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On the Morse and the Maslov index for periodic geodesics of arbitrary causal character

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Abstract. We prove a version of the Morse Index Theorem for periodic geodesics in a stationary Lorentzian manifold. This theorem relates the index of a suitable restriction of the second variation of the Lorentzian action functional to the Maslov index of a periodic geodesic. The Maslov index of a periodic geodesic is a semi-integer defined in terms of the flow of the Jacobi equation along the geodesic, which produces a curve in the symplectic group.

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1. Introduction

The classical Morse index theorem in Riemannian geometry for geodesics between two fixed points (see for instance [4]) gives an equality between the Morse index of the action functional $f(\gamma) = \frac{1}{2}\int_a^b g(\gamma', \gamma')$ at a critical point γ, which is a geodesic, and the number of conjugate points along γ. Recall that the Morse index of f is the index of the second variation of f (called the *index form*), which is a symmetric bilinear form defined on the space of vector fields along γ vanishing at the endpoints.

The theorem generalizes to the case of periodic geodesics, where the Morse index of the action functional is given by the index of the index form defined on the space of vector fields V along γ with $V(a) = V(b)$; the theorem gives an equality

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between the Morse index of f at γ and the sum of the number of conjugate points along γ with a term called by Morse the *order of concavity* of γ (see [10]).

When passing to the case of Lorentzian manifolds, i.e., manifolds endowed with a metric tensor of index 1, it is well known that the Morse index theorem for geodesics between fixed points generalizes in the case of *nonspacelike* geodesics (see [1, 2]), and for this generalization one has to consider the index of the restriction of the index form to the space of vector fields which are everywhere orthogonal to γ. In [8] the authors prove a general formulation of the Lorentzian Morse index theorem in the case of stationary Lorentzian manifolds for geodesics of any causal character. For this generalization, one needs to consider the index of the restriction of the index form to the space of vector fields along γ satisfying a suitable conservation law with respect to a fixed timelike Killing vector field. Moreover, the number of conjugate points along the geodesic (that can be infinite in the spacelike case) is replaced by an integer called the *Maslov index* of the geodesic, which roughly speaking gives an algebraic count of the conjugate points along γ. More precisely, the Maslov index of a geodesic is defined as an intersection number of a curve in the Lagrangian Grassmannian Λ of a symplectic space with the *Maslov cycle* of Λ (see for instance [12] for details). We remark that an even deeper generalization of the Morse Index Theorem for general semi-Riemannian manifolds can be found in [14, 15].

In this paper, following the ideas of [8], we prove a Lorentzian version of the Morse index theorem for periodic geodesics of any causal character in the case of stationary manifolds. In analogy with [8], given a timelike Killing vector field Y, we consider the restriction of the action functional to the space $\mathcal{N}$ of closed curves η satisfying the conservation law $g(\eta', Y) \equiv c_\eta$ (constant). The critical points of the action functional f in $\mathcal{N}$ are precisely the periodic geodesics (Theorem 4.1). Moreover, given a periodic geodesic γ, the Morse index of the restriction of the action functional to $\mathcal{N}$ at γ is finite.

Using a periodic trivialization of the tangent bundle along γ, the flow of the Jacobi equation along γ produces a smooth curve in the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ starting at the identity. Using an embedding of $\mathrm{Sp}(2n, \mathbb{R})$ into the Lagrangian Grassmannian of a $4n$-dimensional symplectic space we define a notion of Maslov index for curves Φ in $\mathrm{Sp}(2n, \mathbb{R})$. This notion of Maslov index gives a sort of algebraic count of the instants t when $\Phi(t)$ is a symplectomorphism having one of its eigenvalues equal to 1; in the geodesic case, these instants play the role of “conjugate points” along a periodic geodesic. For instance, the dimension of the kernel of the index form (defined as the *nullity* of the geodesic) equals the geometric multiplicity of the eigenvalue 1 of $\Phi(b)$.

Observe that each periodic geodesic γ is a degenerate critical point of f; namely, $\dot\gamma$ and $Y \circ \gamma$ are always in the kernel of the index form. There are several notions of Maslov index in the literature; in order to deal with degenerate geodesics, the appropriate notion is the one given in [17] for arbitrary curves in the symplectic group. Given a continuous curve Φ in the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$, the Maslov index $\mathrm{i}_{\mathrm{Maslov}}(\Phi)$ is an intersection number of Φ with the variety Γ_0 consisting of symplectomorphisms having eigenvalue 1. When Φ is obtained from the flow of the Jacobi equation along a geodesic γ, the degeneracy of γ corresponds to the fact that the final endpoint of Φ is in Γ_0, in which case a suitable correction term (a semi-integer number) is added in the computation of $\mathrm{i}_{\mathrm{Maslov}}(\Phi)$.

The main theorem of the paper gives a relation between the Maslov index and the Morse index of a periodic geodesic in a stationary Lorentzian manifold. It should be observed that a different formulation of the periodic index theorem in the Lorentzian case could be given in terms of the order of concavity of the geodesic. However, the authors believe the formulation involving the Maslov index of symplectic paths is more interesting, due to the fact that it allows the development of iteration formulas, in the spirit of [3, 7].

For a better understanding of the material presented we give a short overview of the contents of each section.

In Section 2 we recall some basic properties of the geometry of the symplectic group and the Lagrangian Grassmannian Λ of a symplectic space, and we recall the definition of the Maslov index for curves in Λ. We also give the definition of Maslov index for curves in the symplectic group; the main results of the section are Lemma 2.10 and its Corollary 2.11, where we establish a relation between the two notions of Maslov index.

In Section 3 we describe how to associate to a periodic geodesic a curve in the symplectic group; we use a Hamiltonian formalism which is well suited in the context of symplectic geometry.

Finally, in Section 4 we state and prove our main theorem; its proof uses the index theorem of [8] and the technical Lemma 2.10 with its Corollary 2.11.

2. The Maslov index of curves in the Lagrangian Grassmannian and in the symplectic group

In this section we will recall some basic facts concerning the geometry of the Lagrangian Grassmannian and of the symplectic group of a symplectic space; proofs of such facts can be found for instance in [5, 6, 12, 17].

Let (V, ω) be a *symplectic vector space*, i.e., V is a real vector space with $\dim(V) = 2n$ and $\omega : V \times V \to \mathbb{R}$ is an anti-symmetric nondegenerate bilinear form; a linear operator $T : V \to V$ is called a *symplectomorphism* if it preserves ω. The *symplectic group* $\mathrm{Sp}(V, \omega)$ of (V, ω) is the Lie group of all symplectomorphisms of (V, ω).

We define

$$\Gamma_+ = \big\{T \in \mathrm{Sp}(V, \omega) : \det(T - \mathrm{Id}) > 0\big\},$$

$$\Gamma_- = \big\{T \in \mathrm{Sp}(V, \omega) : \det(T - \mathrm{Id}) < 0\big\}$$

and

$$\Gamma_0 = \big\{T \in \mathrm{Sp}(V, \omega) : \det(T - \mathrm{Id}) = 0\big\};$$

we also set $\Gamma_\pm = \Gamma_+ \cup \Gamma_-$.

We have that Γ_+ and Γ_- are open arc-connected subsets of $\mathrm{Sp}(V, \omega)$.

A subspace $L \subset V$ is said to be a *Lagrangian subspace* if

$$\omega|_L = 0 \quad \text{and} \quad \dim(L) = n.$$

We denote by $\Lambda(V, \omega)$, or more simply by Λ, the set of all Lagrangian subspaces of (V, ω):

$$\Lambda(V, \omega) = \Lambda = \big\{L \subset V : L \text{ is a Lagrangian subspace of } (V, \omega)\big\}.$$

The set Λ is called the *Lagrangian Grassmannian* of the symplectic space (V, ω) and it is a compact, connected real-analytic $\frac{1}{2}n(n+1)$-dimensional embedded submanifold of the Grassmannian of all n-dimensional subspaces of V.

Given a pair L_0, L_1 of complementary Lagrangian subspaces of (V, ω), then every subspace L which is complementary to L_1 is the graph of a unique linear map $T : L_0 \to L_1$. Identifying L_1 with the dual space L_0^* by the map $v \to \omega(v, \cdot)|_{L_0}$, then the subspace L is Lagrangian if and only if T corresponds to a *symmetric* bilinear form on L_0. The map φ_{L_0,L_1} given by:

$$\varphi_{L_0,L_1}(L) = \omega(T\,\cdot\,, \cdot)|_{L_0},$$

where $T : L_0 \to L_1$ is the unique linear map whose graph is L, is a local chart on Λ. The domain of φ_{L_0,L_1} is the set $\Lambda_0(L_1)$ of all Lagrangian subspaces that are complementary to L_1, and it takes values in the vector space $\mathrm{B}_{\mathrm{sym}}(L_0)$ of symmetric bilinear forms on L_0.

The differential of the chart φ_{L_0,L_1} at the point L_0 gives an isomorphism

$$\mathrm{d}\varphi_{L_0,L_1}(L_0) : T_{L_0}\Lambda \longrightarrow \mathrm{B}_{\mathrm{sym}}(L_0);$$

an easy computation shows that such isomorphism does not depend on the choice of the complementary Lagrangian L_1 to L_0. This observation allows us to identify for each $L \in \Lambda$ the tangent space $T_L\Lambda$ with $\mathrm{B}_{\mathrm{sym}}(L)$.

Let $L_0 \in \Lambda$ be fixed. We define the following subsets of Λ:

$$\Lambda_k(L_0) = \big\{L \in \Lambda : \dim(L \cap L_0) = k\big\}, \quad k = 0, \dots, n.$$

Each $\Lambda_k(L_0)$ is a connected real-analytic embedded submanifold of Λ having codimension $\frac{1}{2}k(k+1)$ in Λ; $\Lambda_0(L_0)$ is a dense open contractible subset of Λ, while its complementary set

$$\Lambda_{\geq 1}(L_0) = \bigcup_{k=1}^{n} \Lambda_k(L_0)$$

is *not* a regular submanifold of Λ. It is an analytic subset of Λ and its regular part is given by $\Lambda_1(L_0)$, which is a dense open subset of $\Lambda_{\geq 1}(L_0)$. Observe that $\Lambda_1(L_0)$ has codimension 1 in Λ; moreover, it has a *natural* transverse orientation in Λ, which is canonically associated to the symplectic form ω.

It's well known that $\pi_1(\Lambda) \cong \mathbb{Z}$. Using the Hurewicz's homomorphism, we conclude that the first singular homology group $H_1(\Lambda)$ is isomorphic to $\mathbb{Z}$. The choice of a sign for such isomorphism is canonically associated to the transverse orientation of $\Lambda_1(L_0)$ in Λ. Since $\Lambda_0(L_0)$ is contractible, the inclusion

$$(\Lambda, \emptyset) \to \big(\Lambda, \Lambda_0(L_0)\big)$$

induces an isomorphism in the first relative homology group; we then get an isomorphism

$$\mu_{L_0} : H_1\big(\Lambda, \Lambda_0(L_0)\big) \xrightarrow{\cong} \mathbb{Z}. \tag{2.1}$$

Definition 2.1. Let $\ell : [a, b] \to \Lambda$ be an arbitrary continuous curve with endpoints in $\Lambda_0(L_0)$. We denote by $\mu_{L_0}(\ell) \in \mathbb{Z}$ the integer number that corresponds to the homology class of ℓ in $H_1(\Lambda, \Lambda_0(L_0))$ by the isomorphism (2.1). The number $\mu_{L_0}(\ell)$ is called the *Maslov index* of the curve ℓ relative to the Lagrangian L_0.

Clearly, the Maslov index is additive by concatenation and invariant by homotopies of curves with endpoints in $\Lambda_0(L_0)$.

Given a symmetric bilinear form $\mathfrak{b}$ on a real vector space W, we define the index of $\mathfrak{b}$ by

$$n_-(\mathfrak{b}) = \sup\big\{\dim(W') : W' \text{ is a } \mathfrak{b}\text{-negative subspace of } W\big\}$$

and the co-index of $\mathfrak{b}$ by

$$n_+(\mathfrak{b}) = n_-(-\mathfrak{b});$$

the *signature* $\operatorname{sgn}(\mathfrak{b})$ is defined to be the difference $n_+(\mathfrak{b}) - n_-(\mathfrak{b})$. We have the following method for computing the Maslov index of a curve in Λ with endpoints in $\Lambda_0(L_0)$.

Theorem 2.2. *Let $\ell : [a, b] \to \Lambda$ be a continuous curve with endpoints in $\Lambda_0(L_0)$. Then*

1. *If there exists a Lagrangian $L_1 \in \Lambda$ complementary to L_0 such that the image of ℓ is contained in $\Lambda_0(L_1)$, then*

$$\mu_{L_0}(\ell) = n_+\Big(\varphi_{L_0,L_1}\big(\ell(b)\big)\Big) - n_+\Big(\varphi_{L_0,L_1}\big(\ell(a)\big)\Big);$$

2. *If ℓ intercepts $\Lambda_{\geq 1}(L_0)$ only when $t = t_0$ and if $\ell|_{[t_0-\varepsilon,t_0]}$ and $\ell|_{[t_0,t_0+\varepsilon]}$ are of class C^1, for some $\varepsilon > 0$, then*

$$\mu_{L_0}(\ell) = n_+\big(\ell'(t_0^+)|_{\ell(t_0)\cap L_0}\big) - n_-\big(\ell'(t_0^-)|_{\ell(t_0)\cap L_0}\big)$$

provided that the symmetric bilinear forms $\ell'(t_0^+)|_{\ell(t_0)\cap L_0}$ and $\ell'(t_0^-)|_{\ell(t_0)\cap L_0}$ are both nondegenerate.

Proof. See [12]. □

We say that a C^1 curve ℓ in Λ has a *nondegenerate* intersection with $\Lambda_{\geq 1}(L_0)$ at $t = t_0$ if

$$\ell(t_0) \in \Lambda_{\geq 1}(L_0)$$

and $\ell'(t_0)|_{\ell(t_0)\cap L_0}$ is nondegenerate. It is known that nondegenerate intersections with $\Lambda_{\geq 1}(L_0)$ are isolated.

We will now extend the definition of Maslov index to the class of continuous curves in Λ whose endpoints do not necessarily belong to $\Lambda_0(L_0)$.

Definition 2.3. Let $\ell : [a, b] \to \Lambda$ be any continuous curve; the *Maslov index* $\mu_{L_0}(\ell)$ is the semi-integer defined by:

$$\mu_{L_0}(\ell) = \mu_{L_0}(\bar\ell) - n_+\big(\bar\ell'(a^-)|_{\ell(a)\cap L_0}\big) - n_+\big(\bar\ell'(b^+)|_{\ell(b)\cap L_0}\big) + \frac{\dim\big(\ell(a)\cap L_0\big) + \dim\big(\ell(b)\cap L_0\big)}{2}, \tag{2.2}$$

where $\bar\ell : [a-\varepsilon, b+\varepsilon] \to \Lambda$ is a continuous extension of ℓ satisfying the following properties:

– $\bar\ell|_{[a-\varepsilon,a]}$ and $\bar\ell|_{[b,b+\varepsilon]}$ are of class C^1;
– $\bar\ell'(a^-)|_{\ell(a)\cap L_0}$ and $\bar\ell'(b^+)|_{\ell(b)\cap L_0}$ are nondegenerate;
– $\bar\ell(t) \in \Lambda_0(L_0)$ for $t \in [a-\varepsilon, a[\,\cup\,]b, b+\varepsilon]$.

The right hand side of (2.2) does not depend on the extension $\bar\ell$ satisfying the properties above.

The notion of Maslov index for continuous curves in Λ given in Definition 2.3 coincides with that of Robbin and Salamon in [17]; we summarize below the main properties of μ_{L_0} proven in [17]:

Proposition 2.4. *The quantity $\mu_{L_0}(\ell) \in \frac{1}{2}\mathbb{Z}$ is additive by concatenation and it is invariant by homotopies with fixed endpoints in Λ; moreover, if ℓ is a C^1 curve having only nondegenerate intersections with $\Lambda_{\ge 1}(L_0)$, then the following identity hold:*

$$\mu_{L_0}(\ell) = n_+\big(\ell'(a)|_{\ell(a)\cap L_0}\big) + n_+\big(\ell'(b)|_{\ell(b)\cap L_0}\big) - \frac{\dim\big(\ell(a)\cap L_0\big) + \dim\big(\ell(b)\cap L_0\big)}{2} + \sum_{t\in]a,b[} \operatorname{sgn}\big(\ell'(t)|_{\ell(t)\cap L_0}\big). \tag{2.3}$$

We now pass to the study of curves in the symplectic group. Denote by $\bar\omega$ the symplectic form on $V \oplus V$ given by

$$\bar\omega = \omega \oplus (-\omega); \tag{2.4}$$

observe that $T : V \to V$ is a symplectomorphism if and only if

$$\mathrm{Gr}(T) \subset V \oplus V$$

is $\bar\omega$-Lagrangian, where $\mathrm{Gr}(T)$ denotes the graph of T. We get a map

$$\varphi : \mathrm{Sp}(V, \omega) \ni \Phi \longmapsto \mathrm{Gr}(\Phi) \in \Lambda(V \oplus V, \bar\omega),$$

which is a diffeomorphism onto an open subset of $\Lambda(V \oplus V, \bar\omega)$. Denote by Δ the *diagonal* of $V \oplus V$, which is obviously a Lagrangian subspace of $(V \oplus V, \bar\omega)$. Given

$\Phi \in \mathrm{Sp}(V, \omega)$, it's easily seen that $\Phi \in \Gamma_0$ if and only if $\mathrm{Gr}(\Phi)$ is not transverse to Δ, i.e.:

$$\varphi^{-1}\big(\Lambda_{\geq 1}(\Delta)\big) = \Gamma_0.$$

Using local charts, the differential of the map φ is computed as follows:

$$\mathrm{d}\varphi(\Phi) \cdot A = -\omega(A\pi_1 \cdot, \Phi\pi_1 \cdot) \in \mathrm{B}_{\mathrm{sym}}\big(\mathrm{Gr}(\Phi)\big), \tag{2.5}$$

where $\pi_1 : \mathrm{Gr}(\Phi) \to V$ is the projection onto the first coordinate and $A \in T_\Phi \mathrm{Sp}(V, \omega)$. Now we can give the following

Definition 2.5. Let $\Phi : [a, b] \to \mathrm{Sp}(V, \omega)$ be a continuous curve in the symplectic group; we define the *Maslov index* of Φ to be the semi-integer:

$$\mathrm{i}_{\mathrm{Maslov}}(\Phi) = \mu_\Delta(\varphi \circ \Phi).$$

As observed in [17], for continuous curves Φ with $\Phi(a) = \mathrm{Id}$ and $\Phi(b) \in \Gamma_\pm$, the above definition of Maslov index is equivalent to the one given in [5] and [11].

Remark 2.6. Using Proposition 2.4 and formula (2.5), it is possible to obtain a simple method for computing the Maslov index of a curve Φ in $\mathrm{Sp}(V, \omega)$ in the case that Φ is of class C^1 and $\varphi \circ \Phi$ has only nondegenerate intersections with $\Lambda_{\geq 1}(\Delta)$. To this aim, observe that the projection onto the first coordinate of $V \oplus V$ restricts to an isomorphism from $\varphi(\Phi(t)) \cap \Delta$ to $\mathrm{Ker}(\Phi(t) - \mathrm{Id})$ which carries the restriction of $(\varphi \circ \Phi)'(t)$ to the restriction of $-\omega(\Phi'(t) \cdot, \cdot)$.

From now on, we set $V = \mathbb{R}^n \oplus \mathbb{R}^{n*} \cong \mathbb{R}^{2n}$ and we denote by ω the canonical symplectic form given by

$$\omega\big((v_1, \alpha_1), (v_2, \alpha_2)\big) = \alpha_2(v_1) - \alpha_1(v_2),$$

where $v_1, v_2 \in \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathbb{R}^{n*}$. We also set

$$L_0 = \{0\} \oplus \mathbb{R}^{n*}$$

and we write $\mathrm{Sp}(2n, \mathbb{R})$ instead of $\mathrm{Sp}(\mathbb{R}^{2n}, \omega)$. If

$$\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.6}$$

is a representation of Φ in block matrix form, we have that $\Phi \in \mathrm{Sp}(2n, \mathbb{R})$ if and only if A^*C, B^*D are $n \times n$ symmetric matrices and $D^*A - BC^* = \mathrm{Id}_n$, where Id_n denotes the $n \times n$ identity matrix.

We define a map $\beta : \mathrm{Sp}(2n, \mathbb{R}) \to \Lambda$ by

$$\beta(\Phi) = \Phi(L_0), \qquad \forall \Phi \in \mathrm{Sp}(2n, \mathbb{R}).$$

Using local coordinates, it is easy to show that the differential $\mathrm{d}\beta(\Phi)$ is given by:

$$\mathrm{d}\beta(\Phi) \cdot A = \omega(A\Phi^{-1} \cdot, \cdot)|_{\Phi(L_0)} \in \mathrm{B}_{\mathrm{sym}}\big(\Phi(L_0)\big), \tag{2.7}$$

for all $\Phi \in \mathrm{Sp}(2n, \mathbb{R})$ and for all $A \in T_\Phi \mathrm{Sp}(2n, \mathbb{R})$.

Obviously, $\beta^{-1}(\Lambda_0(L_0)) \subset \mathrm{Sp}(2n, \mathbb{R})$ is open. Writing a symplectomorphism $\Phi \in \mathrm{Sp}(2n, \mathbb{R})$ as in (2.6), we have

$$\Phi \in \beta^{-1}\big(\Lambda_0(L_0)\big) \iff B \text{ is invertible.}$$

Definition 2.7. We define a map

$$\tau : \beta^{-1}\big(\Lambda_0(L_0)\big) \to \mathrm{B}_{\mathrm{sym}}(\mathbb{R}^n)$$

by setting, for $\Phi \in \beta^{-1}(\Lambda_0(L_0))$,

$$\tau(\Phi)(v, w) = (\delta - \alpha)w, \qquad \forall\, v, w \in \mathbb{R}^n,$$

where $\alpha, \delta \in \mathbb{R}^{n*}$ are the (uniquely determined) covectors such that $\Phi(v, \alpha) = (v, \delta)$. Using the block matrix representation of Φ as in (2.6), we have that $\tau(\Phi)$ is given by

$$\tau(\Phi) = C + (D - \mathrm{Id}_n)B^{-1}(\mathrm{Id}_n - A).$$

Observe that the symmetry of $\tau(\Phi)$ is a consequence of the fact that Φ is a symplectomorphism; moreover,

$$\Phi \in \Gamma_0 \iff \tau(\Phi) \text{ is degenerate.} \tag{2.8}$$

The derivative of τ is computed in the next lemma:

Lemma 2.8. *For all $\Phi \in \beta^{-1}(\Lambda_0(L_0))$, the projection onto the first coordinate of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ maps $\mathrm{Ker}(\Phi - \mathrm{Id})$ isomorphically onto $\mathrm{Ker}(\tau(\Phi))$.*

For $A \in T_\Phi \mathrm{Sp}(2n, \mathbb{R})$, this isomorphism carries the restriction of $-\omega(A\,\cdot\,, \cdot\,)$ to the restriction of

$$\mathrm{d}\tau(\Phi) \cdot A \in \mathrm{B}_{\mathrm{sym}}(\mathbb{R}^n).$$

Proof. The fact that the projection onto the first coordinate of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ maps $\mathrm{Ker}(\Phi - \mathrm{Id})$ isomorphically onto $\mathrm{Ker}(\tau(\Phi))$ follows easily from Definition 2.7. We'll now compute the restriction of $\mathrm{d}\tau(\Phi) \cdot A$ to $\mathrm{Ker}(\tau(\Phi))$. To this aim, consider a smooth curve $\Phi(t)$ with $\Phi(0) = \Phi$ and $\Phi'(0) = A$; define $\tau(t) = \tau(\Phi(t))$. For all t and for all $v, w \in \mathrm{Ker}(\tau(\Phi))$, we have

$$\tau(t)(v, w) = \big(\delta(t) - \alpha(t)\big)w, \tag{2.9}$$

where $\alpha(t), \delta(t) \in \mathbb{R}^{n*}$ are (the uniquely determined) smooth functions such that

$$\Phi(t)\big(v, \alpha(t)\big) = \big(v, \delta(t)\big). \tag{2.10}$$

Let $\beta \in \mathbb{R}^{n*}$ be such that $\Phi(w, \beta) = (w, \beta)$; differentiating formula (2.10) at $t = 0$ and applying $\omega(\,\cdot\,, (w, \beta))$ to the result, we obtain:

$$\omega\Big(A\big(v, \alpha(0)\big), (w, \beta)\Big) - \alpha'(0)w = -\delta'(0)w.$$

The conclusion follows by differentiating (2.9) at $t = 0$ and using the above equality. □

We use the map τ to determine the connected components of $\beta^{-1}(\Lambda_0(L_0))\cap\Gamma_\pm$:

Lemma 2.9. *The open set* $\beta^{-1}(\Lambda_0(L_0))\cap\Gamma_\pm$ *in* $\mathrm{Sp}(2n,\mathbb{R})$ *has* $2(n+1)$ *connected components; more explicitly, we have that* $\Phi_1,\Phi_2\in\beta^{-1}(\Lambda_0(L_0))\cap\Gamma_\pm$ *are in the same connected component of* $\beta^{-1}(\Lambda_0(L_0))\cap\Gamma_\pm$ *if and only if* $\tau(\Phi_1)$ *and* $\tau(\Phi_2)$ *have the same index and* $\det(B_1)$, $\det(B_2)$ *have the same sign;* B_i *denotes the* $n\times n$ *right upper block of* Φ_i, $i=1,2$ *here.*

Proof. Write each $\Phi\in\beta^{-1}(\Lambda_0(L_0))$ in block matrix form as in (2.6); then Φ can be written uniquely as a product

$$\Phi=\begin{pmatrix}0 & B\\ -B^{*-1} & SB\end{pmatrix}\begin{pmatrix}\mathrm{Id}_n & 0\\ U & \mathrm{Id}_n\end{pmatrix}\tag{2.11}$$

with S,U symmetric $n\times n$ matrices; namely, take $S=DB^{-1}$ and $U=B^{-1}A$. Using (2.11), we get a diffeomorphism

$$\mathrm{B}_{\mathrm{sym}}(\mathbb{R}^n)\times\mathrm{B}_{\mathrm{sym}}(\mathbb{R}^n)\times GL(n,\mathbb{R})\ni(S,U,B)\longmapsto\Phi\in\beta^{-1}\big(\Lambda_0(L_0)\big).\tag{2.12}$$

In terms of S, U, B, $\tau(\Phi)$ can be written as

$$\tau(\Phi)=S+U-B^{-1}-B^{*-1}.$$

The conclusion now follows easily from (2.8). □

Now we prove the main result of the section:

Lemma 2.10. *Let* $\Phi:[a,b]\to\mathrm{Sp}(2n,\mathbb{R})$ *be a continuous curve in the symplectic group such that* $\Phi(a),\Phi(b)\in\beta^{-1}(\Lambda_0(L_0))\cap\Gamma_\pm$. *If* $\ell=\beta\circ\Phi$, *then*

$$\mathrm{i}_{\mathrm{Maslov}}(\Phi)+\mu_{L_0}(\ell)=n_+\big(\tau(\Phi(b))\big)-n_+\big(\tau(\Phi(a))\big).\tag{2.13}$$

Proof. The proof follows the steps below:

1. $\mathrm{i}_{\mathrm{Maslov}}(\Phi)+\mu_{L_0}(\beta\circ\Phi)=0$ *if* Φ *is closed*. To see this, recall that the unitary group $\mathrm{U}(n)$ is a deformation retract of $\mathrm{Sp}(2n,\mathbb{R})$ and that the determinant map $\det:\mathrm{U}(n)\to S^1$ induces an isomorphism on the fundamental group. Therefore, $H_1(\mathrm{Sp}(2n,\mathbb{R}))\cong\pi_1(\mathrm{Sp}(2n,\mathbb{R}))\cong\mathbb{Z}$; if a loop $\Phi:[a,b]\to\mathrm{U}(n)$ is such that $\det\circ\Phi$ is a generator of $\pi_1(S^1)$ then Φ is a generator of $H_1(\mathrm{Sp}(2n,\mathbb{R}))\cong\mathbb{Z}$. Using Theorem 2.2 and Remark 2.6, one can compute easily that $\mathrm{i}_{\mathrm{Maslov}}(\Phi)+\mu_{L_0}(\beta\circ\Phi)=0$ on one such generator of $H_1(\mathrm{Sp}(2n,\mathbb{R}))$.

2. $\mathrm{i}_{\mathrm{Maslov}}(\Phi)+\mu_{L_0}(\beta\circ\Phi)$ *depends only on the connected components of the open set*

$$\beta^{-1}\big(\Lambda_0(L_0)\big)\cap\Gamma_\pm$$

that contain the endpoints of Φ. From step (1) we see that $\mathrm{i}_{\mathrm{Maslov}}(\Phi)+\mu_{L_0}(\beta\circ\Phi)$ depends only on the endpoints of Φ. The proof of step (2) is concluded with the observation that both sides of (2.13) are additive by concatenation of curves and both sides of (2.13) vanish on curves which are entirely contained in

$$\beta^{-1}\big(\Lambda_0(L_0)\big)\cap\Gamma_\pm.$$

3. *Equality* (2.13) *holds in general*. For this, it suffices to show that for each pair $(\mathcal{C}_1, \mathcal{C}_2)$ of connected components of $\beta^{-1}(\Lambda_0(L_0)) \cap \Gamma_\pm$ there exists a curve Φ with $\Phi(a) \in \mathcal{C}_1$, $\Phi(b) \in \mathcal{C}_2$ and such that (2.13) holds. This is easily done using Lemma 2.9, Theorem 2.2 and Remark 2.6. □

In the proof of the index theorem in Section 4 we will actually need a version of the above result in the case that $\Phi(b)$ may be in Γ_0. This is done in the following:

Corollary 2.11. *Let* Φ *be a curve as in the statement of Lemma* 2.10*, except for the fact that* $\Phi(b)$ *may belong to* Γ_0.

Then, the following equality holds:

$$\mathrm{i}_{\mathrm{Maslov}}(\Phi) + \mu_{L_0}(\ell) = n_+\big(\tau(\Phi(b))\big) - n_+\big(\tau(\Phi(a))\big) + \frac{1}{2}\dim\big(\mathrm{Ker}(\Phi(b) - \mathrm{Id})\big).$$

Proof. Let $\lambda : [b, b+\varepsilon] \to \mathrm{Sp}(2n, \mathbb{R})$ be a C^1 curve such that $\lambda(b) = \Phi(b)$ and such that $\omega(\lambda'(b)\lambda(b)^{-1}\cdot, \cdot)$ is a positive definite symmetric bilinear form in $\mathbb{R}^n \oplus \mathbb{R}^{n*}$. We can clearly assume that $\lambda(t) \in \Gamma_\pm \cap \beta^{-1}(\Lambda_0(L_0))$ for $t \in\,]b, b+\varepsilon]$. The conclusion is obtained by applying Lemma 2.10 to the concatenation of Φ and λ, observing the followings facts:

- $\mathrm{i}_{\mathrm{Maslov}}(\lambda) = -\frac{1}{2}\dim(\mathrm{Ker}(\Phi(b) - \mathrm{Id}))$ (see Remark 2.6);
- $\mu_{L_0}(\beta \circ \lambda) = 0$;
- $n_+(\tau(\lambda(b+\varepsilon))) = n_+(\tau(\Phi(b)))$, see Lemma 2.8.

Actually Lemma 2.8 implies that $(\tau \circ \lambda)'(b)$ is negative definite on $\mathrm{Ker}(\tau(\lambda(b)))$ and therefore the function $n_+(\tau(\lambda(t)))$ is right continuous at $t = b$ (see [12, Lemma 4.2.3]). □

3. Geodesics and the Maslov index

In this section we show how the flow of the Jacobi equation along a geodesic produces a curve in the symplectic group. We start with some generalities on Hamiltonian systems.

Let M be a differentiable manifold and let TM^* be its cotangent bundle. We denote by θ the canonical 1-form of TM^* and by $\omega = -\mathrm{d}\theta$ the canonical symplectic form of TM^*. Using a chart $(q_i)_{i=1}^n$ in M and the corresponding chart $(q_i, p_i)_{i=1}^n$ in TM^*, one has

$$\theta = \sum_{i=1}^n p_i \mathrm{d}q_i, \quad \omega = \sum_{i=1}^n \mathrm{d}q_i \wedge \mathrm{d}p_i.$$

Let $H : TM^* \to \mathbb{R}$ be a smooth map, called the *Hamiltonian*, and let $\vec{H}$ be the symplectic gradient of H which is defined by $\mathrm{d}H = \omega(\vec{H}, \cdot)$; we call $\vec{H}$ *the Hamiltonian vector field* associated to H. By *a solution of* H we mean an integral curve of $\vec{H}$.

We denote by F the flow of the vector field $\vec{H}$, that is, for all $p \in TM^*$ the curve $t \mapsto F(t, p) \in TM^*$ is a maximal solution of the Hamiltonian H, with $F(0, p) = p$, for all $p \in TM^*$; it is well known that $F_t = F(t, \cdot)$ is a symplectomorphism between open subsets of TM^*.

We will consider the special case that M is endowed with a semi-Riemannian metric $\mathfrak{g}$, i.e., $\mathfrak{g}$ is a nondegenerate metric tensor, and $H : TM^* \to \mathbb{R}$ is the *geodesic Hamiltonian* $H(q, p) = \frac{1}{2}\mathfrak{g}^{-1}(p, p)$. Denote by ∇ the Levi-Civita connection and by $\mathcal{R}$ its curvature tensor chosen with sign convention $\mathcal{R}(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$. We have that the solutions of the Hamiltonian H are of the form $t \mapsto (\gamma(t), \mathfrak{g} \cdot \dot{\gamma}(t))$, where $\gamma : [a, b] \to M$ is a geodesic and $\mathfrak{g}$ is thought as a linear map from the tangent space to its dual. For each $(q, p) \in TM^*$, the Levi-Civita connection ∇ induces a decomposition

$$T_{(q,p)}TM^* \cong \mathrm{Hor}_{(q,p)} \oplus \mathrm{Ver}_{(q,p)},$$

where $\mathrm{Ver}_{(q,p)} = \mathrm{Ker}(\mathrm{d}\pi_{(q,p)}) = T_p(T_qM)^*$. We identify $\mathrm{Hor}_{(q,p)}$ with T_qM using $\mathrm{d}\pi_{(q,p)}$ and $\mathrm{Ver}_{(q,p)} \cong T_qM^*$. Therefore, we have

$$T_{(q,p)}TM^* \cong T_qM \oplus T_qM^*.$$

Given $\xi \in T_{(q,p)}TM^*$, we write $\xi = (\mathfrak{v}, \mathfrak{a})$, with $\mathfrak{v} \in T_qM$ and $\mathfrak{a} \in T_qM^*$. Using Cartan's formula to compute $\mathrm{d}\theta$, one can show

$$\omega_{(q,p)}\big((\mathfrak{v}_1, \mathfrak{a}_1), (\mathfrak{v}_2, \mathfrak{a}_2)\big) = \mathfrak{a}_2(\mathfrak{v}_1) - \mathfrak{a}_1(\mathfrak{v}_2). \tag{3.1}$$

Let $\gamma : [a, b] \to M$ be a geodesic. Let $(\mathcal{V}_i)_{i=1}^n$ be a smooth referential of TM along γ, that is, for each $t \in [a, b]$, $(\mathcal{V}_i(t))_{i=1}^n$ is a basis for $T_{\gamma(t)}M$. We denote by $(\mathcal{V}_i^*(t))_{i=1}^n$ the dual basis of $(\mathcal{V}_i(t))_{i=1}^n$.

Let $\Gamma(t) = (\gamma(t), \mathfrak{g} \cdot \dot{\gamma}(t))$ be the solution of H corresponding to γ. Obviously, $\Gamma(t) = F_{t-a}(\Gamma(a))$ for $t \in [a, b]$; since the Jacobi equation is the linearization of the geodesic equation, it follows easily that for any Jacobi field $\mathfrak{v}$ along γ we have

$$\mathrm{d}F_{t-a}\big(\Gamma(a)\big) \cdot \big(\mathfrak{v}(a), \mathfrak{g} \cdot \mathfrak{v}'(a)\big) = \big(\mathfrak{v}(t), \mathfrak{g} \cdot \mathfrak{v}'(t)\big), \tag{3.2}$$

for $t \in [a, b]$, where $\mathfrak{v}'$ denotes the covariant derivative of $\mathfrak{v}$ along γ.

Define a referential of $T(TM^*)$ along Γ by setting $\xi = (\xi_i)_{i=1}^{2n}$, where

$$\xi_i = (\mathcal{V}_i, 0), \quad \xi_{n+i} = (0, \mathcal{V}_i^*), \qquad i = 1, \dots, n.$$

Let $\psi_t : T_{\Gamma(t)}TM^* \longrightarrow \mathbb{R}^n \oplus \mathbb{R}^{n*}$ be the isomorphism that carries the basis $(\xi_i)_{i=1}^{2n}$ into the canonical basis. Using (3.1), it is easy to see that ψ_t is a symplectomorphism. Therefore, the map

$$\Phi_t = \psi_t \circ \mathrm{d}F_{t-a}\big(\Gamma(a)\big) \circ \psi_a^{-1} \tag{3.3}$$

is an element of $\mathrm{Sp}(2n, \mathbb{R})$, for all $t \in [a, b]$. We say that $(\psi_t)_{t\in[a,b]}$ is a *symplectic trivialization* along γ.

If $\mathfrak{v}$ is a Jacobi field along γ and if $\mathfrak{a} = \mathfrak{g} \cdot \mathfrak{v}'$, then setting $(v(t), \alpha(t)) = \psi_t(\mathfrak{v}(t), \mathfrak{a}(t))$, it follows from (3.2) and (3.3) that $(v(t), \alpha(t)) = \Phi(t)(v(a), \alpha(a))$; moreover, it is easy to show that (v, α) is a solution of the first order linear homogeneous system

$$\begin{pmatrix} v \\ \alpha \end{pmatrix}' = X \begin{pmatrix} v \\ \alpha \end{pmatrix}, \tag{3.4}$$

with

$$X = \begin{pmatrix} -\sigma & g^{-1} \\ gR & \sigma^* \end{pmatrix}$$

where σ, g and R are defined by $\mathcal{V}_i{}' = \Sigma_{j=1}^n \sigma_{ji} \mathcal{V}_j$, $g_{ij} = \mathfrak{g}(\mathcal{V}_i, \mathcal{V}_j)$ and

$$\mathcal{R}(\dot\gamma, \mathcal{V}_j)\dot\gamma = \sum_{i=1}^n R_{ij} \mathcal{V}_i.$$

It follows that

$$\Phi'(t) = X(t)\Phi(t), \tag{3.5}$$

for $t \in [a, b]$. Observe that X is a smooth curve in the Lie algebra of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$; in the terminology of [13] and [16] a system of the form (3.4) is called a *symplectic differential system*. Symplectic differential systems are more generally obtained using a symplectic trivialization along a solution of a (possibly time-dependent) Hamiltonian system in an arbitrary symplectic manifold endowed with a Lagrangian distribution, see [13, Section 3] for details.

Assume now that γ is a *periodic* geodesic, i.e., $\gamma(a) = \gamma(b)$ and $\dot\gamma(a) = \dot\gamma(b)$. Choose the referential $(\mathcal{V}_i)_{i=1}^n$ along γ in such a way that $\mathcal{V}_i(a) = \mathcal{V}_i(b)$, $i = 1, \ldots, n$. We'll say, in this case, that the corresponding symplectic trivialization $(\psi_t)_{t\in[a,b]}$ is *periodic*.

Recall that a closed curve $\gamma : [a, b] \to M$ is said to be *orientation preserving* if for some (and hence for any) continuous referential $(\mathcal{V}_i)_{i=1}^n$ along γ the bases $(\mathcal{V}_i(a))_{i=1}^n$ and $(\mathcal{V}_i(b))_{i=1}^n$ determine the same orientation of $T_{\gamma(a)}M = T_{\gamma(b)}M$. For instance, if M is orientable then any γ is orientation preserving. It is easy to prove that if γ is orientation preserving then there exists a smooth referential $(\mathcal{V}_i)_{i=1}^n$ along γ with $\mathcal{V}_i(a) = \mathcal{V}_i(b)$, $i = 1, \ldots, n$.

We define the *nullity* $\mathrm{null}(\gamma)$ of γ to be the dimension of the space of Jacobi fields $\mathfrak{v}$ along γ such that $\mathfrak{v}(a) = \mathfrak{v}(b)$ and $\mathfrak{v}'(a) = \mathfrak{v}'(b)$. Obviously,

$$\mathrm{null}(\gamma) = \dim\Big(\mathrm{Ker}\big(\Phi(b) - \mathrm{Id}\big)\Big). \tag{3.6}$$

Definition 3.1. If γ is an orientation preserving periodic geodesic, define the *Maslov index* of γ by

$$\mathrm{i}_{\mathrm{Maslov}}(\gamma) = \mathrm{i}_{\mathrm{Maslov}}(\Phi),$$

where Φ corresponds to γ by the choice of a periodic symplectic trivialization $(\psi_t)_{t\in[a,b]}$ and $\mathrm{i}_{\mathrm{Maslov}}(\Phi)$ was introduced in Definition 2.5.

It is not difficult to show that $\mathrm{i}_{\mathrm{Maslov}}(\Phi)$ does not depend on the choice of the periodic symplectic trivialization $(\psi_t)_{t\in[a,b]}$.

The curve in $\mathrm{Sp}(2n, \mathbb{R})$ corresponding to the choice of a different periodic symplectic trivialization is of the form $\widetilde{\Phi}_t = \phi_t \Phi_t \phi_a^{-1}$, where

$$\phi_t = \begin{pmatrix} A(t) & 0 \\ 0 & A(t)^{*-1} \end{pmatrix}$$

and A is a loop in the general linear group of $\mathbb{R}^n$. The loop ϕ in $\mathrm{Sp}(2n, \mathbb{R})$ is contractible, which implies that $\mathrm{i}_{\mathrm{Maslov}}(\Phi) = \mathrm{i}_{\mathrm{Maslov}}(\widetilde{\Phi})$.

So $\mathrm{i}_{\mathrm{Maslov}}(\gamma)$ is indeed well defined.

4. The index theorem

In this section we state and prove the main theorem of the paper. As in the previous section, $(M, \mathfrak{g})$ denotes a semi-Riemannian manifold. Let $\Omega(M)$ be the set of closed curves $\eta : [a, b] \to M$ of Sobolev class H^1, i.e.,

$$\Omega(M) = \big\{\eta : [a, b] \xrightarrow{H^1} M : \eta(a) = \eta(b)\big\}.$$

We define the *action functional* $f : \Omega(M) \to \mathbb{R}$ by

$$f(\eta) = \frac{1}{2} \int_a^b \mathfrak{g}\big(\eta'(t), \eta'(t)\big)\mathrm{d}t.$$

It is well known that $\Omega(M)$ has the structure of an infinite dimensional Hilbert manifold and that f is a smooth map; moreover, it is easy to see that the critical points of f in $\Omega(M)$ are the periodic geodesics. If $\gamma \in \Omega(M)$ is a periodic geodesic, the second variation of f at γ, or *periodic index form* (denoted by I^{per}), is defined on the space $T_\gamma\Omega(M) = \{\mathfrak{v}$ vector field along γ of class H^1: $\mathfrak{v}(a) = \mathfrak{v}(b)\}$ and it is given by

$$I^{\mathrm{per}}(\mathfrak{v}, \mathfrak{w}) = \mathrm{d}^2 f_\gamma(\mathfrak{v}, \mathfrak{w}) = \int_a^b \Big[\mathfrak{g}(\mathfrak{v}', \mathfrak{w}') + \mathfrak{g}\big(\mathcal{R}(\gamma', \mathfrak{v})\gamma', \mathfrak{w}\big)\Big]\mathrm{d}t. \tag{4.1}$$

From now on we assume that $(M, \mathfrak{g})$ is a *Lorentzian manifold*, i.e., that the metric tensor $\mathfrak{g}$ has index 1. Recall that a Lorentzian manifold is called *stationary* if it admits a timelike Killing vector field, i.e., a Killing vector field Y with $\mathfrak{g}(Y, Y) < 0$.

Now we have the following:

Theorem 4.1. *Let $(M, \mathfrak{g})$ be a stationary Lorentzian manifold and Y a timelike Killing vector field.*

Set $\mathcal{N} = \{\eta \in \Omega(M) : \mathfrak{g}(\eta', Y) \equiv \mathrm{const}\}$. Then:

1. *$\mathcal{N}$ is a Hilbert submanifold of $\Omega(M)$;*
2. *The critical points of the action functional f restricted to $\mathcal{N}$ are precisely the periodic geodesics;*
3. *Let $\gamma : [a, b] \to M$ be an orientation preserving periodic geodesic. If $\gamma(b)$ is not conjugate to $\gamma(a)$ along γ then the index of I^{per} in $T_\gamma\mathcal{N}$ is given by:*

$$n_-(I^{\mathrm{per}}|_{T_\gamma\mathcal{N}}) = 1 - \mathrm{i}_{\mathrm{Maslov}}(\gamma) - \tfrac{1}{2}\,\mathrm{null}(\gamma).$$

Proof. The proof of (1) and (2) is analogous to the proof of [9, Proposition 3.1, Theorem 3.3], *mutatis mutandis*.

To prove step (3), set $\mathcal{K} = T_\gamma\mathcal{N}$; a simple computation yields

$$\mathcal{K} = \big\{\mathfrak{v} \in T_\gamma\Omega(M) : \mathfrak{g}(\mathfrak{v}', Y) - \mathfrak{g}(\mathfrak{v}, Y') \equiv \text{const}\big\}.$$

Define the spaces

$$\mathcal{K}^0 = \big\{\mathfrak{v} \in \mathcal{K} : \mathfrak{v}(a) = \mathfrak{v}(b) = 0\big\}$$

and

$$\mathcal{P} = \big\{\mathfrak{v}\ \text{Jacobi vector field along}\ \gamma : \mathfrak{v}(a) = \mathfrak{v}(b)\big\}.$$

Since $\gamma(b)$ is not conjugate to $\gamma(a)$ along γ, a Jacobi vector field along γ is uniquely determined by its values at the endpoints. Therefore

$$\mathcal{K} = \mathcal{K}^0 \oplus \mathcal{P}.$$

Integration by parts on (4.1) shows that $\mathcal{K}^0$ and P are I^{per}-orthogonal. Choose a periodic symplectic trivialization $(\psi_t)_{t\in[a,b]}$ along γ and define Φ as in (3.3). Using (3.2) and integration by parts in (4.1), it is easy to see that $I^{\text{per}}|_{\mathcal{P}}$ corresponds by the isomorphism

$$\mathcal{P} \ni \mathfrak{v} \longmapsto \mathfrak{v}(b) \in \mathbb{R}^n$$

to the symmetric bilinear form $\tau(\Phi(b))$ (see Definition 2.7). Hence

$$n_-(I^{\text{per}}|_{\mathcal{K}}) = n_-(I^{\text{per}}|_{\mathcal{K}^0}) + n_-\Big(\tau\big(\Phi(b)\big)\Big). \tag{4.2}$$

Consider the curve $\lambda_0 : [a - \pi/2, a] \to \mathrm{Sp}(2n, \mathbb{R})$ given by:

$$\lambda_0(t) = \begin{pmatrix} \cos(t-a)\,\mathrm{Id}_n & \sin(t-a)\,\mathrm{Id}_n \\ -\sin(t-a)\,\mathrm{Id}_n & \cos(t-a)\,\mathrm{Id}_n \end{pmatrix}.$$

Applying Corollary 2.11 to the concatenation $\lambda_0 \cdot \Phi$, we have that

$$\begin{aligned} &\mathrm{i}_{\text{Maslov}}(\lambda_0\cdot\Phi) + \mu_{L_0}(\beta\circ(\lambda_0\cdot\Phi)) \\ &\quad = n_+\big(\tau(\Phi(b))\big) - n_+\left(\tau\left(\lambda_0\left(a - \frac{\pi}{2}\right)\right)\right) + \frac{1}{2}\dim\big(\mathrm{Ker}(\Phi(b) - \mathrm{Id})\big). \end{aligned} \tag{4.3}$$

We now compute each one of the terms appearing in (4.3). Observe first that the dimension of $\mathrm{Ker}(\Phi(b) - \mathrm{Id})$ is equal to the nullity of γ (recall (3.6)).

By Remark 2.6, we have

$$\mathrm{i}_{\text{Maslov}}(\lambda_0 \cdot \Phi) = \mathrm{i}_{\text{Maslov}}(\Phi) + \mathrm{i}_{\text{Maslov}}(\lambda_0) = \mathrm{i}_{\text{Maslov}}(\Phi) - n. \tag{4.4}$$

Since $\tau(\lambda_0(a - \pi/2)) = 2\mathrm{Id}_n$, we have

$$n_+\left(\tau\left(\lambda_0\left(a - \frac{\pi}{2}\right)\right)\right) = n. \tag{4.5}$$

Setting $\ell = \beta \circ (\lambda_0 \cdot \Phi)$, we obtain:

$$\mu_{L_0}\big(\beta \circ (\lambda_0 \cdot \Phi)\big) = \mu_{L_0}(\ell|_{[a-\pi/2,a+\varepsilon]}) + \mu_{L_0}(\ell|_{[a+\varepsilon,b]}),$$

for $\varepsilon > 0$ small enough. In [8], it is shown that

$$\mu_{L_0}(\ell|_{[a+\varepsilon,b]}) = n_-(I^{\mathrm{per}}|_{\mathcal{K}^0}). \tag{4.6}$$

In order to compute $\mu_{L_0}(\ell|_{[a-\pi/2,a+\varepsilon]})$, we use Theorem 2.2. From (2.7) it follows that $\ell'(a^-)$ is positive definite; then

$$n_-\big(\ell'(a^-)\big) = 0. \tag{4.7}$$

Using (3.5) and (2.7) it follows that $\ell'(a^+)$ is nondegenerate and

$$n_+\big(\ell'(a^+)\big) = n - 1. \tag{4.8}$$

From (4.7) and (4.8), we have

$$\mu_{L_0}(\ell|_{[a-\pi/2,a+\varepsilon]}) = n - 1. \tag{4.9}$$

From (4.6) and (4.9) we get

$$\mu_{L_0}\big(\beta \circ (\lambda_0 \cdot \Phi)\big) = n_-(I^{\mathrm{per}}|_{\mathcal{K}^0}) + n - 1. \tag{4.10}$$

Observe that, using Lemma 2.8, we have the following equality:

$$\begin{aligned} n_+\big(\tau(\Phi(b))\big) &= n - n_-\big(\tau(\Phi(b))\big) - \dim\big(\mathrm{Ker}(\Phi(b) - \mathrm{Id})\big) \\ &= n - n_-\big(\tau(\Phi(b))\big) - \mathrm{null}(\gamma). \end{aligned} \tag{4.11}$$

From (4.3), (4.4), (4.5), (4.10) and (4.11) we conclude that

$$n_-\big(\tau(\Phi(b))\big) = 1 - \mathrm{i}_{\mathrm{Maslov}}(\gamma) - n_-(I^{\mathrm{per}}|_{\mathcal{K}^0}) - \tfrac{1}{2}\,\mathrm{null}(\gamma). \tag{4.12}$$

Substituting (4.12) in (4.2), we complete the proof of the theorem. □

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