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Energy of some Gödel type models[1]

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Abstract. The Komar superpotential is applied to the calculation of the local density of the matter energy and gravitational energy in the Gödel type cosmological models. The possibility of the conversion of the energy in these models is discussed.

Keywords. General relativity, cosmology, conservation law, Lagrangian field theory, Komar superpotential, Gödel model, energy.

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1. Introduction

The problem of the calculation of the energy for some cosmological models has been re-opened recently. Garecki ([2], 1995) calculated the energy of the homogeneous and isotropic cosmological models — so-called Friedmann–Robertson–Walker–Lemâitre (FRWL) models — by means of the Komar superpotential. This contribution extends Garecki's approach to the cosmological models, which are homogeneous, but not isotropic — the Gödel type models.

In the following the Latin indices run over 0, 1, 2, 3, the Greek indices run over $1, 2, 3$ and the index 0 is used for the time coordinate $x^0 = ct$, the symbol '$_{,i}$' denotes partial derivative with respect to x^i and the symbol '$_{;i}$' means the covariant derivative.

The signature of the metric field is $(+, -, -, -)$, Lorentzian metric.

2. Energy in General Relativity (GR)

Various approaches to the conservation laws problem in General Relativity (GR) are discussed in detail in many books and articles, for example in ([1], 1992), ([4],

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

1969), ([6], 1984), ([7], 1974), ([9], 1984). In agreement with Garecki ([2], 1995), we shall adopt the form of the conservation laws based on the Komar superpotential ([5], 1959), which is very natural from geometric and physical point of view.

2.1. Conservation law for energy

A differential form of conservation law in a curved spacetime can be most naturally expressed as

$$S^k{}_{;k} = 0, \tag{1}$$

where S^k is a vector field. Introducing the vector density field $w^k = \sqrt{-g}\, S^k$ we can rewrite (1) as the well know continuity equation

$$\frac{1}{\sqrt{-g}} w^k{}_{,k} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial ct} w^0 = -w^\alpha{}_{,\alpha}. \tag{2}$$

Consequently, it is possible to interpret w^0 as the local value (density) of some conserved quantity. The integral form of conservation law can be obtained from (2) by integration over four-dimensional region of spacelike hypersurface. But in our article we shall work with differential laws and local quantities only.

The automatic fulfilment of (1) is guaranteed by the introduction of an antisymmetric tensor field $U^{kl} = -U^{lk}$ (the so-called superpotential), connected with the conserved field S^k by the relation

$$S^k = U^{kl}{}_{;l}, \quad w^k = \sqrt{-g}\, S^k = \sqrt{-g}\, U^{kl}{}_{;l}. \tag{3}$$

Komar ([5], 1959) introduced the superpotential

$$U^{kl} = \frac{1}{2\kappa}(X^{k;l} - X^{l;k}) \tag{4}$$

(κ is constant), where X^k is an arbitrary vector field — the so-called descriptor. In this way, we can relate a conserved field S^k to every descriptor X^k. According to Komar and other authors, in the case of timelike descriptor, i.e.,

$$g(X, X) = g_{ik} X^i X^k > 0,$$

the S^k can be interpreted as the four-dimensional vector field of energy density flow, connected with a given reference system defined by time translation field X^k. This flow has a very important property that it does not depend on the choice of space coordinates and, consequently, it allows the energy localization in the framework of the reference system. The quantity

$$w^0 = (\sqrt{-g}\, U^{[0\lambda]})_{,\lambda} \tag{5}$$

has the meaning of the *local density of the total energy*, which includes all its forms.

2.2. Matter and gravitational part of the energy

In General Relativity, the flow of matter energy density can be defined as $T^k = T^{kl}X_l$. It is easy to prove (see, for example ([9], 1984)) that this flow is conserved in the case when the descriptor represents a Killing vector field, i.e., it fulfils the Killing equation $X_{i;k} + X_{k;i} = 0$. Consequently, the *matter energy* with the *local density*

$$\sqrt{-g}\,T^0 = \sqrt{-g}\,T^{0l}X_l \tag{6}$$

and the gravitational energy are conserved separately in the case of the Killing vector field. The symmetric energy-momentum tensor T^{ik} of matter (including all particles and fields with the exception of the gravitational field) fulfils the Einstein equations

$$G_{ik} = \kappa T_{ik}, \tag{7}$$

where G_{ik} is Einstein tensor, dependent on the metric tensor g_{ik} and its first and second derivatives. It follows from (7)

$$U^{kl}{}_{;l} = U^{kl}{}_{;l} - \left(\frac{1}{\kappa}G^{kl} - T^{kl}\right)X_l. \tag{8}$$

Consequently, we can introduce the flow of the gravitational energy density

$$t^k = U^{kl}{}_{;l} - \frac{1}{\kappa}G^{kl}X_l = S^k - T^k \tag{9}$$

and the related *local density of the gravitational energy*

$$\sqrt{-g}\,t^0 = w^0 - \sqrt{-g}\,T^0. \tag{10}$$

In the case that X^k is the Killing vector field, the matter energy and the gravitational energy are conserved separately. In general case of arbitrary vector field only the total energy is conserved and the conversion of the matter energy into the gravitational energy (and vice versa) is possible.

In the case of non-zero cosmological constant, also contribution of cosmological term to the local density can be considered. It is clearly

$$T^0_\Lambda = \sqrt{-g}\,\Lambda g^{0k}X_k,$$

where Λ is the cosmological constant.

3. Gödel type cosmological models – expanding and stationary

This section describes the basic properties of the Gödel type models ([8], 2000) and the possible symmetries for these models.

3.1. The metric tensor for the Gödel type models, possible values of parameters

The Gödel type cosmological models are defined on the manifold $\mathbb{R}^4$, the form of the metric tensor field (in the coordinate system ct, x, y, z) is

$$g_{ik} = \begin{pmatrix} 1 & 0 & -\sqrt{\sigma}\, a(ct)e^{mx} & 0 \\ 0 & -a^2(ct) & 0 & 0 \\ -\sqrt{\sigma}\, a(ct)e^{mx} & 0 & -ka^2(ct)e^{2mx} & 0 \\ 0 & 0 & 0 & -a^2(ct) \end{pmatrix}, \tag{11}$$

where $a(ct)$ is the time-dependent scale factor, the parameters σ and m are non-negative and $\sigma + k > 0$ for the Lorentzian signature of the metric. This model is usually called the Gödel type model with rotation and expansion. Coordinate z gives the direction of the global rotation, the magnitude of which

$$\omega(ct) = \sqrt{\tfrac{1}{2}\,\omega_{ik}\omega^{ik}} = \frac{m}{2a(ct)}\sqrt{\frac{\sigma}{\sigma + k}} \tag{12}$$

decreases in the expanding world.

If we put $a(ct) = \text{const}$, for simplicity $a(ct) = 1$, we obtain the form of so-called stationary Gödel type metric:

$$g_{ik} = \begin{pmatrix} 1 & 0 & -\sqrt{\sigma}\, e^{mx} & 0 \\ 0 & -1 & 0 & 0 \\ -\sqrt{\sigma}\, e^{mx} & 0 & -ke^{2mx} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{13}$$

The rotation around the z-axis has the constant magnitude:

$$\omega = \sqrt{\tfrac{1}{2}\,\omega_{ik}\omega^{ik}} = \frac{m}{2}\sqrt{\frac{\sigma}{\sigma + k}} \tag{14}$$

for the stationary model. First model belonging to this class was created by Gödel ([3], 1949); the parameters $m,\ k,\ \sigma$ for this model have the values: $k = -\frac{1}{2}$, $m = 1,\ \sigma = 1$.

If the angular velocity ω of the rotation is equal to zero ($\sigma = 0$ and/or $m = 0$), the metric tensor (11) passes into the metric tensor describing some FRWL (Friedmann–Robertson–Walker–Lemâitre) model with the zero spatial curvature defined on $\mathbb{R}^4$.

3.2. The number of the Killing vector fields for the Gödel type models

The number of Killing vector field for the Gödel type models is determined by the properties of these models. The concrete form of all Killing vector fields for the Gödel type models and the commutation relations between them are written up in ([8], 2000).

The timelike Killing vector field cannot exist for the expanding models; in this case only spacelike Killing vector fields exist.

The timelike Killing vector field for the stationary models has the form of time translation: $X = \partial_{ct}$. If the parameter k has non-zero value, there exist five Killing vector fields including the vector field of time translation. If we put $k = 0$, there exist in addition two spacelike Killing vector fields.

There exist at least six spacelike Killing vector fields describing the spatial rotations and translations for the FRWL models — the concrete form of the vector fields is included in ([2], 1995). The vector field of time translation is the Killing vector field for the static models only.

4. Calculation of energy for the Gödel type models

We can calculate the local density of the total energy for the Gödel type models by means of (5) and we can compare the obtained values with the previous Garecki's ([2], 1995) results for the FRWL cosmological models. The physical meaning of the particular Gödel type models filled by the cosmological fluid and the possibility of the conversion of the energy for these models will be discussed.

4.1. The local density of the total energy for the Gödel type cosmological models

We obtain the local density of the total energy for the Gödel type models by means of (5), where U^{kl} is the Komar superpotential (4). Using the vector field of time translation $X = \partial_{ct}$ as the descriptor, the local density of the total energy (with respect to (11)) has the form

$$w^0 = -\frac{a(ct)\sigma m^2}{2\kappa\sqrt{\sigma + k}} e^{mx} = -\frac{2a^3(ct)}{\kappa}\omega^2\sqrt{\sigma + k}\, e^{mx}, \tag{15}$$

where ω (12) denotes the angular velocity of the rotation. The local density of the total energy is negative, only for the non-rotating model ($m = 0$ or $\sigma = 0$) this quantity is equal to zero.

4.2. The possibility of the conversion of the energy for expanding and stationary Gödel type models

The local density of the total energy *for the expanding model* is time dependent. Because we choice the vector field of time translation $X = \partial_{ct}$ as the descriptor of the energy, which is not the Killing vector field for this model, the conversion of the matter energy into the gravitational energy and vice versa is possible. The local density of the matter energy (6) is dependent on the form of the energy-momentum tensor T^{ik}, which characterizes the matter. The local density of the gravitational energy (10) is also dependent on the parameters m, σ, k and $a(ct)$, describing the gravitational field, and on the energy-momentum tensor T^{ik} characterizing the matter. It is possible to find by means of the Einstein equations (7) the connection between these quantities. This problem is solved in the next subsection.

The local density of the total energy, matter energy and gravitational energy *for the stationary model* can be obtained putting $a(ct) = 1$ in the relations for the expanding models. Because the descriptor of the energy $X = \partial_{ct}$ is the Killing vector field for this model, the conversion of the matter energy into the gravitational energy and vice versa is forbidden (in contrast to the expanding models). The local density of the total energy is time independent, the values for the matter and the gravitational field too. The concrete expression for these quantities are given in the next subsection.

4.3. The possible Gödel type models for the energy-momentum tensor of the cosmological fluid

The energy-momentum tensor of the cosmological fluid in the co-moving coordinate system (plus the term belonging to the cosmological constant Λ)

$$T_{ik} = (\rho c^2 + p)g_{0i}g_{0k} - pg_{ik} + \Lambda g_{ik}, \tag{16}$$

where p is the pressure and ρ the density of the cosmological fluid, describes the simplest choice of the matter content for the spatial homogeneous Gödel type models.

The left hand side of Einstein equations (7) for the Gödel type models depends on the metrical tensor field g_{ik} (11) and the energy-momentum tensor on the right hand side has the form (16). The quantities $p,\ \rho$ and Λ characterizing the matter depend also on the values $m, \sigma, k, a(ct)$. The requirements

$$m = 0 \quad \text{or} \quad \sigma = 0 \quad \text{or} \quad a(ct) = \text{const} = 1 \tag{17}$$

follow from the Einstein equations (7) using (11) and (16).

We must add the form of the equation of state determining the relation between p and ρ (for example $p = \rho = 0$ (the empty space, but Λ may be non-zero), $p = 0,\ \rho > 0$ (the ideal dust) or $p = \frac{1}{3}\rho c^2$ (relativistic particles, radiation)).

1. The expanding model. $\dot{a}(ct) = da(ct)/dct \neq 0$. It must be $m = 0$, that means that *the expanding models filled by the ideal fluid are non-rotating*. Such cosmological models pass into some FRWL non-static model with the topology $\mathbb{R}^4$ and zero spatial curvature. Because the FRWL models are not only homogeneous, but also isotropic, no other energy-momentum tensor as (16) can be used for these models.

If σ is equal to zero, k must be positive and it holds from (7), (11) and (16)

$$\kappa(p - \Lambda) = -\frac{\dot{a}^2(ct) + 2a(ct)\ddot{a}(ct)}{a^2(ct)}, \quad \kappa(\rho c^2 + \Lambda) = \frac{3\dot{a}^2(ct)}{a^2(ct)}. \tag{18}$$

We can choose the equation of the state and determine the concrete form of the scale factor $a(ct)$ for the FRWL models.

2. The stationary model. $\dot{a}(ct) = 0$. It is $m = 0$ or it holds $\sigma = -2k$ (k is negative).

For $m = 0$ it holds $p - \Lambda = \rho c^2 + \Lambda = 0$ for all values σ, k. The reasonable equation of state for this model is $p = \rho = 0,$ also $\Lambda = 0$. This model is in fact the Minkowski spacetime of Special Relativity. Ten Killing vector fields for this model include the vector of time translation $X = \partial_{ct}$.

If m is non-zero and $\sigma = -2k$,

$$\kappa(p - \Lambda) = \frac{m^2}{2} = \omega^2, \qquad \kappa(\rho c^2 + \Lambda) = \frac{m^2}{2} = \omega^2, \tag{19}$$

where ω is the angular velocity of the rotation. If we put $p = 0$ and $\rho > 0,$ it holds $\Lambda = -m^2/2 = -\omega^2$ and $\rho c^2 = m^2 = 2\omega^2$. This solution was obtained by Gödel ([3], 1949).

If we choose $p = \frac{1}{3}\rho c^2,$ it holds $\Lambda = p = -m^2/4 = -\omega^2/2$ and $\rho c^2 = 3m^2/4 = (3\omega^2/2)$. This radiative model is analogical to the previous Gödel solution with ideal dust.

4.4. The matter and gravitational energy for the Gödel type models filled by the cosmological fluid

1. Garecki ([2], 1995) has calculated the total energy of FRWL models by using of the vector of time translation $X = \partial_{ct}$. This energy is equal to zero, which is in accordance with the relation (15) for $m = 0$. It is easy to see from (15), (6) and (10), that the positive local density of the matter energy is in counterbalance with the negative local density of the gravitational energy

$$\sqrt{-g}\,T^0 = \sqrt{k}\,(\rho c^2 + \Lambda) = \frac{3\sqrt{k}}{\kappa}\left(\frac{\dot{a}(ct)}{a(ct)}\right)^2 = -\sqrt{-g}\,t^0. \tag{20}$$

If the matter energy in this model increases with time, the gravitational energy decreases and vice versa. We could obtain the concrete time dependence of the matter and gravitational energy by the aid of the concrete form of scale factor $a(ct)$.

2. The local density of the matter energy for the Minkowski spacetime of Special Relativity is equal to the local density of the gravitational energy and it is equal to zero. This model is static, because the Killing vector field of time translation $X = \partial_{ct}$ is orthogonal to the hypersurface $ct = \text{const}$. The energy conversion is forbidden in such model.

The local density of the total energy for the stationary Gödel type model calculated by (15) is negative

$$w^0 = -\frac{\sigma m^2}{2\kappa\sqrt{\sigma + k}}e^{mx} = -\frac{\sqrt{|k|}\,m^2}{\kappa}e^{mx} = -\frac{\sqrt{|k|}}{\kappa}e^{mx}2\omega^2. \tag{21}$$

The local density of the matter energy for the Gödel model ([3], 1949) is equal to

$$\sqrt{-g}\,T^0 = \frac{\sqrt{|k|}}{\kappa}e^{mx}\frac{m^2}{2} = \frac{\sqrt{|k|}}{\kappa}e^{mx}\omega^2 \tag{22}$$

and it holds for the local density of the gravitational energy

$$\begin{aligned}\sqrt{-g}\,t^0 &= -\frac{\sqrt{|k|}}{\kappa}m^2e^{mx} - \frac{\sqrt{|k|}}{\kappa}e^{mx}\frac{m^2}{2} = -\frac{\sqrt{|k|}}{\kappa}e^{mx}\frac{3m^2}{2}\\ &= -\frac{\sqrt{|k|}}{\kappa}e^{mx}3\omega^2.\end{aligned} \tag{23}$$

The values of total, matter and gravitational energy for the radiative Gödel type model are the same as the values for Gödel model with dust.

5. Conclusion

We have calculated the local density of the total energy for the Gödel type models and the local densities of the matter and the gravitational energy for these models filled by the cosmological fluid. It is shown, that in this case the expanding Gödel type models pass into the FRWL models. The obtained local density of the total energy for these models is equal to zero with the agreement with the Garecki's ([2], 1995) result. On the other hand the local density of the total energy is non-zero (negative) for the Gödel stationary model ([3], 1949). The conversion of the matter energy into the gravitational energy and vice versa is forbidden for this model.

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