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Hamilton equations for non-holonomic systems

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Abstract. Hamilton equations for Lagrangian systems on fibred manifolds, subjected to general non-holonomic constraints (i.e., not necessarily affine in the velocities) are studied. Conditions for existence of a non-holonomic Legendre transformation are discussed, and the corresponding formulas for constraint momenta and Hamiltonian are found.

Keywords. Lagrangian system, non-holonomic constraints, semi-holonomic constraints, constraint distribution, constrained equations of motion, regularity, momenta, Hamiltonian, Legendre transformation, Hamilton equations.

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1. Introduction

Mechanical systems subjected to non-holonomic constraints, i.e., constraints depending on time, positions and velocities, have been recently intensively studied by methods of differential geometry (see for example [1, 4, 7–15], and references therein). A classical description of a constrained motion is based on the idea that the existence of constraints gives rise to an additional force, a *constraint force*, acting on the mechanical system. More precisely, if

$$f^i(t, q^1, \dots, q^m, \dot q^1, \dots, \dot q^m) = 0, \quad 1 \le i \le k, \qquad \text{where} \quad \operatorname{rank}\left(\frac{\partial f^i}{\partial \dot q^\sigma}\right) = k \tag{1.1}$$

are constraint equations, then the motion of a mechanical system represented by a Lagrangian L, and subjected to the above constraints is described by the system of

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$k + m$ equations

$$f^i = 0, \quad \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma} = \lambda_i \frac{\partial f^i}{\partial \dot{q}^\sigma}, \tag{1.2}$$

where λ_i, $1 \leq i \leq k$, are *Lagrange multipliers*. Solving the motion equations one finds Lagrange multipliers as functions of the parameter t, and the trajectories of the constrained system.

The classical procedure of excluding Lagrange multipliers from the motion equations with help of the equations of constraints geometrically means that a constrained system can be represented as *defined on a constraint manifold*, given by the equations $f^i = 0$. With help of such a geometric model one can study important properties of constrained dynamics, and try to develop an appropriate *Lagrangian* and *Hamiltonian formulation*. The latter problem is really non-trivial, since it is known that for a constrained system such fundamental features as, e.g., regularity, or existence of a Lagrangian, are not inherited from the corresponding unconstrained system.

In this paper we use a geometric setting for non-holonomic systems on fibred manifolds proposed in [7] (cf. also [8–10] for further developments). In Sections 2 and 3 we recall main concepts and results leading to an *intrinstric formulation* of both the non-constrained and constrained equations of motion. In the sequel we deal with Hamilton equations for constrained systems. Our approach is different from a usual procedure (see, e.g., [1, 4]) in the following main points: We introduce constrained Hamiton equations in an *intrinsic* form, as equations for generally non-holonomic sections of the constraint (fibred) manifold. Then, Legendre coordinates are *defined* to be coordinates transferring locally the corresponding constrained system (represented by a class of differential 2-forms) to a simple, canonical, form.

Consequently, *new constraint Legendre transformation* formulas are obtained, and the expression of the constraint Hamilton equations in Legendre coordinates is found. These results are valid for *general non-holonomic constraints*. We also show that, in particular, in the case of constraints affine in velocities and that of semi-holonomic constraints (affine in velocities and integrable) our results correspond to [10], and are in accordance with results and comments in [1].

2. Lagrangian systems on fibred manifolds

We consider a fibred manifold $\pi : Y \to X$ with $\dim X = 1$, $\dim Y = m + 1$, and the jet prolongations $\pi_1 : J^1Y \to X$ and $\pi_2 : J^2Y \to X$ of π. Local fibred coordinates on Y are denoted by (t, q^σ) where $1 \leq \sigma \leq m$. The associated coordinates on J^1Y and J^2Y are denoted by $(t, q^\sigma, \dot{q}^\sigma)$ and $(t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma)$, respectively. In calculations we use either a canonical basis of one forms on J^1Y, $(dt, dq^\sigma, d\dot{q}^\sigma)$, or a basis adapted to the contact structure, $(dt,\ \omega^\sigma,\ d\dot{q}^\sigma)$, where

$$\omega^\sigma = dq^\sigma - \dot{q}^\sigma\, dt, \qquad 1 \leq \sigma \leq m.$$

Recall that for every 2-form η on J^1Y, its lift $\pi^*_{2,1}\eta$ admits a unique decomposition into a sum of a 1-contact and 2-contact form. We denote by $p_1\eta$ and $p_2\eta$ the 1-contact and 2-contact part of $\pi^*_{2,1}\eta$ respectively.

A (local) section δ of π_1 is called *holonomic* if $\delta = J^1\gamma$ for a section γ of π.

By a *distribution* on J^1Y we shall mean a mapping assigning to every point $z \in J^1Y$ a vector subspace of the vector space $T_z J^1Y$.

If λ is a Lagrangian on J^1Y, we denote by θ_λ and E_λ its Cartan and Euler–Lagrange form, respectively. Recall that $E_\lambda = p_1 d\theta_\lambda$. In fibred coordinates, where $\lambda = L\,dt$, we have

$$\theta_\lambda = L\,dt + (\partial L/\partial \dot{q}^\sigma)\omega^\sigma,$$

and $E_\lambda = E_\sigma dq^\sigma \wedge dt$, where

$$E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma}.$$

Since the functions E_σ are affine in the variables $\ddot{q}$, we write

$$E_\sigma = A_\sigma + B_{\sigma\nu}\ddot{q}^\nu, \tag{2.1}$$

where

$$B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}, \quad A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\partial^2 L}{\partial t\,\partial \dot{q}^\sigma} - \frac{\partial^2 L}{\partial q^\nu \partial \dot{q}^\sigma}\dot{q}^\nu. \tag{2.2}$$

Distribution Δ_λ annihilated by all the one-forms $i_\xi d\theta_\lambda$ where ξ runs over all *vertical* vector fields on J^1Y is then called *Euler–Lagrange distribution*. Sections (respectively, holonomic sections) of the fibred manifold π_1, which are integral sections of Δ_λ, are called *Hamilton extremals* (respectively, *extremals*) of λ. Hence, the corresponding equations, called *Hamilton* (respectively, *Euler–Lagrange equations*) of λ, read $\delta^* i_\xi d\theta_\lambda = 0$ (respectively, $J^1\gamma^* i_\xi d\theta_\lambda = 0$) for every π_1-vertical vector field ξ on J^1Y, see [3, 5].

A Lagrangian λ is called *regular* if $\operatorname{rank}\Delta_\lambda = \operatorname{corank} d\theta_\lambda = 1$. We have the following equivalent characterizations of a regular Lagrangian (cf. [2, 5, 6]).

Theorem 1. *In case that $d\theta_\lambda$ is not projectable onto Y, regularity is equivalent with any of the following conditions:*

(1) $\det(B_{\sigma\nu}) = \det(\partial^2 L/\partial \dot{q}^\sigma \partial \dot{q}^\nu) \neq 0$.

(2) *The Euler–Lagrange distribution is locally spanned by the following semispray:*

$$\zeta = \frac{\partial}{\partial t} + \dot{q}^\sigma \frac{\partial}{\partial q^\sigma} - B^{\sigma\rho} A_\rho \frac{\partial}{\partial \dot{q}^\sigma}, \tag{2.3}$$

where $(B^{\sigma\rho})$ is the inverse matrix to $(B_{\rho\nu})$.

(3) *The Euler–Lagrange equations have an equivalent form*

$$\ddot{q}^\sigma = -B^{\sigma\rho} A_\rho \quad 1 \le \sigma \le m.$$

Note that condition (2) means that the characteristic distribution of $d\theta_\lambda$ has rank 1, coincides with the Euler–Lagrange distribution, and is locally spanned by semisprays (2.3).

With help of the Euler–Lagrange form, the *Euler–Lagrange equations* can be written equivalently as follows:

$$J^1\gamma^* i_\xi \alpha = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi \text{ on } J^1Y, \tag{2.4}$$

where α is *any* 2-form on J^1Y such that $p_1\alpha = E_\lambda$. Apparently,

$$\alpha = d\theta_\lambda + F, \tag{2.5}$$

where F runs over all 2-contact 2-forms, horizontal with respect to the projection $\pi_{1,0}$. In fibred coordinates we have

$$F = F_{\sigma\nu}\,\omega^\sigma \wedge \omega^\nu,$$

where $F_{\sigma\nu}(t, q^\rho, \dot{q}^\rho)$ are *arbitrary* functions. Hence, (in the notations of (2.1), (2.2)) every admissible α takes the form

$$\alpha = A_\sigma \omega^\sigma \wedge dt + B_{\sigma\nu}\omega^\sigma \wedge d\dot{q}^\nu + F_{\sigma\nu}\omega^\sigma \wedge \omega^\nu. \tag{2.6}$$

Recall from [7] that every 2-form α defined on an open subset $W \subset J^1Y$, satisfying condition (2.5) (i.e., such that, over W, $p_1\alpha = E_\lambda$), is called a *Lepage equivalent* of E_λ; the family of all (local) Lepage equivalents of an Euler–Lagrange form E_λ is then called a *Lepage class* of E_λ, or, a *Lagrangian system*, and denoted by $[\alpha]$. The Euler–Lagrange equations (2.4) now have the meaning of equations for *holonomic integral sections* of a *dynamical distribution* Δ_α, defined on the domain of definition of α, and annihilated by the system of 1-forms $i_\xi\alpha$, where ξ runs over all π_1-vertical vector fields on J^1Y. Note that although the dynamical distributions of different α's are generally different, their holonomic integral sections locally coincide, and are nothing but the *extremals* of E_λ. We denote by $[\Delta_\alpha]$ the family of the dynamical distributions associated with the elements of a Lepage class $[\alpha]$. Clearly, the family $[\Delta_\alpha]$ contains the Euler–Lagrange distribution Δ_λ.

If $\alpha \in [\alpha]$ and Δ_α is its dynamical distribution we call equations for *integral sections* of Δ_α *Hamilton equations*, and the 2-form α itself a *Hamiltonian system* related with the Lagrangian system $[\alpha]$. We stress that *if the Lagrangian λ is regular, then the Euler–Lagrange equations are equivalent with the Hamilton equations of any $\alpha \in [\alpha]$*, see [9]. Moreover, in a neighbourhood of every point in J^1Y one has a local coordinate transformation $(t, q^\sigma, \dot{q}^\sigma) \to (t, q^\sigma, p_\sigma)$, called *Legendre transformation*, such that *every $\alpha \in [\alpha]$* takes the *canonical form*

$$\alpha = -dH \wedge dt + dp_\sigma \wedge dq^\sigma + F, \tag{2.7}$$

where F is a certain 2-contact $\pi_{1,0}$-horizontal 2-form. It holds $d\theta_\lambda = -dH \wedge dt + dp_\sigma \wedge dq^\sigma$, and

$$p_\sigma = \frac{\partial L}{\partial \dot{q}^\sigma}, \qquad H = -L + p_\sigma \dot{q}^\sigma.$$

In what follows, we consider Lagrangian systems subjected to non-holonomic constraints and we shall discuss the existence of a 'Legendre transformation' which puts the corresponding 'constraint Hamilton equations' into a canonical form.

3. Non-holonomic Lagrangian systems

Following [7, 8], by a non-holonomic *constraint structure* on J^1Y we mean a pair $(Q, \mathcal{C})$ where Q is a fibred submanifold of the fibred manifold $\pi_{1,0} : J^1Y \to Y$ of codimension k, where $1 \le k \le m-1$, and $\mathcal{C}$ is a distribution on Q, locally annihilated by the 1-forms $\bar{\varphi}^i = \iota^*\varphi^i$, $1 \le i \le k$, where ι is the canonical embedding of Q into J^1Y, and

(3.1) $$\varphi^i = f^i dt + \frac{\partial f^i}{\partial \dot{q}^\sigma}\omega^\sigma, \quad 1 \le i \le k;$$

here $f^i = 0$, $1 \le i \le k$, are (local) equations of the submanifold Q. Notice that the forms (3.1) are linearly independent, since, by definition of Q,

(3.2) $$\operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = k.$$

Q is called a *constraint submanifold*, the distribution $\mathcal{C}$ is called *canonical distribution* or *Chetaev bundle* ([7, 12]), and the forms $\bar{\varphi}^i$ *canonical constraint one-forms*. The ideal of forms on Q, generated by the constraint 1-forms, is called the *constraint ideal* and is denoted by $\mathcal{I}(\mathcal{C}^0)$.

In view of (3.2), it is always possible to define a constraint locally by equations in *normal form*,

(3.3) $$\dot{q}^{m-k+i} - g^i(t, q^\sigma, \dot{q}^1, \ldots, \dot{q}^{m-k}) = 0.$$

Now, consider on J^1Y a Lagrangian system $[\alpha] = [d\theta_\lambda]$, and a constraint structure $(Q, \mathcal{C})$. Put $k = \operatorname{codim} Q$. In keeping with [7, 8], by a related *constrained system* we shall mean the equivalence class of 2-forms on Q defined by

(3.4) $$\alpha_Q = \iota^*\alpha \mod \mathcal{I}(\mathcal{C}^0),$$

for every $\alpha \in [\alpha]$. For the constrained system we use notation $[\alpha_Q]$. Hence, elements of the class $[\alpha_Q]$ are of the form

(3.5) $$\alpha_Q = \iota^* d\theta_\lambda + \bar{F} + \varphi_{(2)},$$

where $\bar{F}$ and $\varphi_{(2)}$ run over all 2-contact $\pi_{1,0}$-horizontal and constraint 2-forms on Q, respectively. In fibred coordinates, where Q is given by (3.3) and the Euler–Lagrange form of λ is represented by (2.1), (2.2), we have

(3.6) $$\alpha_Q = \bar{A}_l \bar{\omega}^l \wedge dt + \bar{B}_{ls}\bar{\omega}^l \wedge d\dot{q}^s + \bar{F}_{ls}\,\bar{\omega}^l \wedge \bar{\omega}^s + \varphi_{(2)},$$

where $\bar{\omega}^l = \iota^*\omega^l$, $\varphi_{(2)} \in \mathcal{I}(\mathcal{C}^0)$, $\bar{F}_{ls}$ are arbitrary, and

$$\bar{A}_l = \left(A_l + A_{m-k+i} \frac{\partial g^i}{\partial \dot{q}^l} + \left(B_{l,m-k+i} + B_{m-k+j,m-k+i} \frac{\partial g^j}{\partial \dot{q}^l} \right) \frac{\bar{d} g^i}{dt} \right) \circ \iota,$$

$$\bar{B}_{ls} = \left(B_{ls} + B_{l,m-k+i} \frac{\partial g^i}{\partial \dot{q}^s} + B_{s,m-k+i} \frac{\partial g^i}{\partial \dot{q}^l} + B_{m-k+i,m-k+j} \frac{\partial g^i}{\partial \dot{q}^l} \frac{\partial g^j}{\partial \dot{q}^s} \right) \circ \iota, \tag{3.7}$$

and summations run over $l, s = 1, 2, \ldots, m-k$ and $i, j = 1, 2, \ldots, k$, see [7]. Above, we used the notation

$$\frac{\bar{d}}{dt} = \frac{\partial}{\partial t} + \dot{q}^\sigma \frac{\partial}{\partial q^\sigma}. \tag{3.8}$$

Note that among the elements of the class $[\alpha_Q]$ we have *'constraint Poincaré–Cartan 2-forms'* $\iota^* d\theta_\lambda \mod \mathcal{I}(\mathcal{C}^0)$.

Since $(B_{\sigma\nu})$ is a symmetric matrix, the above formula gives us that the matrix $(\bar{B}_{ls})$ is *symmetric*.

The *constraint dynamical distribution* related with a 2-form α_Q is defined to be a subdistribution of the canonical distribution $\mathcal{C}$, annihilated by the 1-forms $i_\xi \alpha_Q$, where ξ runs over all π_1-vertical vector fields on Q *belonging to* $\mathcal{C}$, [7]. We stress that every class $\alpha_Q \mod \mathcal{I}(\mathcal{C}^0)$ is represented by a *unique* constraint dynamical distribution. Indeed, if $\alpha'_Q - \alpha_Q$ is a constraint 2-form then for every admissible vector field ξ,

$$i_\xi \alpha'_Q - i_\xi \alpha_Q$$

is a constraint 1-form. In particular, among constraint dynamical distributions related with a constrained Lagrangian system $[\alpha_Q]$ we have the *constraint Euler–Lagrange distribution*.

Now, *constrained Euler–Lagrange equations* are defined to be equations for *holonomic* integral sections of the constraint Euler–Lagrange distribution. It can be easily seen that *a section γ of the fibred manifold $\pi : Y \to X$ such that $J^1\gamma$ is a section of $Q \to X$ is a solution of the constrained Euler–Lagrange equations if and only if*

$$J^1\gamma^* i_\xi \alpha_Q = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi \in \mathcal{C}, \tag{3.9}$$

where α_Q is any representative of the constrained system $[\alpha_Q]$, see [7].

In analogy with the unconstrained case, a constrained system $[\alpha_Q]$ is called *regular* if the constraint Euler–Lagrange distribution has rank 1 (see [7]).

Theorem 2 ([7]). *Let $[\alpha_Q]$ be the constrained system related with a Lagrangian system $[\alpha]$ and a constraint structure $(Q, \mathcal{C})$ on J^1Y. The following conditions are equivalent:*

(1) $[\alpha_Q]$ *is regular.*

(2) *The* $(m-k)\times(m-k)$*-matrix* $(\overline{B}_{sl})$ *is regular, i.e.,*

$$\det(\overline{B}_{sl}) \neq 0. \tag{3.10}$$

(3) *The constraint Euler–Lagrange distribution is locally spanned by the following constraint semispray:*

$$\zeta = \frac{\partial}{\partial t} + \sum_{l=1}^{m-k} \dot{q}^l \frac{\partial}{\partial q^l} + \sum_{i=1}^{k} g^i \frac{\partial}{\partial q^{m-k+i}} - \overline{B}^{sl}\overline{A}_l \frac{\partial}{\partial \dot{q}^s}, \tag{3.11}$$

where $(\overline{B}^{sl})$ *is the inverse matrix to* $(\overline{B}_{sl})$*.*

(4) *The constrained Euler–Lagrange equations have an equivalent form*

$$\dot{q}^{m-k+i} = g^i(t, q^\sigma, \dot{q}^1, \ldots, \dot{q}^{m-k}), \qquad 1 \le i \le k,$$
$$\ddot{q}^l = -\overline{B}^{lp}\overline{A}_p, \qquad 1 \le l \le m-k.$$

Rewriting the *regularity condition* (3.10) by means of a Lagrangian $\lambda = L\,dt$ according to (2.2) and (3.7) we obtain

$$\det\Bigg(\bigg(\frac{\partial^2 L}{\partial \dot{q}^l \partial \dot{q}^s} + \frac{\partial^2 L}{\partial \dot{q}^{m-k+i}\partial \dot{q}^s}\frac{\partial g^i}{\partial \dot{q}^l} + \frac{\partial^2 L}{\partial \dot{q}^{m-k+i}\partial \dot{q}^l}\frac{\partial g^i}{\partial \dot{q}^s} + \frac{\partial^2 L}{\partial \dot{q}^{m-k+j}\partial \dot{q}^{m-k+r}}\frac{\partial g^j}{\partial \dot{q}^l}\frac{\partial g^r}{\partial \dot{q}^s}\bigg)\circ\iota\Bigg) \neq 0. \tag{3.12}$$

On the other hand, in [1] a constrained Lagrangian system is called *regular* if

$$\det\bigg(\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} - \Big(\frac{\partial L}{\partial \dot{q}^{m-k+i}}\circ\iota\Big)\frac{\partial^2 g^i}{\partial \dot{q}^l \partial \dot{q}^s}\bigg) \neq 0, \tag{3.13}$$

where $\bar{L} = L\circ\iota$. However, by a direct computation we can easily check that the matrices in (3.12) and (3.13) are equal, i.e., the above *regularity conditions are the same*.

It is important to stress that, as one can see from any of the above equivalent regularity conditions, a constrained system of a regular Lagrangian system need not be regular.

4. Legendre transformation and Hamilton equations for non-holonomic Lagrangian systems

Let E be an Euler–Lagrange form defined on J^2Y, and $[\alpha]$ its Lepage class of the first order. Consider a constraint structure $(Q,\mathcal{C})$ on J^1Y. If $\alpha \in [\alpha]$ is a Lepage 2-form defined on an open set $W \subset J^1Y$, we shall call the equivalence class α_Q (3.4), i.e., $\alpha_Q = \iota^*\alpha \bmod \mathcal{I}(\mathcal{C}^0)$, a *constrained Hamiltonian system*, related with the Lagrangian system $[\alpha]$ and the constraint $(Q,\mathcal{C})$. Equations for integral sections of the corresponding constraint dynamical distribution Δ_{α_Q} (on $W \cap Q$) will be called *constrained Hamilton equations*. Hence, Hamilton equations take the following intrinsic form:

A section δ of the fibred manifold $Q \to X$, passing in W, is a solution of the constrained Hamilton equations if and only if δ is an integral section of the canonical distribution $\mathcal{C}$, and

$$\delta^* i_\xi \alpha_Q = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi \in \mathcal{C}. \tag{4.1}$$

Theorem 3. *Let $[\alpha_Q]$ be a regular constrained Lagrangian system. Then for any two Hamiltonian systems α_Q^1, α_Q^2 belonging to the Lepage class $[\alpha_Q]$ and such that* $\operatorname{dom}\alpha_Q^1 \cap \operatorname{dom}\alpha_Q^2 = U \neq \emptyset$, *their constrained dynamical distributions coincide on U, i.e., $\Delta_{\alpha_Q^1} = \Delta_{\alpha_Q^2}$, meaning that Hamilton equations are the same.*

Proof. Since α_Q^1 and α_Q^2 are equivalent, we have by (3.6),

$$\begin{aligned} \alpha_Q^1 &= \bar{A}_l \bar{\omega}^l \wedge dt + \bar{B}_{ls} \bar{\omega}^l \wedge d\dot{q}^s + \bar{F}^1_{ls} \bar{\omega}^l \wedge \bar{\omega}^s \quad \operatorname{mod} \mathcal{I}(\mathcal{C}^0), \\ \alpha_Q^2 &= \bar{A}_l \bar{\omega}^l \wedge dt + \bar{B}_{ls} \bar{\omega}^l \wedge d\dot{q}^s + \bar{F}^2_{ls} \bar{\omega}^l \wedge \bar{\omega}^s \quad \operatorname{mod} \mathcal{I}(\mathcal{C}^0). \end{aligned} \tag{4.2}$$

Computing the annihilators of the distributions $\Delta_{\alpha_Q^1}$ and $\Delta_{\alpha_Q^2}$, and using the assumption that the matrix $\bar{B}_{ls}$ is regular, we can see that both the distributions are annihilated by the following (same) system of one forms:

$$\bar{A}_l dt + \bar{B}_{ls} d\dot{q}^s, \qquad \bar{\omega}^l, \qquad \iota^* \omega^{m-k+i}. \tag{4.3}$$

This completes the proof. $\square$

From the above theorem we immediately obtain the following important property of *regular* constrained Hamiltonian systems.

Corollary 4. *Let $[\alpha_Q]$ be a constrained Lagrangian system. If the regularity condition* (3.10) *is satisfied then constrained Euler–Lagrange equations are equivalent with (any) constrained Hamilton equations.*

This means that every Hamiltonian system $\alpha_Q \in [\alpha_Q]$, defined on an open set $W \subset Q$, possesses the following property: a section δ of the fibred manifold $Q \to X$ passing in W is a solution of the constrained Hamilton equations of α_Q if and only if $\delta = J^1\gamma$ where $\gamma : X \to W$ is a constrained extremal of $[\alpha_Q]$.

Now, we are in position to discuss the concept of *constraint Legendre transformation*.

Theorem 5. *Let $[\alpha_Q]$ be a constrained system related with a Lagrangian λ and a constraint structure $(Q, \mathcal{C})$ on J^1Y. Let $x \in Q$ be a point. Suppose that in a neighbourhood of x,*

$$\frac{\partial \bar{B}_{ls}}{\partial \dot{q}^r} = \frac{\partial \bar{B}_{lr}}{\partial \dot{q}^s}, \qquad 1 \leq l, r, s \leq m - k. \tag{4.4}$$

Then there exists a neighbourhood $U \subset Q$ of x, and, on U, functions P_a, $1 \le a \le m-k$ and a 1-form η, such that the class $[\alpha_Q]$ has a representative of the form

$$\alpha'_Q = \eta \wedge dt + dP_a \wedge dq^a. \tag{4.5}$$

If, moreover, the constrained system $[\alpha_Q]$ is regular, then $(t, q^\sigma, \dot q^a) \to (t, q^\sigma, P_a)$ is a coordinate transformation on U.

Proof. In a neighbourhood of x, let us consider the elements of the equivalence class $[\alpha_Q]$ in the form (3.6). From the Poincaré Lemma we get a neighbourhood $U \subset Q$ of x and functions P_a, $1 \le a \le m-k$, on U such that

$$\bar B_{al} = -\partial P_a / \partial \dot q^l. \tag{4.6}$$

In the class $[\alpha_Q]$ there is a local representative of the form

$$\begin{aligned}
\alpha_Q &= \bar A_l \bar\omega^l \wedge dt + \frac{\partial P_a}{\partial \dot q^l} d\dot q^l \wedge \bar\omega^a \\
&= \bar A_l \bar\omega^l \wedge dt + dP_l \wedge \bar\omega^l - \frac{\partial P_l}{\partial t} dt \wedge \bar\omega^l - \frac{\partial P_l}{\partial q^s} dq^s \wedge \bar\omega^l \\
&\quad - \frac{\partial P_l}{\partial q^{m-k+i}} dq^{m-k+i} \wedge \bar\omega^l \\
&= \left(\bar A_l + \frac{\partial P_l}{\partial t} \right) dq^l \wedge dt + dP_l \wedge dq^l - \dot q^l \, dP_l \wedge dt - \frac{\partial P_l}{\partial q^s} (\bar\omega^s + \dot q^s dt) \wedge \bar\omega^l \\
&\quad - \frac{\partial P_l}{\partial q^{m-k+i}} (\bar\omega^{m-k+i} + g^i dt) \wedge \bar\omega^l \\
&= \left(\bar A_l + \frac{\partial P_l}{\partial t} + \frac{\partial P_l}{\partial q^a} \dot q^a + \frac{\partial P_l}{\partial q^{m-k+i}} g^i \right) dq^l \wedge dt - \dot q^l dP_l \wedge dt + dP_l \wedge dq^l \\
&\quad + \frac{\partial P_l}{\partial q^s} \bar\omega^l \wedge \bar\omega^s - \frac{\partial P_l}{\partial q^{m-k+i}} \bar\omega^{m-k+i} \wedge \bar\omega^l.
\end{aligned}$$

Hence, we have a representative

$$\begin{aligned}
\alpha'_Q &= \left(\bar A_l + \frac{\partial P_l}{\partial t} + \frac{\partial P_l}{\partial q^s} \dot q^s + \frac{\partial P_l}{\partial q^{m-k+i}} g^i \right) dq^l \wedge dt - \dot q^s dP_s \wedge dt \\
&\quad + dP_l \wedge dq^l \\
&= \left(\bar A_l + \frac{\partial P_l}{\partial t} + \left(\frac{\partial P_l}{\partial q^s} - \frac{\partial P_s}{\partial q^l} \right) \dot q^s + \frac{\partial P_l}{\partial q^{m-k+i}} g^i \right) dq^l \wedge dt \\
&\quad - \frac{\partial P_s}{\partial q^{m-k+i}} \dot q^s dq^{m-k+i} \wedge dt - \frac{\partial P_s}{\partial \dot q^l} \dot q^s d\dot q^l \wedge dt + dP_l \wedge dq^l.
\end{aligned} \tag{4.7}$$

We can write $\alpha'_Q = \eta \wedge dt + dP_a \wedge dq^a$ with

$$\eta = \bar{\eta}_0 dt + \bar{\eta}_l dq^l + \bar{\eta}_{m-k+i} dq^{m-k+i} + \tilde{\eta}_l d\dot{q}^l, \tag{4.8}$$

where $\bar{\eta}_0$ is an arbitrary function on U, and

$$\begin{aligned} \bar{\eta}_l &= \bar{A}_l + \frac{\partial P_l}{\partial t} + \left(\frac{\partial P_l}{\partial q^s} - \frac{\partial P_s}{\partial q^l}\right)\dot{q}^s + \frac{\partial P_l}{\partial q^{m-k+i}} g^i, \\ \tilde{\eta}_l &= -\frac{\partial P_a}{\partial \dot{q}^l}\dot{q}^a, \qquad \bar{\eta}_{m-k+i} = -\frac{\partial P_a}{\partial q^{m-k+i}}\dot{q}^a, \end{aligned} \tag{4.9}$$

belongs to $[\alpha_Q]$, as desired. Finally, the regularity condition for the transformation $(t, q^\sigma, \dot{q}^a) \to (t, q^\sigma, P_a)$ coincides with (3.10), i.e. (3.13). □

Integrability condition (4.4) for the $\bar{B}_{sl}$'s ensures that one can express the functions P_a explicitly. To this purpose we consider a mapping $\chi : [0, 1] \times W \to W$ defined by $(u, t, q^\sigma, \dot{q}^a) \to (t, q^\sigma, u\dot{q}^a)$, where $W \subset Q$ is an appropriate open set. Then Poincaré Lemma gives us a solution

$$\begin{aligned} P_a &= -\dot{q}^b \int_0^1 (\bar{B}_{al} \circ \chi)\, du \\ &= \frac{\partial \bar{L}}{\partial \dot{q}^a} - \dot{q}^b \int_0^1 \left(\left(\frac{\partial L}{\partial \dot{q}^{m-k+i}} \circ \iota\right) \frac{\partial^2 g^i}{\partial \dot{q}^a \partial \dot{q}^b}\right) \circ \chi\, du. \end{aligned} \tag{4.10}$$

We shall call the above functions P_a the *constraint momenta*, and the corresponding coordinate transformation *constraint Legendre transformation*. The 1-form η in (4.5) is *determined up to a constraint* 1*-form*, and *need not be closed*. We shall call it a *constraint energy* 1*-form*. In view of (4.7), in constraint Legendre coordinates we can write

$$\eta = \eta_0\, dt + \eta_a\, dq^a + \eta^a\, dP_a \quad \mod \mathcal{I}(\mathcal{C}^0). \tag{4.11}$$

It is easy to see that *if* $[\alpha_Q]$ *is regular,* we obtain for the constraint Euler–Lagrange equations *equivalent* constraint Hamilton equations in the following *canonical form*:

$$\frac{d}{dt}(P_a \circ \delta) = -\eta_a, \quad \frac{d}{dt}(q^a \circ \delta) = \eta^a, \quad \frac{d}{dt}(q^{m-k+i} \circ \delta) = g^i, \tag{4.12}$$

for $1 \le a \le m - k$, $1 \le i \le k$.

The above results generalize corresponding results of [1, 4, 10] to general (nonlinear) non-holonomic constraints.

We shall finish by discussing the case of *constraints affine in the velocities*. First, notice that condition (4.4) rewritten in terms of a Lagrangian reads

$$\left(\frac{\partial}{\partial \dot{q}^b}\left(\frac{\partial L}{\partial \dot{q}^{m-k+i}} \circ \iota\right)\right)\frac{\partial^2 g^i}{\partial \dot{q}^c \partial \dot{q}^a} = \left(\frac{\partial}{\partial \dot{q}^c}\left(\frac{\partial L}{\partial \dot{q}^{m-k+i}} \circ \iota\right)\right)\frac{\partial^2 g^i}{\partial \dot{q}^b \partial \dot{q}^a}. \tag{4.13}$$

Now, the following assertion is easily obtained.

Theorem 6. *Suppose that the functions* g^i, $1 \le i \le k$, *are affine in the velocities. Then* (4.4) *is fulfilled identically and the constraint momenta become* $P_a = \partial \bar{L}/\partial \dot{q}^a$. *Regularity condition then takes the form*

$$\det\left(\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s}\right) \neq 0. \tag{4.14}$$

Moreover, if for any Γ,

$$\Gamma = \frac{\partial}{\partial t} + \sum_{l=1}^{m-k} \dot{q}^l \frac{\partial}{\partial q^l} + \sum_{i=1}^{k} g^i \frac{\partial}{\partial q^{m-k+i}} + \sum_{l=1}^{m-k} \Gamma^l \frac{\partial}{\partial \dot{q}^l}, \tag{4.15}$$

the functions g^i *satisfy the condition*

$$\partial_\Gamma \frac{\partial g^i}{\partial \dot{q}^a} - \frac{\partial g^i}{\partial q^a} - \frac{\partial g^i}{\partial q^{m-k+j}} \frac{\partial g^j}{\partial \dot{q}^a} = 0, \tag{4.16}$$

then the family of energy 1*-forms* (4.11) *contains a closed* 1*-form equal to* $-d\bar{H}$, *where* $\bar{H} = -\bar{L} + P_a \dot{q}^a$.

Proof. The only non-trivial part of the proof is to show that (4.16) implies $-d\bar{H} - \eta \in \mathcal{I}(\mathcal{C}^0)$. According to (3.5), the form α'_Q (4.5) is equivalent to $\iota^* d\theta_\lambda$ (we shall write $\alpha'_Q \sim \iota^* d\theta_\lambda$). We have $\iota^* d\theta_\lambda = d\iota^*\theta_\lambda$, and

$$\begin{aligned}
\iota^*\theta_\lambda &= \bar{L}\,dt + \left(\left(\frac{\partial L}{\partial \dot{q}^a} + \frac{\partial L}{\partial \dot{q}^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^a}\right) \circ \iota\right) \bar{\omega}^a + \left(\frac{\partial L}{\partial \dot{q}^{m-k+i}} \circ \iota\right) \bar{\varphi}^i \\
&= -\bar{H}\,dt + P_a dq^a + \left(\frac{\partial L}{\partial \dot{q}^{m-k+i}} \circ \iota\right) \bar{\varphi}^i.
\end{aligned}$$

Now,

$$\begin{aligned}
\alpha'_Q \sim \iota^* d\theta_\lambda &\sim -d\bar{H} \wedge dt + dP_a \wedge dq^a + \left(\frac{\partial L}{\partial \dot{q}^{m-k+i}} \circ \iota\right) d\bar{\varphi}^i \\
&\sim -d\bar{H} \wedge dt + dP_a \wedge dq^a \\
&\quad - \left(\frac{\partial L}{\partial \dot{q}^{m-k+i}} \circ \iota\right)\left(\partial_\Gamma \frac{\partial g^i}{\partial \dot{q}^a} - \frac{\partial g^i}{\partial q^a} - \frac{\partial g^i}{\partial q^{m-k+j}} \frac{\partial g^j}{\partial \dot{q}^a}\right) dq^a \wedge dt,
\end{aligned}$$

and we can see that under the assumption (4.16), $\alpha'_Q \sim -d\bar{H} \wedge dt + dP_a \wedge dq^a$, as desired. □

Note that, as proved in [10], condition (4.16) means that the constraint is *semi-holonomic*.

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