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The 1881 problem of Morera revisited[1]

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Abstract. The equivalence between the Liouville problem and separability of the Hamilton–Jacobi equation established by Morera is reconsidered from the perspective of the group invariants of the associated Killing tensors. As a consequence, an apparently new algebraic method of determining separable coordinates for Hamiltonian systems with two degrees of freedom is obtained.

Keywords. Liouville systems, Hamilton–Jacobi equation, separation of variables, Killing tensors, first integrals.

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1. Introduction

This survey paper aims to review the results pertinent to, undoubtedly, one of the most studied problems of Classical Mechanics, namely, the problem of solving the system of differential equations describing the motion of particle on a curved surface through separation of variables. The study was initiated in the 1840's by J. Liouville [5], who was the first to consider a particle moving on a curved surface defined by a metric in isothermal coordinates. He demonstrated that if both the metric and potential part of the corresponding Hamiltonian (total energy) function admitted special *separable* forms in some system of coordinates, the system in question then could be solved by quadratures. Such metrics and Hamiltonian functions were later named after Liouville. This setting considered by Liouville also implied additive separation of variables for the associated Hamilton–Jacobi equation.

About thirty-five years later in 1881, the converse problem was considered and ultimately solved by G. Morera [8]. More specifically, he showed that if a Hamiltonian system with two degrees of freedom could be solved within the framework

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

of the Hamilton–Jacobi theory of separation of variables, then its metric and Hamiltonian took the Liouville forms with respect to special separable coordinates. We note that this conclusion does not exclude the possibility that the Hamiltonian system in question separates in some other system of coordinates with respect to which the metric and Hamiltonian are not in the Liouville forms. In addition to demonstrating the equivalence between the separation of variables and the Liouville representations for the metric and Hamiltonians, the results due to Morera also show that in the Euclidean plane separation of variables occurs in Cartesian, polar, parabolic and elliptic-hyperbolic coordinates. These four cases represent the situation when the separable coordinates appear as the parameters of confocal ellipses and hyperbolas. Note that the first three cases, namely Cartesian, polar and parabolic coordinates, are nothing but various degeneracies of elliptic-hyperbolic coordinates.

However, this remarkable equivalence does not provide a mechanism for determining separable coordinates for a Hamiltonian system with two degrees of freedom defined by a natural Hamiltonian, or a criterion indicating when such coordinates exist.

Another dimension to the equivalence demonstrated by Morera was brought about in a form of the result now known as the *Bertrand–Darboux–Whittaker theorem*. This theorem characterizes the equivalence from the point of view of the existence, for a given Hamiltonian system, of an additional first integral quadratic in momenta which is functionally independent of the Hamiltonian function. Thus, the existence of such an integral for a given Hamiltonian system with two degrees of freedom defined in a (pseudo)-Riemannian space guarantees that the Hamiltonian function takes on the Liouville form in some system of coordinates and so, in view of the equivalence established by Morera, the corresponding system is separable.

Moreover, since its discovery in the XIX century this result has been used as a practical tool to derive separable coordinates for Hamiltonian systems with two degrees of freedom. Omitting the many details, the method can be briefly described as based on bringing the Hamiltonian function to the Liouville form in which case the corresponding system of coordinates provides integrability by quadratures through separation of variables. However, this analytical method, which involves solving a partial differential equation by the method of characteristics (used in this context for the first time by Darboux), does not tell us whether or not the system is separable in more than *one* system of coordinates (i.e., if it is *super-separable*). Another drawback is that due to the many steps involved, it is hard to devise an algorithm based on the Bertrand–Darboux–Whittaker theorem for solving such problems that can be used in the design of a computer algebra package. Quoting the famous saying of J. Hadamard [4] (made in the context of a problem related to Huygens' principle), one can assert that "…we give *an* answer but not *the* answer, to our question: for it is clear that we can wish it to be *plus résolu* than it has been in the above."

The authors of the present paper have considered this problem from the point of view of the invariants of the associated Killing tensors under the group of rigid motions of the Euclidean plane and presented a new purely *algebraic* method of determining separable coordinate systems for Hamiltonian systems with two degrees of freedom defined in Euclidean space [7]. The method has also been extended to the case when the Hamiltonian system in question is defined on the Minkowski plane.

These results will be published in a forthcoming paper. Thus, we believe that this classical problem is now *plus résolu.* This is the subject of the considerations that follow.

2. From Liouville to Morera

In his first papers on separation of variables (see [6] and the relevant references therein) Liouville was very much inspired by Jacobi's famous result concerning geodesics on ellipsoids, where the author employed the Hamilton–Jacobi equation to determine the geodesic lines. This example still can be found in any book on Classical Mechanics (see, for instance [2]). However, Liouville's first use of separation of variables was related to a problem with a more physical context. More specifically, he considered a particle moving on a surface of arbitrary curvature determined by a metric of the form

$$ds^2 = \lambda(u, v)(du^2 + dv^2),$$

given in some *isothermal* system of coordinates (u, v) under the influence of an external potential force. He showed that the equations of motion of the particle can be integrated by quadratures if the corresponding Hamiltonian $H = T + V$ takes the following form:

$$H = \big(A(u) + B(v)\big)^{-1}\big[\tfrac{1}{2}(p_u^2 + p_v^2) + C(u) + D(v)\big], \tag{1}$$

where $A(u), C(u)$ and $B(v), D(v)$ are arbitrary smooth functions of u and v respectively. Hence, the metric is of the form

$$ds^2 = \big(A(u) + B(v)\big)(du^2 + dv^2), \tag{2}$$

which today, along with (1), bears the name of Liouville. It follows that the corresponding Hamilton–Jacobi equation

$$\tfrac{1}{2} g^{ij}(u, v)\partial_i W \partial_j W + V(u, v) = E, \tag{3}$$

is separable under the standard additive *separation ansatz* for the complete integral W, that is

$$W(u, v) = W_1(u; c_1, c_2) + W_2(v; c_1, c_2),$$

where g^{ij} denote the contravariant components of the metric, V is the potential part of the Hamiltonian (1) and c_1, c_2 are the constants of integration.

Therefore the solution to the equations of motion can easily be obtained by quadratures. Thus, if the Hamiltonian is in the form (1), the solution of the equations of motion is straightforward. In conclusion, the Liouville form (1) for the

Hamiltonian function is a *sufficient* condition for solving the equations of motion via separation of variables. However, it is quite a rare occurrence that the Hamiltonian function H is defined in the right coordinates, namely, that it is in the form (1) with respect to the separable coordinates (u, v). In most physical models the Hamiltonian is defined in position-momenta (Cartesian) coordinates (q^1, q^2, p_1, p_2) which often are *not* the separable coordinates (u, v, p_u, p_v).

In 1881 Morera [8] made the decisive step of considering the *converse problem*. This problem can be stated as follows:

> *Given a Hamiltonian system with two degrees of freedom defined by a natural Hamiltonian*
>
> $$H(\mathbf{q}, \mathbf{p}) = \tfrac{1}{2}\, g^{ij}(\mathbf{q}) p_i p_j + V(\mathbf{q}), \quad i, j = 1, 2, \tag{4}$$
>
> *determine the form of g^{ij} and V under the assumption that the corresponding Hamilton–Jacobi equation is solvable by separation of variables in the given system of coordinates.*

Proving the converse was no less rewarding and Morera demonstrated that the Hamiltonian function (4) took the Liouville form (1) in separable coordinates. It can happen, however, that the Hamilton–Jacobi separation of variables takes place with respect to some other coordinates, for which the Hamiltonian (4) does not assume the form (1) (for instance, it can take the *Gantmakher* form). Nevertheless, it is always possible to bring the Hamiltonian function to the form (1) via a point transformation which *preserves the separability property*. Therefore the condition that the Hamiltonian function (4) takes the Liouville form (1) is also *necessary* for solving the equations of motion via the corresponding Hamilton–Jacobi equation.

3. Bertrand–Darboux–Whittaker theorem

In 1852, Bertrand considered the problem of determining the forces acting on a system (or equivalently, the form of the potential V) when a first integral of the system was known. These results were applied by Darboux in 1901 to Hamiltonian systems with two degrees of freedom defined by a natural Hamiltonian where an additional independent first integral quadratic in the momenta was known. In the case where such an integral contains no first degree terms in the momenta, the resulting partial differential equation satisfied by V (see [9] for more details) can be integrated using a technique developed by Darboux.

The solution indicates that V must be in Liouville form in some canonical system of coordinates (which in general are of elliptic-hyperbolic type), and hence the corresponding Hamilton–Jacobi equation is integrable by quadratures. Given a Hamiltonian system defined by

$$H = \tfrac{1}{2}\,(p_x^2 + p_y^2) + V(x, y), \tag{5}$$

the Bertrand–Darboux–Whittaker theorem establishes the following equivalence:

H has an additional independent first integral quadratic in the momenta if and only if the Hamilton–Jacobi equation associated to (5) *is separable in Cartesian, polar, parabolic, or elliptic-hyperbolic coordinates.* The details of the elliptic-hyperbolic case are provided in [9], and the degenerate cases (Cartesian, polar, and parabolic) are treated in [1].

Once an additional first integral quadratic in the momenta is known, an analytical procedure was developed for finding separable coordinates which involves using the method of characteristics to bring the associated partial differential equation into a form where its solution yields a potential of Liouville type. The drawback in this approach is its complexity. Moreover, it does not give any information regarding separation in other distinct separable coordinate systems—i.e., whether or not the system is super-separable.

4. Benenti's theorem and the method of moving frames

The Bertrand–Darboux–Whittaker theorem was a practical tool for deriving separable coordinates for Hamiltonian systems with two degrees of freedom defined by a natural Hamiltonian. Recently, however, a new existence criterion was developed by Benenti [3], which in two dimensions reads:

> *A Hamiltonian* (4) *on a (pseudo-)Riemannian manifold M is separable in orthogonal coordinates if and only if there exists a Killing tensor* $\mathbf{K}$ *on M with pointwise (real and) simple eigenvalues that satisfies the potential separability condition*
>
> $$\mathrm{d}(\widehat{\mathbf{K}}\,\mathrm{d}V) = 0, \quad \text{where} \quad \widehat{\mathbf{K}} := \mathbf{K}\mathbf{g}. \tag{6}$$

Whereas the existence criterion of the Bertrand–Darboux–Whittaker theorem focused on the potential V, Benenti's theorem focuses on a coordinate-free object, a Killing tensor $\mathbf{K}$ compatible with the Hamiltonian H. Recall that a Killing tensor $\mathbf{K}$ is a symmetric tensor satisfying the Killing tensor equation

$$[\mathbf{g}, \mathbf{K}] = 0, \tag{7}$$

where $[\,\cdot\,,\,\cdot\,]$ denotes the Schouten bracket, which generalizes the usual Lie bracket of vector fields.

The Killing tensor equations (7) can be integrated using the coordinate-free moving frames method. (For more details, see the relevant references in [7].)

In the moving frame of normalized eigenvectors of $\mathbf{K}$, the general solution has the form

$$\mathbf{K} = \ell\mathbf{g} + m\mathbf{K}^{(\mathbf{L})}, \tag{8}$$

where ℓ and m are arbitrary constants. By requiring that the Riemann curvature tensor vanish, we can classify the four separable coordinate systems in the Euclidean plane according to the canonical forms of $\mathbf{K^{(L)}}$ *in the separable coordinates*:

$$\text{Cartesian:}\begin{cases} ds^2 = du^2 + dv^2 \\ x = u, \ \ y = v \\ K_{ij}^{(C)} = \operatorname{diag}(1, 0) \end{cases} \tag{9}$$

$$\text{Polar:}\begin{cases} ds^2 = du^2 + u^2\, dv^2 \\ x = u \cos v, \ \ y = u \sin v \\ K_{ij}^{(P)} = \operatorname{diag}(0, u^4) \end{cases} \tag{10}$$

$$\text{Parabolic:}\begin{cases} ds^2 = (u^2 + v^2)(du^2 + dv^2) \\ x = \frac{1}{2}(u^2 - v^2), \ \ y = uv \\ K_{ij}^{(PB)} = (u^2 + v^2) \operatorname{diag}(v^2, -u^2) \end{cases} \tag{11}$$

$$\begin{matrix}\text{Elliptic-}\\ \text{Hyperbolic:}\end{matrix}\begin{cases} ds^2 = k^2(\cosh^2 u - \cos^2 v)(du^2 + dv^2) \\ x = k \cosh u \cos v, \ \ y = k \sinh u \sin v \\ K_{ij}^{(EH)} = k^4(\cosh^2 u - \cos^2 v) \operatorname{diag}(\cos^2 v, \cosh^2 u) \end{cases} \tag{12}$$

where k in (12) is a positive parameter which can be interpreted as half the distance between the foci of the elliptic-hyperbolic coordinate system.

Thus, up to scalings and addition by the metric, there are only four distinct Killing tensors that can be compatible with H, each characterizing separation in one of Cartesian, polar, parabolic, or elliptic-hyperbolic coordinates.

5. Invariants of the group of rigid motions

The general solution of the Killing tensor equations in Cartesian coordinates is

$$K_{ij} = \begin{pmatrix} A + 2\alpha q^2 + \gamma (q^2)^2 & C - \alpha q^1 - \beta q^2 - \gamma q^1 q^2 \\ C - \alpha q^1 - \beta q^2 - \gamma q^1 q^2 & B + 2\beta q^1 + \gamma (q^1)^2 \end{pmatrix}, \tag{13}$$

where $A, B, C, \alpha, \beta, \gamma$ are arbitrary parameters. Any particular Killing tensor with distinct eigenvalues (i.e., $\mathbf{K}$ is not a multiple of the metric) which satisfies (6) indicates separability in one of Cartesian, polar, parabolic, or elliptic-hyperbolic coordinates. However, each of these coordinate systems is uniquely defined only up to metric-preserving transformations.

The approach of the present authors, as described in [7], was to consider the invariants of (13) under the group of rigid motions, i.e., functions $F(A,B,C,\alpha,\beta,\gamma)$, which are invariant under the induced action of the Euclidean group $E(2)$, consisting of translations and rotations.

These considerations led to the use of the tools from the Lie group theory which yielded two smooth invariants

$$\gamma \quad \text{and} \quad \Delta = \big(\alpha^2 - \beta^2 + \gamma(B - A)\big)^2 + 4(\gamma C + \alpha\beta)^2,$$

which can be used to provide a classification of all (non-metric) Killing tensors.

This classification is established by using the tensor transformation laws to find the components of the canonical Killing tensors (9)–(12) with respect to Cartesian coordinates and evaluating γ and Δ in each case. Upon establishing the coordinate type, the amount of translation and rotation required (with respect to given Cartesian coordinates) can be expressed in terms of the given parameters A, B, C, α, β, γ, and hence the transformation to separable coordinates is obtained. A summary of these results is presented in Table 1 (see p. 399).

This establishes the following fact: *Once a compatible Killing tensor, or equivalently a first integral quadratic in the momenta, is known, the task of finding the transformation to separable coordinate is essentially complete.* The appropriate parameters need only be located in Table 1 (see p. 399). *No additional computations are necessary.* The computational difficulty is reduced to finding the general solution of the compatibility equation (6). However, since the generic form (13) with respect to Cartesian coordinates is known, this computation only amounts to solving linear equations in at most six variables. This algorithm is *completely algebraic*. Moreover, super-separability of the system can be characterized as follows: *A system is super-separable if and only if the dimension of the general solution to* (6) *is greater than two.*

6. Conclusions

We have shown how the equivalence established by Morera sheds new light on the classical Liouville problem. Since the time of its discovery, a number of different facets have been added to this remarkable equivalence. The Bertrand–Darboux–Whittaker theorem provided the first practical tool for finding separable coordinates for Hamiltonian systems of Liouville type defined in the Euclidean plane. More recently Benenti's theorem allows the equivalence to be cast in an intrinsic form. Finally, the balanced combination of the methods of differential geometry and Lie group theory employed by the present authors has made it possible to improve the procedure significantly by making it both *purely algebraic* and applicable to a large class of systems, including super-separable Hamiltonian systems and systems defined in Minkowski space.

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