Differential Geometry and Its Applications
Proc. Conf., Opava (Czech Republic), August 27–31, 2001
Silesian University, Opava, 2001, 383–391

Multidimensional Euler–Poincaré equations[1]

M. Castrillón López and P.L. García Pérez

Abstract. Given a Lagrangian $L : J^1P \to \mathbb{R}$, with $P = M \times G \to M$, invariant under the natural action of G on J^1P, we deduce the analog of the Euler–Poincaré equations. The geometry of the reduced variational problem as well as its link with the Noether Theorem and an example are also given.

Keywords. Euler–Lagrange, Euler–Poincaré, group of symmetries, Lagrangian, reduction, variational calculus.

MS classification. 49Q20, 58E30, 49S05, 53C43.

1. Introduction

There are basically two geometrical approaches to Classical Mechanics: the Hamiltonian picture (on the cotangent bundle) and the Lagrangian picture (on the tangent bundle). One of the great achievements of these geometrical approaches is the theory of reduction of a system under a group of symmetries. It is the well known theory of cotangent symplectic reduction in its Lagrangian counterpart. See, for example, [3] for a nice reference for the latter.

In the same spirit, one could ask for a theory of reduction for Field Theories. We think that such a issue must be worked out by stages. We begin with the Lagrangian picture of Field Theories, the first step of which must be the analog of Euler–Poincaré reduction. These equations are the paradigm of Lagrangian reduction in Mechanics, as they deal with a Lie group as the configuration space and the group of symmetries at the same time. More precisely, the Lagrangian $L : TG \to \mathbb{R}$ defining the variational problem is invariant under the action of G on its tangent bundle. The translation in Field Theories leads one to work with variational problems for mappings from an arbitrary manifold M to the Lie group G defined by

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.
Both authors have been partially supported by DGSIC project BFM2000-1314, Ministerio de Ciencia y Tecnología, Spain, and project SA0 85-01, Junta de Castilla León, Spain.

a Lagrangian $L : J^1(M, G) \to \mathbb{R}$. For $M = \mathbb{R}$ one recovers the Classical Mechanics. The goal of this paper is to present the reduction process for this case. The principal results are the Theorems 1 and 2. The first reference where this result is found is ([2]). The next section gives the reconstruction process for obtaining solutions of the original variational problem from solutions of the reduced one. Indeed, Theorem 3 shows that some compatibility conditions are needed for the reconstruction, which is something new for Field Theories as they do not appear in Classical Mechanics. Finally, in the Corollary 1 it is proved that the Euler–Poincaré equations is just the Noether conservation law defined by the G-symmetry. Hence, the Euler–Lagrange equations of the Lagrangian L are equivalent to a conservation law plus the compatibility condition.

Due to the lack of space, we only give the statements of the results without the proofs.

2. Preliminaries

The main topic of our discussion is the first-order variational calculus on the space of mappings $\gamma : M \to G$, between a compact (and possibly with border) manifold and an arbitrary group. In order to take advantage of the classical theory of variational calculus for Field Theories (see for example [4, 6]), we rather see γ as a section of the trivial bundle $\mathrm{pr}_1 : P = M \times G \to M$. We now outline some basic properties of this principal bundle.

An automorphisms $\Phi : M \times G \to M \times G$ is a diffeomorphism such that $\Phi \circ R_g = \Phi$, for any $g \in G$, where $R_g : G \to G$ represents the right action of G onto itself. Every automorphisms can be thus written as $\Phi(x, g) = (\varphi(x), \phi(x) \cdot g)$, with $\varphi \in \mathrm{Diff}(M)$, and $\phi \in C^\infty(M, G)$. Hence, the group of automorphisms $\mathrm{Aut}P$ is isomorphic to $\mathrm{Diff}(M) \times C^\infty(M, G)$. The subgroup $\{\mathrm{Id}\} \times C^\infty(M, G)$ is called the gauge group $\mathrm{Gau}P$. These groups can be seen as infinite dimensional Lie groups and it is easy to see that the Lie algebras of $\mathrm{Aut}P$ is the algebra of vector fields X on P invariant under $(R_g)_*$, for any $g \in G$. This algebra is denoted by $\mathrm{aut}P$. In fact, as

$$T_{(x,g)}P = T_x M \oplus T_g G,$$

any $X \in \mathrm{aut}P$ can be uniquely written as

$$X_{(x,g)} = X'_x + \widetilde{B(x)}_g, \tag{2.1}$$

where $X' \in \mathfrak{X}(M)$, $B : M \to \mathfrak{g}$ is a smooth map, and $\widetilde{B(x)}$ is the infinitesimal generator of the flow $(x, g) \mapsto (x, \exp(tB(x))g)$. In particular, the elements of the Lie algebra $\mathrm{gau}P$ of the gauge group are those vector fields $X_{(x,g)} = \widetilde{B(x)}_g$, $B \in C^\infty(M, \mathfrak{g})$.

The bundle of connections $C \to M$ of a principal bundle $P \to M$ is the affine bundle whose global sections correspond to principal connections on C (see, for example, [1]). For the trivial bundle $\mathrm{pr}_1 : P = M \times G \to M$, the bundle of connections can be naturally identified with $T^*M \times \mathfrak{g} \to M$. Indeed, for every section $\sigma \in \Gamma(T^*M \otimes \mathfrak{g})$, we define the connection form

$$\omega^\sigma : TP = TM \oplus TG \to \mathfrak{g}$$

as

$$\omega^\sigma_{(x,g)} = \mathrm{Ad}_{g^{-1}} \circ \sigma \circ (\mathrm{pr}_1)_*,$$

for any $(x, g) \in M \times G$. It is straightforward to see that the correspondence $\sigma \leftrightarrow \omega_\sigma$ between the space of sections $\Gamma(T^*M \otimes \mathfrak{g})$ and the space of connections is bijective.

The group $\mathrm{Aut}P$ acts on the space of connections $C \to M$ by pulling-back connection forms (see, [7, II. Proposition 6.2-(b)]). We thus have a morphism $\mathrm{Aut}P \to \mathrm{Diff}(C)$, $\Phi \mapsto \Phi_C$, such that

$$(\Phi_C^{-1})^*\omega^\sigma = \omega^{\Phi_C \circ \sigma}.$$

At the level of Lie algebras, we obtain a Lie algebra morphism $\mathrm{aut}P \to \mathfrak{X}(C)$, $X \mapsto X_C$. For the trivial case $P = M \times G$ and $C = T^*M \otimes \mathfrak{g}$, we fix a basis $\{B_1, ..., B_m\}$ of the Lie algebra $\mathfrak{g}$. A coordinate system $(x^i)_{i=1}^n$ on $U \subset M$ induces a system $(x^i, p_j^\alpha)_{i,j=1}^n$ on $T^*M \otimes \mathfrak{g}$ by setting $\xi = p_j^\alpha(\xi)dx^j \otimes B_\alpha$, for any $\xi \in T^*M \otimes \mathfrak{g}$. For any $X \in \mathrm{aut}P$, we can write

$$X = f^i \frac{\partial}{\partial x^i} + g^\alpha \widetilde{B}_\alpha,$$

with $f^i, g^\alpha \in C^\infty(U)$ and then (see, for example [1,5])

$$X_C = f^i \frac{\partial}{\partial x^i} - \left(\frac{\partial g^\alpha}{\partial x^j} - c^\alpha_{\beta\gamma} g^\beta p^\gamma_j + \frac{\partial f^k}{\partial x^j} p^\alpha_k \right) \frac{\partial}{\partial p^\alpha_j}.$$

3. Reduction of the variational problem

Given a bundle $E \to M$, let $J^1E \to M$ be its bundle of jets of local sections. A first order Lagrangian density is a fiber mapping $\mathcal{L} : J^1E \to \bigwedge^n T^*M$. From now on we will consider that M is oriented by a fixed volume form v. Then, there exists a function $L : J^1E \to \mathbb{R}$, called the Lagrangian, such that $\mathcal{L} = Lv$. Given a section s of $E \to M$, the variational action defined by $\mathcal{L}$ is

$$\mathcal{S}_\mathcal{L}(s) = \int_M \mathcal{L} \circ j^1 s = \int_M L \circ j^1 s \; v.$$

A section is called critical if for every uniparametrical variation $\{s_\varepsilon\}_{\varepsilon\in\mathbb{R}}$, with $s_0 = s$ and $s_\varepsilon|_{\partial M} = s|_{\partial M}$, one has

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{S}_\mathcal{L}(s_\varepsilon) = 0.$$

Every variation defines a vector field δs along the section s, called infinitesimal variation, by

$$\delta s = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} s_\varepsilon .$$

In particular, $\delta s|_{\partial M} = 0$.

For the trivial case $P = M \times G$, we have a natural identification $J^1 P \simeq J^1(M, G) \simeq T^*M \otimes_M \mathrm{pr}_2^*(TG)$, where $\mathrm{pr}_2 : M \times G \to G$ is the projection onto the second factor. More precisely, the elements $j^1_x s \in J^1 P$ can be seen as the linear mapping $T_x s : T_x M \to T_{s(x)} P$, s being a local section. As $s(x) = (x, \gamma(x))$ for certain $\gamma \in C^\infty(M, G)$, and $T_{s(x)} P = T_x M \oplus T_{\gamma(x)} G$, we can thus identify $j^1_x s$ with $T_x \gamma \in T^*_x M \otimes T_{g(x)} G$. For the sake of simplicity we will directly write $J^1 P \simeq T^*M \otimes TG$. Hence, writing $L : T^*M \otimes TG \to \mathbb{R}$, the action on $C^\infty(M, G)$ is

$$\mathcal{S}_L(\gamma) = \int_M L \circ T\gamma \; v.$$

The group G naturally acts on $J^1 P = T^*M \otimes TG$ by tangent lift action on TG, more precisely

$$(\omega \otimes X) \cdot g \mapsto \omega \otimes ((R_g)_* X),$$

for any $g \in G$ and $\omega \otimes X \in T^*M \otimes TG$. The quotient manifold is

$$(T^*M \otimes TG)/G \simeq T^*M \otimes ((TG)/G) \simeq T^*M \otimes \mathfrak{g},$$

where $\mathfrak{g}$ represents the Lie algebra of G. We will identify $\mathfrak{g}$ with $T_e G$ along the paper. Hence, the quotient manifold is the bundle of connections of $M \times G \to M$, see [1].

We now consider that the Lagrangian $L : T^*M \otimes TG \to \mathbb{R}$ is invariant under the action of G. Then, L naturally drops to a function $l : T^*M \otimes \mathfrak{g} \to \mathbb{R}$ called the *reduced Lagrangian*. The symmetries given by the group G can be used to reduce the Euler–Lagrange equations of L for mappings $\gamma : M \to G$ to a new kind of equations for the section $\sigma : M \to T^*M \otimes \mathfrak{g}$, $\sigma(x) = T_{\gamma(x)} R_{\gamma(x)^{-1}} \circ T_x \gamma$. This reduced equations are called the Euler–Poincaré equations of the reduced Lagrangian l on C. Before giving the precise statement of this result we give the following notation:

For every section $\sigma \in \Gamma(T^*M \otimes \mathfrak{g})$, we define $\delta l/\delta\sigma : T^*M \otimes \mathfrak{g} \to \mathbb{R}$ as the vertical derivative of l along σ, i.e.,

$$\frac{\delta l}{\delta \sigma}(\xi) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} l(\sigma(x) + \varepsilon \xi),$$

for $\xi \in T^*_x M \otimes \mathfrak{g}$, $x \in M$. The operator $\delta l/\delta\sigma$ can be thus seen as a section of the dual bundle $TM \otimes \mathfrak{g}^*$, that is, a vector field taking values in $\mathfrak{g}$.

Finally, given $\sigma \in \Gamma(T^*M \otimes \mathrm{ad}P)$, we define the operator $\mathrm{ad}^*_\sigma : TM \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ as the pairing of T^*M and TM and the natural coadjoint operator in the $(\mathrm{ad}P)^*$ part.

Theorem 1. *Let $\pi : P = M \times G \to M$ be a trivial principal G-fiber bundle over a manifold M with a volume form υ and let $L : J^1P = T^*M \otimes TG \to \mathbb{R}$ be a G-invariant Lagrangian. Let $l : T^*M \otimes \mathfrak{g} \to \mathbb{R}$ be the mapping defined by L in the quotient. For a mapping $\gamma : M \to G$ of let $\sigma : M \to \mathfrak{g}$ be defined by $\sigma(x) = T_{\gamma(x)}R_{\gamma(x)^{-1}} \circ T_x\gamma$. Then the following conditions are equivalent:*

1. *the variational principle $\delta \int_M L(T_x\gamma)\upsilon = 0$ holds for variations δg along g with $\delta g|_{\partial M} = 0$,*
2. *the mapping $\gamma : M \to G$ satisfies the Euler–Lagrange equations for L,*
3. *the variational principle $\delta \int_M l(\sigma(x))\upsilon = 0$ holds, using variations of the form*

$$\delta\sigma = \mathrm{d}\eta - [\sigma, \eta], \tag{3.1}$$

for any smooth function $\eta : M \to \mathfrak{g}$ with $\eta|_{\partial M} = 0$,

4. *the Euler–Poincaré equations hold:*

$$\mathcal{EP}(l)(\sigma) = \mathrm{div}\frac{\delta l}{\delta\sigma} + \mathrm{ad}^*_\sigma \frac{\delta l}{\delta\sigma} = 0, \tag{3.2}$$

where div *stands for the divergence operator defined by the volume form υ.*

Remark 1. Note that the operator $\delta l/\delta\sigma$ is a vector field on M taking values on the vector space $\mathfrak{g}^*$. Hence the divergence of $\delta l/\delta\sigma$ yields a function belonging to $C^\infty(M, \mathfrak{g}^*)$ as $\mathrm{ad}^*_\sigma(\delta l/\delta\sigma)$, thus making the formula (3.2) meaningful.

Theorem 2. *The reduced problem defined by l is a zero-order variational problem on the bundle $C = T^*M \otimes \mathfrak{g}$ with constraints in the space of infinitesimal variations: Given a section $\sigma : M \to T^*M \otimes \mathfrak{g}$ is critical if and only if the action defined by l vanishes for infinitesimal variations of the type $\delta\sigma = X_C|_{\sigma(M)}$, with $X \in \mathrm{gau}P$.*

4. Reconstruction of the variational problem

Given a solution $\gamma : M \to G$ of the variational problem defined by the Lagrangian L, Theorem 1 claims that the mapping $\sigma : M \to T^*M \otimes \mathfrak{g}$, with $\sigma(x) = T_{\gamma(x)}R_{\gamma(x)^{-1}} \circ T_x\gamma$ is a solution of the Euler–Poincaré equations. But, a natural question arises, namely: how can one recover a solution γ from a mapping σ satisfying (3.2)?

We first remark the following property:

Lemma 1. *Given a mapping $\gamma : M \to G$, the section $\sigma : M \to T^*M \otimes \mathfrak{g}$ represents a flat connection on the bundle $M \times G \to M$ whose integral leaves are the sets*

$$\big\{(x, \gamma(x) \cdot g) | x \in M\big\}_{g \in G}.$$

Hence not every section $\sigma \in \Gamma(T^*M \otimes \mathfrak{g})$ solution of (3.2) is necessarily induced by a mapping γ. We need to impose a compatibility condition, namely, the vanishing of the curvature of the connection described by σ. More precisely:

Theorem 3. *Let $L : T^*M \otimes TG \to M$ be a G-invariant Lagrangian over a compact manifold. A solution σ of the Euler–Poincaré equations* (3.2) *comes from a solution γ of the original variational problem defined by L if and only if σ is a flat connection with trivial holonomy. If σ satisfies this compatibility condition, every integral leaf is a global section $x \mapsto (x, \gamma(x))$ of the principal bundle $M \times G \to M$. Then we reconstruct a family of mappings by setting $\gamma^g = \gamma \cdot g$, $g \in G$.*

Roughly speaking, we have the equivalence

$$\mathcal{EL}(L)(s) = 0 \Longleftrightarrow \begin{cases} \mathcal{EP}(l)(\sigma) = 0, \\ \sigma \text{ is flat.} \end{cases}$$

Remark 2. Of course, the flatness of σ can be also written as

$$\mathrm{Curv}(\sigma) = \mathrm{d}\sigma + [\sigma, \sigma] = 0.$$

Remark 3. The condition $\mathrm{Curv}(\sigma) = 0$ always holds true for $\dim M = 1$. For that reason we never find such a condition in Classical Mechanics for the reconstruction process.

5. Euler–Poincaré as a conservation law

The symmetries of a variational problem produces conservation laws by means of the Noether's Theorem (see, for example, [4, 6]). For a G-invariant Lagrangian L and every element $B \in \mathfrak{g}$, the vector field B^* on $P = M \times G$, generator of the flow $(x, g \cdot \exp(tB))$, is an infinitesimal symmetry of L, that is,

$$(B^*)^{(1)}(L) = 0,$$

where $(B^*)^{(1)}$ represents the 1-jet lift of B^* to $J^1P = T^*M \otimes TG$. Let $\Theta_{\mathcal{L}}$ be the Poincaré–Cartan n-form defined by $\mathcal{L} = Lv$ on J^1P. We also have that

$$L_{(B^*)^{(1)}} \Theta_{\mathcal{L}} = 0,$$

and the Noether conservation law just reads

$$\mathrm{d}(j^1 s)^* i_{(B^*)^{(1)}} \Theta_{\mathcal{L}} = 0, \tag{5.1}$$

for any critical section s.

We define a $\mathfrak{g}^*$-valued $(n-1)$-form $\mathcal{J}$ on $J^1P = T^*M \otimes TG$ by setting

$$\mathcal{J}(B) := i_{(B^*)^{(1)}} \Theta_{\mathcal{L}}, \qquad \forall B \in \mathfrak{g}.$$

The conservation law given in formula (5.1) can be thus rewritten as

$$\mathrm{d}(j^1 s)^* \mathcal{J} = 0.$$

The form $\mathcal{J}$ can be understood as a sort of current form, as it generates a family of closed forms on the base manifold M.

Proposition 1. *The* $\mathfrak{g}$*-valued* $(n-1)$*-form* $\mathcal{J}$ *satisfies the following properties:*
(i) $i_X \mathcal{J} = 0$, *for any vertical tangent vector of the projection* $J^1 P \to M$;
(ii) $(R_g)^* \mathcal{J} = \mathrm{Ad}_g^* \circ \mathcal{J}$, *for any* $g \in G$.

Then $\mathcal{J}$ is a tensorial form of the coadjoint type (see [7, II]) of the principal bundle $T^*M \otimes TG \to T^*M \otimes \mathfrak{g}$. It thus defines a form $\mathbf{J}$ on $T^*M \otimes \mathfrak{g}$ taking values in the coadjoint bundle. As we are dealing with trivial bundles, the coadjoint bundle is nothing but $(T^*M \otimes \mathfrak{g}) \times \mathfrak{g}^*$, that is, $\mathbf{J}$ is a $\mathfrak{g}^*$-valued form, which can be defined as follows: Considering $T^*M \otimes \mathfrak{g}$ as a submanifold of $T^*M \otimes TG$, $\mathbf{J}$ is the pull-back of $\mathcal{J}$ by the inclusion.

Theorem 4. *Given a section* $\sigma : M \to T^*M \otimes \mathfrak{g}$, *consider the form* $\sigma^* \mathbf{J}$. *If the section* σ, *comes from a mapping* γ *(i.e.,* $\sigma = TR_{\gamma^{-1}} \circ T\gamma$*) which is a critical value of the variational problem defined by* $\mathcal{L} = Lv$, *then the Noether conservation law can be rewritten as*

$$\nabla^\sigma (\sigma^* \mathbf{J}) = 0, \tag{5.2}$$

where ∇^σ *is the covariant derivative with respect to the connection* σ, *that is,*

$$\nabla^\sigma (\cdot) = \mathrm{d} \cdot + \mathrm{ad}_\sigma^* \cdot .$$

By means of the identification between the space of $(n-1)$*-forms on* M *and the space of vector fields* $\mathfrak{X}(M)$, *we can understand* $\sigma^* \mathbf{J}$ *as a* $\mathfrak{g}^*$*-valued vector field* $\bar{\mathbf{J}}_\sigma$ *on* M. *Then* $\bar{\mathbf{J}}_\sigma$ *coincides with* $\delta l / \delta \sigma$.

Corollary 1. *With the identification between* $\Omega^{n-1}(M)$ *and* $\mathfrak{X}(M)$ *given by* v, *the equation* (5.2) *can be written as*

$$\mathrm{d}\, \bar{\mathbf{J}}_\sigma + \mathrm{ad}_\sigma^* \bar{\mathbf{J}}_\sigma = 0,$$

which is exactly the Euler–Poincaré equation (3.2). *In other words, the Euler–Poincaré equation is equivalent to the Noether conservation law given by the G--symmetry.*

6. Example

Let (M, g) be a compact oriented Riemannian manifold, and let (G, h) be a Lie group equipped with a right invariant Riemannian metric. We define the energy Lagrangian $L : T^*M \otimes TG \to \mathbb{R}$ as

$$L(T\gamma) = \frac{1}{2} \langle T\gamma, T\gamma \rangle_{g,h},$$

where $\langle \cdot, \cdot \rangle_{g,h}$ is the induced metric on $T^*M \otimes TG$ defined by g and h. The Euler–Lagrange equations for this Lagrangian are given by

$$\mathrm{Tr}\, \nabla \gamma = 0,$$

where ∇ is the induced Riemannian covariant derivative on $\Gamma(T^*M \otimes TG)$ and Tr is the trace defined by g. The critical mappings are called harmonic mappings. The reduced Lagrangian $l : T^*M \otimes \mathfrak{g} \to \mathbb{R}$ is

$$l(\sigma) = \frac{1}{2} \langle \sigma, \sigma \rangle_{g,h},$$

where h is seen as a metric on $\mathfrak{g}$, and the Euler–Poincaré equations are

$$\langle \mathrm{d}^* \sigma, \cdot \rangle_h + \mathrm{ad}^*_\sigma \langle \sigma, \cdot \rangle_h = 0,$$

where $\mathrm{d}^* = *\mathrm{d}*$ stands for the codifferential. If the metric h is also left invariant, we have

$$\mathrm{ad}^*_A \langle B, \cdot \rangle_h + \mathrm{ad}^*_B \langle A, \cdot \rangle_h = 0,$$

for any $A, B \in \mathfrak{g}$, and then the Euler–Poincaré equation simply reads

$$\mathrm{d}^* \sigma = 0.$$

This equation first appears in the literature in ([9]), and is widely used (for example in [8] and [10]).

References

[1] M. Castrillón López and J. Muñoz Masqué, The geometry of the bundle of connections, *Math. Z.* 236 (2001) (4) 797–811.

[2] M. Castrillón López, T.S. Ratiu and S. Shkoller, Reduction in principal fiber bundles: covariant Euler–Poincaré equations, *Proc. Amer. Math. Soc.* 128 (2000) (7) 2155–2164.

[3] H. Cendra, J. Marsden and T. Ratiu, Lagrangian reduction by stages, *Mem. Amer. Math. Soc.* 152 (2001) (722).

[4] P.L. García, The Poincaré–Cartan invariant in the calculus of variations, *Symposia Math.* 14 (1974) 219–246.

[5] P.L. García Pérez and J. Muñoz Masqué, Lagrangian densities invariant under the full Lie algebra of the infinitesimal automorphisms of a principal bundle, in: *Differential Geometry and its Applications*, Proc. Conf. Opava (Czechoslovakia), 1992, (Silesian University at Opava, Opava, 1993) 423–437.

[6] H. Goldschmidt and S. Sternberg, The Hamiltonian–Cartan formalism in the calculus of variations, *Ann. Inst. Fourier, Grenoble* 23 (1973) (1) 203–267.

[7] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (John Wiley & Sons, Inc. New York, 1963).

[8] F.C. Park and R.W. Brockett, Kinematic dexterity of robotic mechanism, *Int. J. Robotics Res.* 13 (1994) 1–15.

[9] A.I. Pluzhnikov, Some properties of harmonic mappings in the case of spheres and Lie groups, *Soviet Math. Dokl.* 27 (1983) (1).

[10] K. Uhlenbeck, Harmonic maps into Lie groups (Classical solutions of the chiral model), *J. Diff. Geom.* 30 (1989) 1–50.

M. Castrillón López
Dept. Geometría y Topología
Universidad Complutense de Madrid
Avda. Complutense S/N
28040 Madrid
Spain
E-mail: mcastri@mat.ucm.es

P.L. García Pérez
Dept. Matemáticas
Universidad de Salamanca
Plaza de la Merced S/N
37008 Salamanca
Spain
E-mail: pgarcia@usal.es