrary bosonic matter case, *J. Math. Phys.* 38 (1997) (8) 3953–3967.

[6] L. Fatibene, M. Ferraris, M. Francaviglia and M. Godina, A geometric definition of Lie derivative for spinor fields, in: *Differential Geometry and its Applications*, Proc. Conf., Brno, 1995, Ed. I. Kolář (Masaryk University, Brno, Czech Republic, 1996) 549–558.

[7] L. Fatibene, M. Ferraris, M. Francaviglia and M. Godina, Gauge formalism for general relativity and fermionic matter, *Gen. Rel. Grav.* 30 (1998) (9) 1371–1389.

[8] L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, Remarks on Nöther charges and black holes entropy, *Ann. Physics* 275 (1999) (1) 27–53.

[9] L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, Remarks on conserved quantities and entropy of BTZ black hole solutions, II. BCEA theory, *Phys. Rev. D* 60 (1999) (124013) 1–10.

[10] L. Fatibene and M. Francaviglia, Deformations of spin structures and gravity, in: Gauge theories of gravitation, Jadwisin 1997, *Acta Phys. Polon. B* 29 (1998) (4) 915–928.

[11] L. Fatibene and M. Francaviglia, Spin structures on manifolds, in: *Geometry Seminars, 1996–1997*, ed. S. Coen (Univ. Stud. Bologna, Bologna, 1998) 69–98.

[12] L. Fatibene and M. Francaviglia, Natural and gauge natural formalism for classical Field Theory; a geometric perspective including spinors and gauge fields, to appear.

[13] L. Fatibene, M. Francaviglia and M. Godina, Gauge-Natural Structure of Spin Bundles, in print.

[14] L. Fatibene, M. Francaviglia and R.G. McLenaghan, Generalized Symmetries in Mechanics and Field Theories, *J. Math. Phys.*, in print.

[15] M. Ferraris, Fibered connections and global Poincaré–Cartan forms in higher-order calculus of variations, in: *Geometric Methods in Physics*, Proc. Conf. on Diff. Geom. Appl., Vol. 2, Nové Město na Moravě (Czechoslovakia), Sept. 1983, D. Krupka, ed. (J. E. Purkyně University, Brno, 1984) 61–91.

[16] M. Ferraris and M. Francaviglia, *On the Theory of Formal Connections and Fibered Connections in Fibered Manifolds*, Quaderni di Matematica, Quaderno 86 (Università di Torino, Torino, 1984).

[17] M. Ferraris and M. Francaviglia, The Lagrangian Approach to Conserved Quantities in General Relativity, in: *Mechanics, Analysis and Geometry: 200 Years after Lagrange*, Ed. M. Francaviglia (Elsevier, Amsterdam, 1991) 451–488.

[18] P.L. García and J. Muñoz, On the geometrical structure of higher order variational calculus, in: *Proceedings of the IUTAM-ISIMM Symposium on Modern Developments in Analytical Mechanics*, Torino, July 7–11, 1982; S. Benenti, M. Francaviglia and A. Lichnerowicz eds. (Tecnoprint, Bologna, 1983) 127–147.
[19] H. Goldschmidt and D. Spencer, On the nonlinear cohomology of Lie equations, IV. *J. Differential Geom.* 13 (1978) (4) 455–526.
[20] I. Kolář, Prolongations of generalized connections, in: *Differential geometry*, Colloq. Math. Soc. János Bolyai 31, Budapest 1979 (North-Holland, Amsterdam, 1982) 317–325.
[21] I. Kolář, P.W. Michor and J. Slovák, *Natural Operations in Differential Geometry* (Springer, New York, 1993).
[22] D. Krupka, Regular Lagrangians and Lepagean forms, in: *Differential Geometry and its Applications,* Proc. Conf., Brno (Czechoslovakia), August 1986, D. Krupka and A. Švec eds. (J.E. Purkyně University, Brno, 1987) 111–148.
[23] D. Krupka, Variational sequences on finite order jet spaces, in: *Proc. Conf. on Diff. Geom. and its Appl.*, August 1989, J. Janyška and D. Krupka eds. (World Scientific, Singapore, 1990, 236–254.
[24] D. Krupka, Lepagean forms in higher order variational theory, in: *Proceedings of the IUTAM-ISIMM Symposium on Modern Developments in Analytical Mechanics,* Torino, July 1982, S. Benenti, M. Francaviglia and A. Lichnerowicz eds. (Tecnoprint, Bologna, 1983) 197–238.
[25] P. Ramond, *Field Theory: A Modern Primer* (Benjamin/Cummings Publishing Co., Reading, 1981).
[26] A.M. Vinogradov, A spectral sequence that is connected with a nonlinear differential equation, and the algebraic-geometry foundations of the Lagrange field theory with constrain, *Soviet Math. Dokl.* 19 (1978) (1) 144–148.
[27] J. Wess and B. Zumino, Supergauge transformations in four dimensions, *Nuclear Phys.* B70 (1974) 39–50.

L. Fatibene, M. Francaviglia and M. Raiteri
Dipartimento di Matematica
Università degli Studi di Torino
Via Carlo Alberto 10
10123 Torino
Italy
E-mail: fatibene@dm.unito.it, francaviglia@dm.unito.it, raiteri@dm.unito.it