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Covariant pre-quantum operators[1]

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Abstract. We study distinguished operators in the covariant formulation of Quantum Mechanics over a curved spacetime with absolute time. Moreover, we classify all second-order covariant pre-quantum operators on the quantum bundle.

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1. Introduction

The standard formulation of quantum theories is highly based on concepts and methods strictly related to a flat spacetime and inertial observers, which conflict with general covariance on a curved spacetime. So, a consistent formulation of quantum theories and general relativity is still an open problem.

At the beginning of nineties A. Jadczyk and M. Modugno, see [4, 5], proposed a manifestly covariant formulation of quantum mechanics over curved spacetime with absolute time (referred to as "Galilei quantum theory"). One of the results of this theory is the natural isomorphism between the Lie algebra of Hermitian vector fields of the quantum bundle and the new Lie algebra of quantisable functions of the classical phase space; this theorem yields a procedure for a covariant quantisation.

In this paper we complete the above result, by classifying all covariant pre--quantum operators associated with a given quantisable function.

So, "covariance" is a leading concept of this paper. The standard concept of "covariance", widely used in the literature of physics, has been refined by mathematicians. Nowadays, it appears in literature of mathematics under the name of

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"naturality" and "gauge-naturality". A modern comprehensive reference for the geometry of naturality and gauge-naturality is the book [10]. Further information can be found in [2, 11]. Here, we still use the word "covariance" according to the above concept of "naturality".

According to the theory of covariant morphisms, [10, 11], in order to classify covariant morphisms it is sufficient to classify G-equivariant maps on type fibers. For this reasons we shall use the "orbit reduction theorem", [10, Proposition 28.1].

We stress that our concept of covariance involves independence from the choice of coordinates and of units of measurement on the same footing.

2. Classical structure

Covariance requires a rigorous treatment of units of measurement. Therefore, we assume the following "positive 1-dimensional semi-vector spaces" over $\mathbb{R}^+$ as fundamental unit spaces: the space $\mathbb{T}$ of *time intervals*, the space $\mathbb{L}$ of *lengths* and the space $\mathbb{M}$ of *masses*.

Moreover, we assume the *Planck constant* to be an element $\hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}$. We refer to particles with mass $m \in \mathbb{M}$ and charge $q \in \mathbb{T}^* \otimes \mathbb{L}^{\frac{3}{2}} \otimes \mathbb{M}^{\frac{1}{2}}$. Moreover, we define a *time unit* to be an element $u_0 \in \mathbb{T}$ or to its dual $u^0 \in \mathbb{T}^*$.

2.1. Classical spacetime

We start with our postulate concerning the classical spacetime.

We assume *(absolute) time* to be an affine space $\boldsymbol{T}$ associated with the vector space $\bar{\mathbb{T}} := \mathbb{T} \otimes \mathbb{R}$. We assume *spacetime* to be an oriented $(n+1)$-dimensional manifold $\boldsymbol{E}$ fibered over time by the *absolute time map*

$$t : \boldsymbol{E} \to \boldsymbol{T}.$$

Thus, the time fibering yields the *time form* $dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$.

We shall refer to charts of spacetime $(x^\lambda) = (x^0, x^i)$ adapted to the time fibering, to the affine structure of time, and to the orientation of time and of spacetime (and to a time unit of measurement u_0).

We define a *spacetime automorphism* to be a local, orientation preserving fibered diffeomorphism $\phi : \boldsymbol{E} \to \boldsymbol{E}$ over an orientation preserving affine diffeomorphism $\underline{\phi} : \boldsymbol{T} \to \boldsymbol{T}$. If ϕ is a spacetime automorphism and (x^λ) a spacetime chart, then we set

$$\phi^\lambda_{\mu_1 \cdots \mu_r} := \frac{\partial^r \phi^\lambda}{\partial x^{\mu_1} \cdots \partial x^{\mu_r}} \in \text{map}(\boldsymbol{E}, \mathbb{R})$$

and obtain

$$\phi^0 = \phi^0_0 x^0 + b^0, \quad b^0 \in \mathbb{R}, \quad \phi^0_0 \in \mathbb{R}^+, \quad \phi^0_i = 0, \quad \det\,(\phi^i_j) > 0.$$

If (x^λ) and $(\bar{x}^\lambda)$ are two spacetime charts, then we set

$$\sigma^\lambda_{\mu_1\cdots\mu_r} := \frac{\partial^r x^\lambda}{\partial \bar{x}^{\mu_1} \cdots \partial \bar{x}^{\mu_r}} \in \text{map}(\boldsymbol{E},\mathbb{R}),$$

$$\bar{\sigma}^\lambda_{\mu_1\cdots\mu_r} := \frac{\partial^r \bar{x}^\lambda}{\partial x^{\mu_1} \cdots \partial x^{\mu_r}} \in \text{map}(\boldsymbol{E},\mathbb{R})$$

and obtain

$$\bar{x}^0 = \bar{\sigma}^0_0 x^0 + \bar{\sigma}^0, \quad \bar{\sigma}^0 \in \mathbb{R}, \quad \bar{\sigma}^0_0 \in \mathbb{R}^+, \quad \bar{\sigma}^0_i = 0, \quad \det(\bar{\sigma}^i_j) > 0.$$

If $x \equiv (x^\lambda)$ and $\bar{x} \equiv (\bar{x}^\lambda)$ are two spacetime charts, then we obtain the spacetime automorphism $\phi := x^{-1} \circ \bar{x}$.

Hence, we obtain $x \circ \phi \circ x^{-1} = \bar{x} \circ x^{-1}$, which yields

$$\phi^\lambda_{\mu_1\cdots\mu_r} = \bar{\sigma}^\lambda_{\mu_1\cdots\mu_r}.$$

Let G^r_{n+1} be the group of r-jets $j_r\varphi(0)$ of diffeomorphisms $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ which preserve the origin, in particular, $G^1_{n+1} = Gl(n+1, \mathbb{R})$. We are involved with the subgroup

$$G^r_{(1,n)} \subset G^r_{n+1}$$

generated by the diffeomorphisms φ's, which respect the projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and project over an orientation preserving affine map $\underline{\varphi} : \mathbb{R} \to \mathbb{R}$. Thus, the elements of $G^r_{(1,n)}$ are of the type $(a^0_0, a^i_\mu, \ldots, a^i_{\mu_1\cdots\mu_r})$, where $a^0_0 > 0$, det $(a^i_j) > 0$. We denote the inverse of an element $(a^0_0, a^i_\mu, \ldots, a^i_{\mu_1\cdots\mu_r})$ by $(\bar{a}^0_0, \bar{a}^i_\mu, \ldots, \bar{a}^i_{\mu_1\cdots\mu_r})$.

Thus, the group $G^r_{(1,n)}$ acts on the type fiber $F[J_r T\boldsymbol{E}]$. Let us consider a point $e \in \boldsymbol{E}$ and a spacetime chart (x^λ). Then, according to the above results, the r-jet of the tangent prolongation of a spacetime automorphism ϕ and of a transition to a spacetime chart $(\bar{x}^\lambda)$ yield the elements

$$(a^0_0, a^i_\mu, \ldots, a^i_{\mu_1\cdots\mu_r}) := (\phi^0_0, \phi^i_\mu, \ldots, \phi^i_{\mu_1\cdots\mu_r})(e) \in G^r_{(1,n)},$$
$$(a^0_0, a^i_\mu, \ldots, a^i_{\mu_1\cdots\mu_r}) := (\bar{\sigma}^0_0, \bar{\sigma}^i_\mu, \ldots, \bar{\sigma}^i_{\mu_1\cdots\mu_r})(e) \in G^r_{(1,n)}.$$

If (x^λ) is a spacetime chart, then the induced local bases of $T\boldsymbol{E}$, $V\boldsymbol{E}$, $T^*\boldsymbol{E}$ and $V^*\boldsymbol{E}$ are denoted respectively by (∂_λ), (∂_i), (d^λ) and $(\check{d}^i)$.

The coordinate expression of the time form is $dt = u_0 \otimes d^0$.

An *observer* is defined to be a (local) section $o \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E})$, which projects on $\mathbf{1} \in \mathbb{T}^* \otimes \mathbb{T}$. The charts (x^λ) for which $o^i_0 = 0$ are said to be *adapted* to o. Each chart (x^λ) determines the observer $o := u^0 \otimes \partial_0$. Each observer o determines the fibered morphism

$$\nu[o] = (d^i - o^i_0\, d^0) \otimes \partial_i \in \text{fib}(\boldsymbol{E}, T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}).$$

If o and $\bar{o}$ are two observers, then we can write $\bar{o} = o + v$, with $v \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes V\boldsymbol{E})$.

2.2. Metric field

Then, we discuss the metric structure of spacetime.

We define a *spacelike metric* to be a scaled Riemannian metric of the fibers of $\boldsymbol{E}$

$$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes S^2 V^* \boldsymbol{E}.$$

With a reference to a mass m, we define the *re-scaled spacelike metric*

$$G := \frac{m}{\hbar} g : \boldsymbol{E} \to \mathbb{T} \otimes S^2 V^* \boldsymbol{E}, \qquad G = G^0_{ij} u_0 \otimes \check{d}^i \otimes \check{d}^j.$$

We denote the contravariant spacelike metric and the contravariant re-scaled spacelike metric by $\bar{g} : \boldsymbol{E} \to \mathbb{L}^{2*} \otimes S^2 V \boldsymbol{E}$ and $\bar{G} := (\hbar/m)\bar{g} : \boldsymbol{E} \to \mathbb{T}^* \otimes S^2 V \boldsymbol{E}$.

We denote the natural vector bundle of re-scaled spacelike metrics by

$$\operatorname{Met} \boldsymbol{E} \subset \mathbb{T} \otimes S^2 V^* \boldsymbol{E}.$$

The spacelike metric g and the spacetime orientation naturally yield the scaled *spacelike volume form* and the *spacetime volume form*

$$\eta : \boldsymbol{E} \to \mathbb{L}^n \otimes \bigwedge^n V^* \boldsymbol{E}, \qquad \eta = \sqrt{|g|}\, \check{d}^1 \wedge \cdots \wedge \check{d}^n$$

$$\upsilon := dt \wedge \eta : \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{L}^n) \otimes \bigwedge^{n+1} T^* \boldsymbol{E},$$

$$\upsilon = \sqrt{|g|}\, u_0 \otimes d^0 \wedge d^1 \wedge \cdots \wedge d^n.$$

We set $\upsilon^0 := \upsilon(u^0) = \sqrt{|g|}\, d^0 \wedge d^1 \wedge \cdots \wedge d^n$.

For each vector field X of $\boldsymbol{E}$, we define the *spacetime divergence*

$$\operatorname{div}_\upsilon X \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}), \quad \text{by the equality} \quad L[X]\,\upsilon = (\operatorname{div}_\upsilon X)\,\upsilon.$$

For each projectable vector field X of $\boldsymbol{E}$, we define the *spacelike divergence* and the *timelike divergence*, respectively,

$$\operatorname{div}_\eta X \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}), \quad \text{by the equality} \quad L[X]\,\eta = (\operatorname{div}_\eta X)\,\eta,$$

$$\operatorname{div}_{\mathrm{dt}} X \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}), \quad \text{by the equality} \quad L[X]\,dt = (\operatorname{div}_{\mathrm{dt}} X)\,dt.$$

The re-scaled spacelike metric G naturally yields:

– the fiber-wise Riemannian connection

$$\varkappa[G] : V\boldsymbol{E} \to V^*\boldsymbol{E} \underset{V\boldsymbol{E}}{\otimes} VV\boldsymbol{E},$$

– the fiber-wise curvature tensor

$$R[\varkappa] : V\boldsymbol{E} \to \bigwedge^2 V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E},$$

– the fiber-wise Ricci tensor

$$\operatorname{Ricci}[\varkappa] := C^1_1 R[\varkappa] : \boldsymbol{E} \to V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E},$$

– the fiber-wise scaled scalar curvature

$$r[\varkappa] := \langle \bar{G}, \mathrm{Ricci}[\varkappa] \rangle : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{R}.$$

2.3. Galileian connections

In view of next developments, we need some preliminary concepts and results.

Definition 2.1. We define a *spacetime connection* to be a connection of the vector bundle $T\boldsymbol{E} \to \boldsymbol{E}$, which is linear, torsion free and such that $\nabla dt = 0$. A spacetime connection K is said to be *metric* if $\nabla G = 0$. A metric spacetime connection K is said to be *Galileian* if its curvature tensor fulfills the condition $R^{j}_{0\lambda}{}^{i}{}_{\mu} = R^{i}_{0\mu}{}^{j}{}_{\lambda}$.

The coordinate expression of a spacetime connection is of the type

$$K = d^\lambda \otimes \partial_\lambda + K_\lambda{}^\mu{}_\nu \, \dot{x}^\nu \, d^\lambda \otimes \dot{\partial}_\mu, \qquad K_\mu{}^i{}_\nu = K_\nu{}^i{}_\mu, \quad K_\lambda{}^0{}_\nu = 0.$$

We denote the second-order natural bundle of spacetime connections by $\mathrm{Con}\,\boldsymbol{E}$. Now, let us consider an observer o. The covariant differential of o is the section

$$\nabla o \in \mathrm{sec}\big(\boldsymbol{E}, \, \mathbb{T}^* \otimes (T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E})\big),$$

with coordinate expression, in adapted coordinates, $\nabla o = -K_\lambda{}^i{}_0 \, u^0 \otimes d^\lambda \otimes \partial_i$.

Then, we obtain the section

$$G^\flat(\nabla o) \in \mathrm{sec}(\boldsymbol{E}, \, T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E}),$$

whose expression, in adapted coordinates, is $G^\flat(\nabla o) = -K_{\lambda}{}^{0}_{j0} \, d^\lambda \otimes \check{d}^j$, where $K_{\lambda}{}^{0}_{j0} = G^0_{ij} K_\lambda{}^i{}_0$.

By anti-symmetrising $\nu^*[o]\big(G^\flat(\nabla o)\big)$ we obtain $\Phi[o] \in \mathrm{sec}(\boldsymbol{E}, \bigwedge^2 T^*\boldsymbol{E})$.

Proposition 2.2 ([9]). *Let us consider a spacetime connection K and refer to an observer o and to adapted coordinates. Then, the following conditions are equivalent.*

1. *K is Galileian.*
2. *$\Phi[o]$ is closed and we have*

$$K_k{}^i{}_h = K_h{}^i{}_k = -\tfrac{1}{2} \, G^{ij}_0 \, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}),$$

$$(2.1) \qquad K_0{}^i{}_h = K_h{}^i{}_0 = -\tfrac{1}{2} \, G^{ij}_0 \, (\partial_h A_j - \partial_j A_h + \partial_0 G^0_{hj}),$$

$$K_0{}^i{}_0 = -G^{ij}_0 \, (\partial_0 A_j - \partial_j A_0),$$

where $A[o] = A_0 \, dx^0 + A_j \, dx^j$ is a local potential of $\Phi[o]$ defined up to a local gauge of the type df, with $f \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$.

2.4. Natural bundle of classical potentials

The above expression of Galileian connections can be reformulated in a more compact way as follows.

We define a *metric extension* to be a section $\widetilde{G} \in \sec(\boldsymbol{E}, \mathbb{T} \otimes S^2 T^* \boldsymbol{E})$, whose vertical restriction is G. Thus, the coordinate expression of a metric extension is of the type $\widetilde{G} = u_0 \otimes (G^0_{ij}\, d^i \otimes d^j + G^0_{0j}\, d^0 \otimes d^j + G^0_{i0}\, d^i \otimes d^0 + G^0_{00}\, d^0 \otimes d^0)$, where $G^0_{0j}, G^0_{i0}, G^0_{00} \in \operatorname{map}(\boldsymbol{E}, \mathbb{R})$ and $G^0_{0i} = G^0_{i0}$.

In particular, each observer o yields the metric extension $\widetilde{G} := \nu^*[o]G$, whose expression, in adapted coordinates, is $\widetilde{G} = G^0_{ij}\, u_0 \otimes d^i \otimes d^j$.

Lemma 2.3. *A metric extension $\widetilde{G}$ and an observer o determine the local section $A[\widetilde{G}, o] := o \lrcorner \widetilde{G} - \frac{1}{2} o \lrcorner o \lrcorner \widetilde{G} \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$, with coordinate expression, in adapted coordinates,*

$$A[\widetilde{G}, o] \equiv A_\lambda\, d^\lambda = \tfrac{1}{2}\, G^0_{00}\, d^0 + G^0_{i0}\, d^i .$$

Clearly, we have $o \lrcorner A[\widetilde{G}, o] = \frac{1}{2} o \lrcorner o \lrcorner \widetilde{G}$.

Corollary 2.4. *Let us consider a metric extension $\widetilde{G}$ and an observer o. Then, we obtain $\widetilde{G} = dt \otimes A[\widetilde{G}, o] + A[\widetilde{G}, o] \otimes dt + \nu^*[o]G$.*

Corollary 2.5. *Let us consider a metric extension $\widetilde{G}$ and two observers o and $\bar{o} + v$. Then, we obtain $A[\widetilde{G}, \bar{o}] = A[\widetilde{G}, o] - \frac{1}{2} G(v, v) + \nu[o] \lrcorner G^\flat(v)$, i.e., in a chart adapted to $\bar{o}$, we have $A[\widetilde{G}, \bar{o}] = A[\widetilde{G}, o] + \frac{1}{2} G^0_{ij}\, v^i_0\, v^j_0\, d^0 + G^0_{ij}\, v^j_0\, d^i$.*

Lemma 2.6. *A local form $A \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$ and an observer o determine the metric extension $\widetilde{G}[A, o] := dt \otimes A + A \otimes dt + \nu^*[o]G$.*

Clearly, we have $\frac{1}{2} o \lrcorner o \lrcorner \widetilde{G}[A, o] = o \lrcorner A$.

Corollary 2.7. *Let us consider an observer o. Then, the maps $\widetilde{G} \mapsto A[\widetilde{G}, o]$ and $A \mapsto \widetilde{G}[A, o]$ are inverse bijections.*

The above constructions yields the following result.

Proposition 2.8. *The metric extensions turn out to be the sections of a bundle* $\operatorname{Ext} \boldsymbol{E} \to \boldsymbol{E}$.

Its type fiber is the subset $F[\operatorname{Ext} \boldsymbol{E}] \subset (\mathbb{R} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}) \times (\mathbb{R} \times \mathbb{R}^n)$, whose elements (G^0_{ij}, A_0, A_i) fulfill the conditions $G^0_{ij} = G^0_{ji}$ and $\det(G^0_{ij}) \neq 0$.

The bundle $\operatorname{Ext} \boldsymbol{E}$ turns out to be a first-order natural bundle and we obtain the following action of the group $G^1_{(1,n)}$ on the type fiber $F[\operatorname{Ext} \boldsymbol{E}]$

$$\overline{G}{}^0_{ij} = a^0_0\, G^0_{pq}\, \bar{a}^p_i \bar{a}^q_j ,$$

$$\overline{A}_i = A_p\, \bar{a}^p_i + a^0_0\, G^0_{pq}\, \bar{a}^p_0\, \bar{a}^q_i ,$$

$$\overline{A}_0 = A_\mu\, \bar{a}^\mu_0 + \tfrac{1}{2}\, a^0_0\, G^0_{pq}\, \bar{a}^p_0\, \bar{a}^q_0 .$$

Then, Proposition 2.2 can be reformulated in the following way.

Theorem 2.9. *Let us consider a spacetime connection K. Then, the following conditions are equivalent.*

1. *K is Galileian.*
2. *The coordinate expression of K is of the type*

$$K_{\lambda}{}^{i}{}_{\mu} = -\tfrac{1}{2}\, G_0^{ij}\,(\partial_\lambda G^0_{j\mu} + \partial_\mu G^0_{j\lambda} - \partial_j G^0_{\lambda\mu}),$$

where $\widetilde{G} = G^0_{\lambda\mu}\, d^\lambda \otimes d^\mu$ is a metric extension.

Thus, the metric extensions $\widetilde{G}$ can be regarded as potentials of Galileian connections. Each observer o splits $\widetilde{G}$ into the components $\nu^*[o]G$ and $A[\widetilde{G}, o]$.

Remark 2.10. Each potential $A[o]$ of $\Phi[o]$ is defined up to a gauge. On the other hand, given a metric extension $\widetilde{G}$ and an observer o, the form $A[o] = A[\widetilde{G}, o]$ turns out to be a distinguished potential of $\Phi[o]$. Thus, the choice of a metric extension determines the gauge of the potentials $A[o]$, for each observer o.

Corollary 2.11. *Theorem* 2.9 *yields a first-order covariant operator*

$$K : \mathrm{Ext}\ \boldsymbol{E} \to \mathrm{Con}\, \boldsymbol{E}.$$

2.5. Gravitational and electromagnetic fields

In the Galilei theory of classical spacetime we describe the gravitational and electromagnetic fields as follows.

The *gravitational field* and the *electromagnetic field* are defined to be a Galileian spacetime connection and a scaled closed 2-form of spacetime respectively,

$$K^{\natural} : T\boldsymbol{E} \to T^*\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} TT\boldsymbol{E} \quad \text{and} \quad f : \boldsymbol{E} \to (\mathbb{L}^{\frac{1}{2}} \otimes \mathbb{M}^{\frac{1}{2}}) \otimes \overset{2}{\bigwedge} T^*\boldsymbol{E}.$$

With a reference to a charge q, we define the *re-scaled electromagnetic field*

$$F = \frac{q}{\hbar}\, f : \boldsymbol{E} \to \overset{2}{\bigwedge} T^*\boldsymbol{E}.$$

Then, it is convenient to introduce the *total connection*

$$K := K^{\natural} + K^{e} = K^{\natural} + (dt \otimes \widehat{F} + \widehat{F} \otimes dt),$$

where $\widehat{F} = G_0^{ih}\, F_{h\mu}\, u^0 \otimes \partial_i \otimes d^\mu$.

Proposition 2.12. *The total connection K turns out to be Galileian.*

Hence, in what follows we shall refer to a total connections K.

2.6. Classical phase space

The *phase space* is defined to be the first jet space $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$ of spacetime [13]. We denote the fibered charts of $J_1\boldsymbol{E}$ by (x^0, x^i, x^i_0).

We recall that $J_1\boldsymbol{E} \to \boldsymbol{E}$ is an affine bundle associated with the vector bundle $\mathbb{T}^* \otimes V\boldsymbol{E}$. Hence, the vertical space of $J_1\boldsymbol{E}$ with respect to $\boldsymbol{E}$ turns out to be $V_{\boldsymbol{E}} J_1\boldsymbol{E} = \mathbb{T}^* \otimes V\boldsymbol{E}$.

We recall the natural *contact maps*

$$
\begin{aligned}
д &= u^0 \otimes (\partial_0 + x^i_0\, \partial_i) : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E},\\
\theta &= (d^i - x^i_0\, d^0) \otimes \partial_i : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}.
\end{aligned}
$$

2.7. Distinguished phase fields

A total connection K and a metric G yield in a covariant way, [3, 4, 8], the following objects:

– the torsion free affine connection of the affine bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$, called *phase connection*, $\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} T J_1\boldsymbol{E}$, with coordinate expression

$$
\Gamma_{\lambda}{}^{i}_{0} := \Gamma_{\lambda}{}^{i}{}^{0}_{0j}\, x^j_0 + \Gamma_{\lambda}{}^{i}{}^{0}_{00}, \qquad \Gamma_{\lambda}{}^{i}{}^{0}_{0\mu} = K_{\lambda}{}^{i}{}_{\mu};
$$

– the *second-order connection* $\gamma := д \lrcorner \Gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T J_1\boldsymbol{E}$, with coordinate expression

$$
\begin{aligned}
\gamma &= u^0 \otimes (\partial_0 + x^i_0\, \partial_i + \gamma_0{}^i_0\, \partial^0_i),\\
\gamma_0{}^i_0 &= \Gamma_h{}^{i}{}^{0}_{0k}\, x^h_0\, x^k_0 + 2\, \Gamma_h{}^{i}{}^{0}_{00}\, x^h_0 + \Gamma_0{}^{i}{}^{0}_{00};
\end{aligned}
$$

– the *phase* 2*-form* $\Omega := \nu[\Gamma] \bar{\wedge} \theta : J_1\boldsymbol{E} \to \bigwedge^2 T^* J_1\boldsymbol{E}$, with coordinate expression

$$
\Omega = G^0_{ij}\, (d^i_0 - \gamma_0{}^i_0\, d^0 - \Gamma_h{}^i_0\, \theta^h) \wedge \theta^j,
$$

where $\nu[\Gamma]$ is the vertical valued form associated with Γ and $\bar{\wedge}$ is the wedge product followed by the contraction through G;

– the *phase 2-vector* $\Lambda := \check{\Gamma} \mathbin{\bar{\wedge}} \nu : J_1\boldsymbol{E} \to \bigwedge^2 T J_1\boldsymbol{E}$, with coordinate expression

$$
\Lambda = G^{ij}_0\, (\partial_i + \Gamma_i{}^h_0\, \partial^0_h) \wedge \partial^0_j,
$$

where

$$
\check{\Gamma} : J_1\boldsymbol{E} \to V^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} V J_1\boldsymbol{E}
$$

is the vertical restriction of Γ and ν is the covariant tensor

$$
\nu : J_1\boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} V_{\boldsymbol{E}} J_1\boldsymbol{E}),
$$

with coordinate expression $\nu = u_0 \otimes \check{d}^i \otimes \partial^0_i$.

Clearly, each of the above objects splits into gravitational and electromagnetic components.

Proposition 2.13 ([3]). *Let K be a spacetime connection. Then,*

$$d\Omega = 0$$

if and only if K is Galileian.

Proposition 2.14 ([4, 8]). *If K is a Galileian connection, then we obtain*

$$dt \wedge \Omega^n \not\equiv 0, \quad i(\gamma)\,\Omega = 0, \quad d\Omega = 0,$$
$$L[\gamma]\,\Lambda = 0, \quad [\Lambda,\ \Lambda] = 0.$$

Hence, $(J_1\boldsymbol{E},\ dt, \Omega)$ turns out to be a (scaled) cosymplectic manifold, γ the associated (scaled) Reeb vector field and Λ the associated 2-vector field.

Proposition 2.15. *For each observer o and classical potential $A[o]$, we obtain*

$$dA[o] = o^*\Omega.$$

3. Special quadratic functions

Next, we analyse the special quadratic functions, which constitute an important aspect of the covariant quantum theory.

After a few recalls, we prove new results on covariant transformations of special quadratic functions.

3.1. Hamiltonian lift of functions

We start with a few recalls, [3].

Definition 3.1. The *vertical Hamiltonian lift* of a function $f \in \mathrm{map}(J_1\boldsymbol{E},\mathbb{R})$ is defined to be the vector field

$$\Lambda^\sharp(df) \in \sec(J_1\boldsymbol{E}, VJ_1\boldsymbol{E}).$$

Given a *time scale* $\tau \in \mathrm{map}(J_1\boldsymbol{E},\ \mathbb{T}\otimes\mathbb{R})$, we define the *$\tau$-Hamiltonian lift* of a function $f \in \mathrm{map}(J_1\boldsymbol{E},\mathbb{R})$ to be the vector field

$$X^{\uparrow}{}_{\mathrm{Ham}}[\tau,\ f] := \gamma(\tau) + \Lambda^\sharp(df) \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}).$$

For every $f \in \mathrm{map}(J_1\boldsymbol{E},\mathbb{R})$, we have the distinguished time scale, called the *time component* of f,

$$f'' := \frac{1}{n}\langle \bar{G},\, D^2 f\rangle \in \mathrm{map}(J_1\boldsymbol{E},\ \mathbb{T}\otimes\mathbb{R}),$$

where

$$D^2 f \in \mathrm{fib}(J_1\boldsymbol{E}, \mathbb{T}\otimes\mathbb{T}\otimes V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E})$$

is the second fiber derivative of f with respect to the affine fiber of the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$. Thus, we obtain the coordinate expression

$$f'' \equiv f^0 u_0 = \frac{1}{n} G_0^{ij} \partial_i^0 \partial_j^0 f \, u_0.$$

We define the *Hamiltonian lift* of $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R})$ to be the vector field

$$X^{\uparrow}{}_{\mathrm{Ham}}[f] := X^{\uparrow}{}_{\mathrm{Ham}}[f'', f] = \gamma(f'') + \Lambda^{\sharp}(df) \in \mathrm{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}).$$

3.2. Special quadratic functions

We pursue with further recalls, [3].

Definition 3.2. A *special quadratic function* is defined to be a phase function $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R})$, such that

$$D^2 f = \tau \otimes G, \qquad \tau \in \mathrm{map}(\boldsymbol{E}, \mathbb{T} \otimes \mathbb{R}).$$

Clearly, if $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R})$ is a special quadratic function, then we obtain $\tau = f''$, hence $D^2 f = f'' \otimes G$, with $f'' \in \mathrm{map}(\boldsymbol{E}, \mathbb{T} \otimes \mathbb{R})$.

The coordinate expression of special quadratic functions is of the type

$$f = \tfrac{1}{2} f^0 G^0_{ij} x^i_0 x^j_0 + f^i G^0_{ij} x^j_0 + f^\circ,$$

where $f^0, f^i, f^\circ \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$, $f'' = f^0 u_0$. The subsheaf of special quadratic functions is denoted by

$$\mathrm{spec}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R}).$$

It turns out to be a sheaf of modules over the sheaf of rings $\mathrm{map}(\boldsymbol{E}, \mathbb{R})$.

Theorem 3.3 ([4]). *Let* $\tau \in \mathrm{map}(J_1\boldsymbol{E}, \bar{\mathbb{T}})$ *and* $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R})$. *Then, the following conditions are equivalent:*

1. $X^{\uparrow}{}_{\mathrm{Ham}}[\tau, f] \in \mathrm{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ *is projectable on* $X[\tau, f] \in \mathrm{sec}(\boldsymbol{E}, T\boldsymbol{E})$*;*
2. $f \in \mathrm{spec}(J_1\boldsymbol{E}, \mathbb{R})$ *and* $\tau = f''$.

Thus, if the above conditions are fulfilled, then we obtain

$$X^{\uparrow}{}_{\mathrm{Ham}}[\tau, f] = X^{\uparrow}{}_{\mathrm{Ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df).$$

Definition 3.4. If $f \in \mathrm{spec}(J_1\boldsymbol{E}, \mathbb{R})$, then the vector field

$$X[f] := X[f'', f] = f^0 \partial_0 - f^i \partial_i$$

is called the *tangent lift* of f.

Theorem 3.5 ([4]). *The sheaf* $\mathrm{spec}(J_1\boldsymbol{E}, \mathbb{R})$ *turns out to be an* $\mathbb{R}$*-Lie algebra through the* special bracket

$$[\![f, g]\!] := \{f, g\} + \gamma(f'').g - \gamma(g''). f,$$

where $\{f, g\}$ *is the Poisson bracket.*

We shall be involved with the following subalgebras related to the affine structure of the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$:

– the subalgebra $\mathrm{quan}(J_1\boldsymbol{E},\mathbb{R}) \subset \mathrm{spec}(J_1\boldsymbol{E},\mathbb{R})$ consisting of functions, called *quantisable*, whose time component $f'' \in \mathrm{map}(\boldsymbol{T}, \bar{\mathbb{T}})$ depends only on $\boldsymbol{T}$;

– the subalgebra $\mathrm{fine}(J_1\boldsymbol{E},\mathbb{R}) \subseteq \mathrm{quan}(J_1\boldsymbol{E},\mathbb{R})$ consisting of quantisable functions whose time component $f'' \in \bar{\mathbb{T}}$ is constant;

– the subalgebra $\mathrm{aff}(J_1\boldsymbol{E},\mathbb{R}) \subset \mathrm{fine}(J_1\boldsymbol{E},\mathbb{R})$ consisting of affine functions with respect to the affine fibers of the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$;

– the subalgebra $\mathrm{map}(\boldsymbol{E},\mathbb{R}) \subset \mathrm{aff}(J_1\boldsymbol{E},\mathbb{R})$ consisting of functions which depend only on $\boldsymbol{E}$.

3.3. Bundle of special quadratic functions

We can describe the special quadratic functions through a bundle in the following way.

Proposition 3.6. *The special quadratic functions turn out to be the local sections of a vector bundle* $\mathrm{Spec}\,\boldsymbol{E}$ *over* $\boldsymbol{E}$*, whose fibers are the spaces of special quadratic functions defined on the fibers of the affine bundle* $J_1\boldsymbol{E} \to \boldsymbol{E}$*, and whose type fiber is*

$$F[\mathrm{Spec}\,\boldsymbol{E}] = \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}.$$

Lemma 3.7. *Let us consider two spacetime charts* (x^λ) *and* $(\bar{x}^\lambda)$ *and a special quadratic function* f*, whose expression in the above charts is*

$$\begin{aligned} f &= f^0\, \tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + f^i\, G^0_{ij}\, x^j_0 + f^\circ \\ &= \bar{f}^0\, \tfrac{1}{2}\, \bar{G}^0_{ij}\, \bar{x}^i_0\, \bar{x}^j_0 + \bar{f}^i\, \bar{G}^0_{ij}\, \bar{x}^j_0 + \overline{f^\circ}. \end{aligned}$$

Then, we obtain

$$\begin{aligned} \bar{f}^0 &= \bar{\sigma}^0_0\, f^0, \\ \bar{f}^i &= \bar{\sigma}^i_j\, f^j - \bar{\sigma}^i_0\, f^0, \\ \overline{f^\circ} &= f^\circ + f^0\, \tfrac{1}{2}\, G^0_{ij}\, \sigma^i_h\, \sigma^j_k\, \bar{\sigma}^h_0\, \bar{\sigma}^k_0 - f^i\, G^0_{ij}\, \sigma^j_k\, \bar{\sigma}^k_0. \end{aligned}$$

Proposition 3.8. *The bundle* $\mathrm{Spec}\,\boldsymbol{E}$ *turns out to be a first-order natural bundle and we obtain the following action of the group* $G^1_{(1,n)}$ *on the type fiber* $F[\mathrm{Spec}\,\boldsymbol{E}]$

$$\begin{aligned} \bar{f}^0 &= a^0_0\, f^0, \\ \bar{f}^i &= a^i_j\, f^j - a^i_0\, f^0, \\ \overline{f^\circ} &= f^\circ + f^0\, \tfrac{1}{2}\, G^0_{ij}\, \bar{a}^i_h\, \bar{a}^j_k\, a^h_0\, a^k_0 - f^i\, G^0_{ij}\, \bar{a}^j_k\, a^k_0. \end{aligned}$$

Proposition 3.9. *The affine functions and the spacetime functions are the sections of vector subbundles* $\mathrm{Aff}\,\boldsymbol{E} \subset \mathrm{Spec}\,\boldsymbol{E}$ *and* $\mathrm{Map}\,\boldsymbol{E} \subset \mathrm{Spec}\,\boldsymbol{E}$*, respectively.*

Remark 3.10. The quantisable functions and the fine functions cannot be regarded as sections of subbundles of Spec $\boldsymbol{E}$.

However, if we fix a time scale $\tau : \boldsymbol{T} \to \bar{\mathbb{T}}$, then the quantisable functions f such that $f'' = \tau$ are the sections of an affine subbundle of Spec $\boldsymbol{E}$.

Analogously, if we fix a time scale $\tau \in \bar{\mathbb{T}}$, then the fine functions f such that $f'' = \tau$ are the sections of an affine subbundle of Spec $\boldsymbol{E}$.

On the other hand, the most interesting fine functions, with non vanishing time component, are the classical Hamiltonian $\mathcal{H}_0$ and Lagrangian $\mathcal{L}_0$. Indeed, they are the components $\mathcal{F}_0$ of horizontal forms of the type $\mathcal{F} = \mathcal{F}_0\, d^0$, whose time component is $\mathcal{F}^0_0 = 1$. In other words, they are elements of $\mathrm{spec}(J_1\boldsymbol{E},\mathbb{R}) \otimes \mathbb{T}^*$, which are projectable on $1 \in \mathbb{T} \otimes \mathbb{T}^*$. Actually, these $\mathbb{T}^*$-scaled fine functions are the sections of an affine subbundle of Spec $\boldsymbol{E} \otimes \mathbb{T}^*$.

3.4. Covariant transformations of special quadratic functions

Every special quadratic function yields other special quadratic functions by means of covariant transformations.

We have a first immediate result.

Proposition 3.11. *Each special quadratic function f yields the function*

$$f_{Д} := Д \lrcorner\, d f'' = \partial_i f^0\, x^i_0 + \partial_0 f^0 \in \mathrm{aff}(J_1\boldsymbol{E},\mathbb{R}). \tag{3.1}$$

The operator $f \mapsto f_{Д}$ is a first-order covariant operator Spec $\boldsymbol{E} \to$ Aff $\boldsymbol{E}$.

Proof. In fact, the operator $f \mapsto f_{Д}$ is a composition of covariant operators. □

Then, we have a family of further covariant operators.

Definition 3.12. Let us consider a special quadratic function f.

The *covariant differential* of f, with respect to the total connection Γ, is defined to be the fibered morphism over $\boldsymbol{E}$

$$\nabla^1 f := Tf \circ \Gamma \in \mathrm{fib}(J_1\boldsymbol{E},\ T^*\boldsymbol{E}).$$

The 2*-covariant differential* of f, with respect to the total connection Γ and the dual K^* of the total connection K, is defined to be the fibered morphism over $\boldsymbol{E}$

$$\nabla^2 f := \nu[K^*] \circ T\nabla^1 f \circ \Gamma \in \mathrm{fib}\Big(J_1\boldsymbol{E}, \bigotimes^2 T^*\boldsymbol{E}\Big).$$

For each integer $k \geq 2$, the *k-covariant differential* of f, with respect to the total connection Γ and the dual K^* of the total connection K, is defined, by iteration with respect to k, to be the fibered morphism over $\boldsymbol{E}$

$$\nabla^k f := \bigotimes^{k-1} \nu[K^*] \circ T\nabla^{k-1} f \circ \Gamma \in \mathrm{fib}\Big(J_1\boldsymbol{E}, \bigotimes^k T^*\boldsymbol{E}\Big).$$

Proposition 3.13. *Let us consider a special quadratic function* f*. Then, we obtain the coordinate expression* $\nabla^1 f \equiv \nabla_\lambda f \, d^\lambda$*, where*

$$\nabla_\lambda f = \partial_\lambda f + \Gamma_\lambda{}^i_0 \, \partial^0_i f$$

turns out to be the special quadratic function given by

$$\begin{aligned}
&(\nabla_\lambda f)^0 = \partial_\lambda f^0 \\
(3.2) \qquad &(\nabla_\lambda f)^i = \partial_\lambda f^i - K_\lambda{}^i{}_j \, f^j + K_\lambda{}^i{}_0 \, f^0 \\
&(\nabla_\lambda f)^\circ = \partial_\lambda f^\circ + K_\lambda{}^0_{j0} \, f^j.
\end{aligned}$$

Proposition 3.14. *Let us consider a special quadratic function* f*. Then, we obtain the coordinate expression* $\nabla^2 f \equiv \nabla_{\lambda\mu} f \, d^\lambda \otimes d^\mu$*, where*

$$\nabla_{\lambda\mu} f = \partial_\lambda \nabla_\mu f + \Gamma_\lambda{}^i_0 \, \partial^0_i \nabla_\mu f + K_\lambda{}^i{}_\mu \, \nabla_i f$$

turns out to be the special quadratic function given by

$$\begin{aligned}
&(\nabla_{\lambda\mu} f)^0 = \partial_\lambda \nabla_\mu f^0 + K_\lambda{}^j{}_\mu \, \nabla_i f^0 \\
(3.3) \qquad &(\nabla_{\lambda\mu} f)^i = \partial_\lambda \nabla_\mu f^i - K_\lambda{}^i{}_j \, \nabla_\mu f^j + K_\lambda{}^i{}_0 \, \nabla_\mu f^0 + K_\lambda{}^j{}_\mu \, \nabla_j f^i \\
&(\nabla_{\lambda\mu} f)^\circ = \partial_\lambda \nabla_\mu f^\circ + K_\lambda{}^0_{j0} \, \nabla_\mu f^j + K_\lambda{}^j{}_\mu \, \nabla_j f^\circ.
\end{aligned}$$

Proposition 3.15. *Let us consider a special quadratic function* f*. Then, we obtain the coordinate expression*

$$\nabla^k f \equiv \nabla_{\lambda\mu_1\cdots\mu_{k-1}} f \, d^\lambda \otimes (d^{\mu_1} \otimes \cdots \otimes d^{\mu_{k-1}}),$$

where

$$\begin{aligned}
\nabla_{\lambda\mu_1\cdots\mu_{k-1}} f &= \partial_\lambda \nabla_{\mu_1\cdots\mu_{k-1}} f + \Gamma_\lambda{}^i_0 \, \partial^0_i \nabla_{\mu_1\cdots\mu_{k-1}} f \\
&\qquad + K_\lambda{}^i{}_{\mu_1} \, \nabla_{i\mu_2\cdots\mu_{k-1}} f + \cdots + K_\lambda{}^i{}_{\mu_{k-1}} \, \nabla_{\mu_1\cdots\mu_{k-2}i} f
\end{aligned}$$

turns out to be a special quadratic function.

Corollary 3.16. *For each integer* $k \geq 1$*, the* k*-covariant differential* $\nabla^k f$ *turns out to be a section of the bundle*

$$\operatorname{Spec} \boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} (\otimes^k T^*\boldsymbol{E}).$$

Corollary 3.17. *For each integer* $k \geq 1$*, the above map* ∇^k *yields a covariant fibered morphism over* $\boldsymbol{E}$

$$\nabla^k : J_{k-1} \operatorname{Con} \boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_k \operatorname{Met} \boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_k \operatorname{Spec} \boldsymbol{E} \to \operatorname{Spec} \boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \Big(\bigotimes^k T^*\boldsymbol{E} \Big).$$

For each integer $k \geq 0$, let us define the first-order natural bundle over $\boldsymbol{E}$

$$\operatorname{Spec}^{(k)} \boldsymbol{E} := \operatorname{Spec} \boldsymbol{E} \underset{\boldsymbol{E}}{\times} \Big(\operatorname{Spec} \boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \Big(\bigotimes^1 T^*\boldsymbol{E} \Big) \Big) \underset{\boldsymbol{E}}{\times} \cdots \underset{\boldsymbol{E}}{\times} \Big(\operatorname{Spec} \boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \Big(\bigotimes^k T^*\boldsymbol{E} \Big) \Big).$$

Corollary 3.18. *For each integer $k \geq 0$, we obtain the covariant fibered morphism over $\boldsymbol{E}$, denoted*

$$\nabla^{(k)} := (\mathrm{id}\,, \nabla^1, \dots, \nabla^k) : J_{k-1}\,\mathrm{Con}\,\boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_k\,\mathrm{Met}\,\boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_k\,\mathrm{Spec}\,\boldsymbol{E} \to \mathrm{Spec}^{(k)}\,\boldsymbol{E}.$$

Next, we classify all finite order covariant operators transforming linearly a special quadratic function, the normalized vertical metric and a total connection into special quadratic functions.

Lemma 3.19. *All covariant fibered morphisms over $\boldsymbol{E}$*

$$\mathfrak{d}_{\boldsymbol{E}} : J_{k-1}\,\mathrm{Con}\,\boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_k\,\mathrm{Met}\,\boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_k\,\mathrm{Spec}\,\boldsymbol{E} \to \mathrm{Spec}\,\boldsymbol{E}, \tag{3.4}$$

which are linear on $J_k\,\mathrm{Spec}\,\boldsymbol{E}$, are of the type

$$\begin{aligned} \mathfrak{d}^0_{\boldsymbol{E}} &= A\, f^0 \\ \mathfrak{d}^i_{\boldsymbol{E}} &= A\, f^i + B\, G^{ij}_0\, \partial_j\, f^0 \\ \mathfrak{d}^\circ_{\boldsymbol{E}} &= A\, f^\circ + B\, \partial_0\, f^0 + C\, (\partial_0\, f^0 - f^0\, K_0{}^i{}_i - \partial_i\, f^i + f^j\, K_j{}^i{}_i) \\ &\quad + D\, (G^{ij}_0\, \partial_i \partial_j\, f^0 + G^{ij}_0\, K_i{}^h{}_j\, \partial_h\, f^0) + E\, f^0\, G^{ij}_0\, R_{hi}{}^h{}_j, \end{aligned} \tag{3.5}$$

where A, B, C, D, E are real numbers and $R_{hk}{}^i{}_j$ are the components of the curvature tensor of the vertical connection $\varkappa[G]$.

Proof. Let us denote by $\rho_k : J_{k+1}\,\mathrm{Con}\,\boldsymbol{E} \to \mathrm{Curv}_k\,\boldsymbol{E}$ the covariant operator given by the curvature of K and its covariant derivatives up to the order k. Then, in virtue of [10, Proposition 28.9] and $\nabla G = 0$, there is a unique first-order covariant operator

$$\mathfrak{h}_{\boldsymbol{E}} : \mathrm{Curv}_{k-2}\,\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \mathrm{Met}\,\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \mathrm{Spec}^{(k)}\,\boldsymbol{E} \to \mathrm{Spec}\,\boldsymbol{E} \tag{3.6}$$

such that $\mathfrak{d}_{\boldsymbol{E}} = \mathfrak{h}_{\boldsymbol{E}} \circ (\rho_{k-2}, \nabla^{(k)})$.

We denote the $G^1_{(1,n)}$-equivariant map, induced by $\mathfrak{h}_{\boldsymbol{E}}$ on the type fibers, by

$$\mathfrak{h} : F[\mathrm{Curv}_{k-2}\,\boldsymbol{E}] \times F[\mathrm{Met}\,\boldsymbol{E}] \times F[\mathrm{Spec}^{(k)}\,\boldsymbol{E}] \to F[\mathrm{Spec}\,\boldsymbol{E}].$$

Moreover, the "homogeneous function theorem" ([10]) and the linearity in f imply the following expression

$$\begin{aligned} \mathfrak{h}^0 &= A\, f^0, \\ \mathfrak{h}^i &= B_1\, f^i + B_2\, G^{ij}_0\, f^0{}_{;j}, \\ \mathfrak{h}^\circ &= C_1\, f^\circ + C_2\, f^0{}_{;0} + C_3\, f^i{}_{;i} + C_4\, G^{ij}_0\, f^0{}_{;i;j} + C_5\, f^0\, G^{ij}_0\, R_{hi}{}^h{}_j, \end{aligned}$$

where A, B_i, C_j are real numbers and

$$(f^\lambda, f^\circ, f^\lambda_{;\mu}, f^\circ_{;\mu}, \dots, f^\lambda_{;\mu_1;\dots;\mu_k}, f^\circ_{;\mu_1;\dots;\mu_k})$$

are the coordinates on the type fiber $F[\mathrm{Spec}^{(k)}\,\boldsymbol{E}]$.

Furthermore, by considering the equivariancy with respect to the subgroup in $G^1_{(1,n)}$ consisting of the elements $(\delta^\lambda_\mu, a^p_0)$, we obtain

$$A = B_1 = C_1, \qquad C_2 = B_2 - C_3.$$

Then, we obtain

$$\mathfrak{h}^0 = A\, f^0, \quad \mathfrak{h}^i = A\, f^i + B\, G^{ij}_0\, f^0_{;j},$$
$$\mathfrak{h}^\circ = A\, f^\circ + B\, f^0{}_{;0} + C\,(f^0{}_{;0} - f^j{}_{;j}) + D\, G^{ij}_0\, f^0{}_{;i;j} + E\, f^0\, G^{ij}_0\, R_{hi}{}^h{}_j,$$

where we have put

$$B \equiv B_2,\ C \equiv -C_3,\ D \equiv C_4,\ E \equiv C_5.$$

Eventually, by inserting (3.2) and (3.3) into the above formulas, we get the $G^{k+1}_{(1,n)}$-equivariant map which corresponds to the morphism (3.5). □

Theorem 3.20. *All k-order covariant differential operators transforming a special quadratic function f, a vertical metric G and a total connection K into special quadratic functions, and which are linear with respect to f, are of maximal order* 2 *and are of the type*

$$f \mapsto A\ f + B\ f_{Д} + C\ \mathrm{div}_\upsilon X[f] + D\ \mathrm{div}_\eta X[f_{Д}] + E\ f'' \lrcorner\ r[\varkappa], \tag{3.7}$$

where A, B, C, D, E are real numbers.

Proof. By taking into account the following equalities

$$\mathrm{div}_\upsilon X[f] = \frac{\partial_0(f^0\sqrt{|g|})}{\sqrt{|g|}} - \frac{\partial_i(f^i\sqrt{|g|})}{\sqrt{|g|}}$$
$$= \partial_0 f^0 - f^0 K_0{}^i{}_i - \partial_i f^i + f^j K_j{}^i{}_i,$$
$$\mathrm{div}_\eta X[f_{Д}] = \frac{\partial_i\big(G^{ij}_0\,(\partial_j f^0)\sqrt{|g|}\big)}{\sqrt{|g|}}$$
$$= G^{ij}_0\,(\partial_i\partial_j f^0 + K_i{}^h{}_j\,\partial_h f^0),$$

it is easy to see that the morphism given by Lemma 3.19 corresponds to the operator (3.7). □

Corollary 3.21. *All k-order covariant differential operators, transforming a quantisable function, or a fine function, or an affine function, or a spacetime function f, a vertical metric G and a total connection K into special quadratic func-*

tions, and which are linear with respect to f, are, respectively, of the type

$$\begin{aligned}
&f \mapsto A\, f + B\, f_{Д} + C\, \mathrm{div}_\eta X[f] + E\, f'' \lrcorner\, r[\varkappa], && f \in \mathrm{quan}(J_1\boldsymbol{E}, \mathbb{R}),\\
&f \mapsto A\, f + C\, \mathrm{div}_\eta X[f] + E\, f'' \lrcorner\, r[\varkappa], && f \in \mathrm{fine}(J_1\boldsymbol{E}, \mathbb{R}),\\
&f \mapsto A\, f + C\, \mathrm{div}_\eta X[f], && f \in \mathrm{aff}(J_1\boldsymbol{E}, \mathbb{R}),\\
&f \mapsto A\, f, && f \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}).
\end{aligned}$$

4. Quantum structure

4.1. Quantum bundle

We start with the framework of the quantum theory.

We assume, according to [3], the *quantum bundle* to be a 1-dimensional complex bundle over spacetime

$$\pi : \boldsymbol{Q} \to \boldsymbol{E}\,,$$

equipped with a fibered Hermitian product with values in vertical volume forms

$$\mathfrak{h} : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{C} \otimes \overset{n}{\bigwedge} V^*\boldsymbol{E}.$$

A *quantum section* is a (local) section $\Psi \in \sec\,(\boldsymbol{E},\ \boldsymbol{Q})$.

We shall refer to a *quantum basis*, i.e., to a (local) section $\mathfrak{b} \in \sec(\boldsymbol{E}, \mathbb{L}^{n/2} \otimes \boldsymbol{Q})$ such that $\mathfrak{h}(\mathfrak{b}, \mathfrak{b}) = \eta$. Moreover, we shall refer to a *quantum chart*, i.e., to a fibered chart (x^λ, z) where (x^λ) is a spacetime chart and $z \in \mathrm{map}(\boldsymbol{Q}, \mathbb{C} \otimes \mathbb{L}^{*n/2})$ is the complex fiber coordinate associated with a quantum basis $\mathfrak{b}$.

We write $\Psi = \psi \mathfrak{b}$ and obtain $\mathfrak{h}(\Phi, \Psi) = \bar{\phi}\,\psi\,\eta$, where $\bar{\phi}$ is the complex conjugate of ϕ.

Let us consider two quantum bases $\mathfrak{b}$ and $\bar{\mathfrak{b}}$ and the associated fiber coordinates z and $\bar{z}$. Then we have

$$\bar{\mathfrak{b}} = e^{\mathrm{i}\vartheta}\,\mathfrak{b}, \quad \bar{z} = e^{-\mathrm{i}\vartheta}\,z, \tag{4.1}$$

where $\vartheta \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$.

Thus, the quantum bundle turns out to be [2, 10] a vector bundle associated with a principal bundle Ref $\boldsymbol{Q}$ over $\boldsymbol{E}$, whose structure group is $U(1, \mathbb{C})$.

We deal with the semidirect products of groups $W^{(s,r)}_{(1,n)} := G^s_{(1,n)} \rtimes T^r_{n+1} U(1, \mathbb{C})$, which are central extensions of $G^s_{(1,n)}$ by $T^r_{n+1}U(1, \mathbb{C})$, for each $0 \le r$. The charts of $W^{(s,r)}_{(1,n)}$ will be denoted by $(a^\lambda_\mu, \dots, a^\lambda_{\mu_1 \cdots \mu_s}, \vartheta, \vartheta_j, \dots, \vartheta_{j_1 \cdots j_r})$.

4.2. Quantum connection

We proceed by completing the quantum framework.

A linear connection $\mathrm{Ч}$ of $\boldsymbol{Q}$ is said to be *Hermitian* if it preserves the Hermitian product $\mathfrak{h}$.

Proposition 4.1. *Let us refer to a quantum basis* $\mathfrak{b}$ *and to an adapted quantum chart* (x^λ, z)*. Then, the Hermitian connections* Ч *are of the type,* [3],

$$Ч = \chi[\mathfrak{b}] + \mathfrak{i}A[\mathfrak{b}] \otimes \mathbb{I} = d^\lambda \otimes \partial_\lambda + \mathfrak{i}A_\lambda \, d^\lambda \otimes \mathbb{I}, \tag{4.2}$$

where $\chi[\mathfrak{b}]$ *is the flat Hermitian connection induced by* $\mathfrak{b}$, $\mathbb{I} = z \otimes \mathfrak{b}$ *is the Liouville vector field of* $\boldsymbol{Q}$ *and* $A[\mathfrak{b}] \equiv A_\lambda \, d^\lambda \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$ *is a spacetime form.*

Proposition 4.2. *Let us consider two quantum charts* (x^λ, z) *and* $(\bar{x}^\lambda, \bar{z})$*. Then, the transition maps for the components of the Hermitian connection* Ч *are*

$$\begin{aligned} \bar{A}_i &= (A_j - \partial_j \vartheta) \, \sigma_i^j, \\ \bar{A}_0 &= (A_0 - \partial_0 \vartheta) \, \sigma_0^0 + (A_j - \partial_j \vartheta) \, \sigma_0^j. \end{aligned} \tag{4.3}$$

Corollary 4.3. *The Hermitian connections of* $\boldsymbol{Q}$ *turn out to be the sections of a bundle* $\mathrm{Con}\,\boldsymbol{Q} \to \boldsymbol{E}$ *associated with a principal bundle with structure group* $W^{1,1}_{(1,n)}$.
The type fiber of $\mathrm{Con}\,\boldsymbol{Q}$ *is* $F[\mathrm{Con}\,\boldsymbol{Q}] = \mathbb{R}^{*(n+1)}$.

Next, let us consider the *extended quantum bundle* defined as the pull-back bundle

$$\pi^\uparrow : \boldsymbol{Q}^\uparrow := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E}$$

of the quantum bundle with respect to the map $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$. Here, the extended base space $J_1\boldsymbol{E}$ plays the role of space of observers.

The extended quantum bundle inherits a fibered Hermitian product from the quantum bundle by pullback. We can define a Hermitian connection

$$Ч^\uparrow : \boldsymbol{Q}^\uparrow \to T^*J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} T\boldsymbol{Q}^\uparrow$$

of the extended quantum bundle analogously to an Hermitian connection of the quantum bundle and obtain analogous results.

Proposition 4.4 ([3]). *Let us consider a family of Hermitian connections of the quantum bundle* $\{Ч^\circ\}$ *parametrised by the observers o. Then there is a unique Hermitian connection* $\{Ч^\uparrow\}$ *of the extended quantum bundle, called* universal*, such that, for each observer o,*

$$Ч^\circ = o^* Ч^\uparrow. \tag{4.4}$$

The universal Hermitian connections Ч *are of the type,* [3],

$$Ч^\uparrow = \chi^\uparrow[\mathfrak{b}] + \mathfrak{i}A^\uparrow[\mathfrak{b}] \otimes \mathbb{I}^\uparrow = d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + \mathfrak{i}A^\uparrow{}_\lambda \, d^\lambda \otimes \mathbb{I}^\uparrow, \tag{4.5}$$

where $\chi^\uparrow[\mathfrak{b}]$ *is the flat Hermitian connection induced by* $\mathfrak{b}$, $\mathbb{I}^\uparrow = z \otimes \mathfrak{b}$ *is the Liouville vector field of* $\boldsymbol{Q}^\uparrow$ *and* $A^\uparrow[\mathfrak{b}] \equiv A^\uparrow{}_\lambda \, d^\lambda \in \mathrm{fib}(J_1\boldsymbol{E}, T^*\boldsymbol{E})$ *is a horizontal phase form.*

Hence, for each quantum basis $\mathfrak{b}$ *and observer o, the form associated with* $Ч^\circ$ *is*

$$A[\mathfrak{b}, o] = A^\uparrow[\mathfrak{b}] \circ o.$$

Conversely, an Hermitian connection $\Upsilon^{\uparrow}$ *of* $\boldsymbol{Q}^{\uparrow}$ *of the type* (4.5) *yields a system of Hermitian connections of* $\boldsymbol{Q}$*, whose universal connection is* $\Upsilon^{\uparrow}$.

Definition 4.5. We define ([3, 4]) a *quantum connection* to be a connection $\Upsilon^{\uparrow}$ of the extended quantum bundle which is Hermitian, universal and whose curvature is $R[\Upsilon^{\uparrow}] = -2\,\mathrm{i}\Omega \otimes \mathbb{I}$.

Proposition 4.6. *A quantum connection* $\Upsilon^{\uparrow}$ *exists locally. Indeed, with reference to a quantum basis* $\mathfrak{b}$*, its local coordinate expression is of the type*

$$A^{\uparrow}[\mathfrak{b}] \equiv A^{\uparrow}{}_{\lambda}\, d^{\lambda} = -(\tfrac{1}{2}\ G^{0}_{ij}\, x^{i}_{0}\, x^{j}_{0} - A_{0})\, d^{0} + (G^{0}_{ij}\, x^{j}_{0} + A_{i})\, d^{i},$$

where

$$d A^{\uparrow}[\mathfrak{b}] = \Omega.$$

Corollary 4.7. *The quantum connections turn out to be the sections of a bundle* $\mathrm{Con}\, \boldsymbol{Q}^{\uparrow} \to \boldsymbol{E}$ *associated with a principal bundle with structure group* $W^{1,1}_{(1,n)}$.

The type fiber of $\mathrm{Con}\, \boldsymbol{Q}^{\uparrow}$ *is*

$$F[\mathrm{Con}\, \boldsymbol{Q}^{\uparrow}] = \mathbb{R}^{*(n+1)}$$

and, in virtue of (4.3)*, we obtain the following action of* $W^{1,1}_{(1,n)}$ *on* $F[\mathrm{Con}\, \boldsymbol{Q}^{\uparrow}]$

$$\begin{aligned} \bar{A}_{i} &= (A_{j} + a^{0}_{0}\, G^{0}_{jh}\, \bar{a}^{h}_{0} - \vartheta_{j})\, \bar{a}^{j}_{i}, \\ \bar{A}_{0} &= (A_{\lambda} - \vartheta_{\lambda})\, \bar{a}^{\lambda}_{0} + \tfrac{1}{2}\ a^{0}_{0}\, G^{0}_{hk}\, \bar{a}^{h}_{0}\, \bar{a}^{k}_{0}. \end{aligned} \tag{4.6}$$

Corollary 4.8. *For each quantum basis* $\mathfrak{b}$ *and observer* o*, we have the fibered isomorphism*

$$\mathrm{Con}\, \boldsymbol{Q}^{\uparrow} \to T^{*}\boldsymbol{E} : \Upsilon^{\uparrow} \mapsto A[\mathfrak{b}, o].$$

Corollary 4.9. *Given a quantum basis* $\mathfrak{b}$*, if*

$$\Upsilon^{\uparrow} \in \mathrm{sec}(\boldsymbol{E}, \mathrm{Con}\, \boldsymbol{Q}^{\uparrow}),$$

then

$$A[\mathfrak{b}, o] := A^{\uparrow}[\mathfrak{b}] \circ o = A_{\lambda}\, d^{\lambda} \in \mathrm{sec}(\boldsymbol{E}, T^{*}\boldsymbol{E})$$

turns out to be a classical potential associated with the observer o*, whose gauge is determined by* $\mathfrak{b}$.

Thus, the choice of a quantum basis $\mathfrak{b}$ turns out to be another way to control the gauge of classical potentials. Indeed, each quantum basis $\mathfrak{b}$ and observer o yield a fibered isomorphism $\mathrm{Ext}\, \boldsymbol{E} \to \mathrm{Con}\, \boldsymbol{Q}^{\uparrow}$ over $\boldsymbol{E}$.

Corollary 4.10. *Let us consider a quantum connection* $\Upsilon^{\uparrow}$*. Then, for each quantum basis* $\mathfrak{b}$ *and observer* o*, we have*

$$d A[\mathfrak{b}, o] = o^{*}\Omega.$$

Hence, we obtain a covariant fibered morphism over $\boldsymbol{E}$*,*

$$d : J_1 \operatorname{Con} \boldsymbol{Q}^{\uparrow} \to \bigwedge^2 T^* \boldsymbol{E}.$$

A quantum connection $\text{Ч}^{\uparrow}$ exists globally if and only if the cohomology class of Ω is an integer, [15]. In the theory of covariant quantum mechanics we suppose that our quantum bundle admit a quantum connection and assume one.

4.3. Canonical covariant pre-quantum operators

In this section we introduce distinguished operators of the quantum bundle.

Definition 4.11. We define a kth-order *covariant pre-quantum operator* to be an operator on the quantum bundle given by a covariant fibered morphism over $\boldsymbol{E}$

(4.7) $$\mathfrak{q}_{\boldsymbol{Q}} : J_k \operatorname{Met} \boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_{k-1} \operatorname{Con} \boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_k \operatorname{Spec} \boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_{k-1} \operatorname{Con} \boldsymbol{Q}^{\uparrow} \underset{\boldsymbol{E}}{\times} J_k \boldsymbol{Q} \to \boldsymbol{Q}$$

which is linear with respect to $J_k \operatorname{Spec} \boldsymbol{E}$.

We obtain a distinguished first-order covariant pre-quantum operator in the following way.

Lemma 4.12. *For each special quadratic function* f*, the vector field*

(4.8) $$Y[f] := \text{Ч}^{\circ}\big(X[f]\big) + \mathfrak{i}\, f^{\circ} \mathbb{I} = f^0\, \partial_0 - f^i\, \partial_i + \mathfrak{i}(f^{\circ} + f^0 A_0 - f^i A_i)\, \mathbb{I},$$

does not depend on the choice of the observer o*,* [9].

We call $Y[f]$ the *quantum lift* of f.

Proposition 4.13. *Each special quadratic function* f *yields the first-order pre-quantum operator*

(4.9) $$L_{Y[f]} \Psi = (-\mathfrak{i}\, f^{\circ} + f^0 \nabla^{\circ}_0 - f^i \nabla^{\circ}_i)\psi\, \mathfrak{b},$$

where $\nabla^{\circ}_{\lambda}\psi := (\partial_{\lambda} - \mathfrak{i} A_{\lambda})\, \psi$ *is the covariant differential associated with* Ч°.

We obtain a distinguished second-order covariant pre-quantum operator in the following way.

A *generalised covariant Schroedinger fibered morphism* is defined to be a covariant fibered morphism over $\boldsymbol{E}$

$$\mathrm{O} : J_2 \operatorname{Met} \boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_1 \operatorname{Con} \boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_1 \operatorname{Con} \boldsymbol{Q}^{\uparrow} \underset{\boldsymbol{E}}{\times} J_2 \boldsymbol{Q} \to \mathbb{T}^* \otimes \boldsymbol{Q}.$$

Lemma 4.14. *The second-order operator*

(4.10) $$\mathrm{S}(\Psi) = u^0 \otimes \big(\nabla^{\circ}_0 + \tfrac{1}{2} (\operatorname{div}_{\eta} o)_0 - \mathfrak{i}\, \tfrac{1}{2}\, \Delta^{\circ}_0\big)\psi\, \mathfrak{b},$$

where

$$\Delta^{\circ}_0 \psi := G^{ij}_0 (\nabla^{\circ}_i \nabla^{\circ}_i + K_i{}^h{}_j \nabla^{\circ}_h)\psi$$

is the Laplacian associated with Υ° *and*

$$\operatorname{div}_\eta o := \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}}\, u^0$$

is the divergence of the observer, does not depend on the choice of the observer o, [9].

Theorem 4.15 ([7]). *All second-order generalised covariant Schroedinger operators can be deduced from covariant quantum Lagrangians and are of the type*

$$\mathrm{O}(\Psi) = a\, \mathrm{S}(\Psi) + b\, r[\varkappa](\Psi), \qquad a, b \in \mathbb{C}. \tag{4.11}$$

Proposition 4.16. *Let* O *be a generalised covariant Schroedinger operator. Then, each special quadratic function* f *yields the second-order pre-quantum operator*

$$f'' \lrcorner\, \mathrm{O}(\Psi) = f^0\big(\nabla^\circ_0 + \tfrac{1}{2}\,(\operatorname{div}_\eta o)_0 - \mathfrak{i}\,\tfrac{1}{2}\,\Delta^\circ_0 + br_0\big)\psi\,\mathfrak{b}.$$

4.4. Classification of second-order pre-quantum operators

Eventually, we classify all covariant second-order pre-quantum operators.

Let us consider the vector bundle over $\boldsymbol{E}$

$$\operatorname{Der}_2 \boldsymbol{Q} := \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} (T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}) \underset{\boldsymbol{E}}{\times} (S^2 T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}),$$

with type fiber

$$F[\operatorname{Der}_2 \boldsymbol{Q}] = (\mathbb{C}\otimes\mathbb{L}^{*(n/2)}) \times (\mathbb{C}\otimes\mathbb{L}^{*(n/2)}\otimes\mathbb{R}^{*n}) \times (\mathbb{C}\otimes\mathbb{L}^{*(n/2)}\otimes\mathbb{R}^{*(n(n+1)/2)}).$$

We denote the charts on this type fiber by $(z,\ z_{;\lambda},\ z_{;\lambda;\mu})$.

Lemma 4.17. *For each observer o,* $\Upsilon^\uparrow \in \operatorname{Con} \boldsymbol{Q}^\uparrow$ *and* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$*, we have*

$$\operatorname{Sym}(\nabla^\circ\nabla^\circ\Psi) = \nabla^\circ\,\nabla^\circ\,\Psi + \mathfrak{i}\nu^*[o]\big(G^\flat(\nabla o)\big)\,\Psi,$$

where $\nabla^\circ\,\nabla^\circ\Psi$ *is the second covariant differential induced by* Υ° *and* K.

Proof. It follows from (4.3) and (2.1). □

Lemma 4.18. *The map given, for each observer o, by*

$$\Psi \mapsto \big(\Psi,\ \nabla^\circ\Psi,\ \operatorname{Sym}(\nabla^\circ\nabla^\circ\Psi)\big)$$

yields a covariant fibered morphism over $\boldsymbol{E}$

$$\operatorname{Der}_2 : J_1 \operatorname{Con} \boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_1 \operatorname{Con} \boldsymbol{Q}^\uparrow \underset{\boldsymbol{E}}{\times} J_2 \boldsymbol{Q} \to \operatorname{Der}_2 \boldsymbol{Q}, \tag{4.12}$$

hence the $W^{2,2}_{(1,n)}$*-equivariant map between type fibers*

$$F[\operatorname{Con} \boldsymbol{E}] \times F[J_1 \operatorname{Con} \boldsymbol{Q}^\uparrow] \times F[J_2 \boldsymbol{Q}] \to F[\operatorname{Der}_2 \boldsymbol{Q}] \tag{4.13}$$

given by

$$
(4.14)\qquad
\begin{aligned}
z &= z,\\
z_{;\lambda} &= z_{\lambda} - \mathrm{i}A_{\lambda} z,\\
z_{;0;\lambda} &= (z_{;0})_{,\lambda} - \mathrm{i}A_{\lambda}\, z_{;0} + K_{\lambda}{}^{p}{}_{0}\, z_{;p},\\
z_{;i;\lambda} &= (z_{;i})_{,\lambda} - \mathrm{i}A_{\lambda}\, z_{;i} + K_{\lambda}{}^{p}{}_{i}\, z_{;p} - \mathrm{i}zK_{\lambda}{}^{p}{}_{0}\, G^{0}_{pi}.
\end{aligned}
$$

Proof. The above fibered morphism is covariant because the covariant differentials ∇° and $\mathrm{Sym}(\nabla^{\circ}\nabla^{\circ})$ are performed with respect to the system of connections $\{Ч^{\circ}\}$ associated with the same quantum connection $Ч^{\uparrow}$. □

Lemma 4.19. *We obtain the following action of the group $W^{(1,0)}_{(1,n)}$ on the type fiber $F[\mathrm{Der}_2\, \boldsymbol{Q}]$*

$$
(4.15)\qquad
\begin{aligned}
\bar{z} &= e^{-\mathrm{i}\vartheta}\, z,\\
\bar{z}_{;0} &= e^{-\mathrm{i}\vartheta}\left(z_{;\mu}\, \bar{a}^{\mu}_{0} - (\mathrm{i}/2)\, zG^{0}_{hk}\, \bar{a}^{h}_{0}\bar{a}^{k}_{0}a^{0}_{0}\right),\\
\bar{z}_{;i} &= e^{-\mathrm{i}\vartheta}(z_{;j} + \mathrm{i}zG^{0}_{hj}\, \bar{a}^{h}_{r}a^{r}_{0})\bar{a}^{j}_{i},\\
\bar{z}_{;0;0} &= e^{-\mathrm{i}\vartheta}(z_{;\mu;\nu}\bar{a}^{\mu}_{0}\bar{a}^{\nu}_{0} - \mathrm{i}z_{;\mu}G^{0}_{pq}\bar{a}^{\mu}_{0}\bar{a}^{p}_{0}\bar{a}^{q}_{0}a^{0}_{0}\\
&\qquad - \tfrac{1}{4}\, zG^{0}_{pq}G^{0}_{rs}\, \bar{a}^{p}_{0}\bar{a}^{q}_{0}\bar{a}^{r}_{0}\bar{a}^{s}_{0}a^{0}_{0}a^{0}_{0}),\\
\bar{z}_{;0;i} &= e^{-\mathrm{i}\vartheta}\big(z_{;\mu;s}\bar{a}^{\mu}_{0}\bar{a}^{s}_{i} - \mathrm{i}z_{;\mu}G^{0}_{pq}\, \bar{a}^{\mu}_{0}\bar{a}^{p}_{i}\bar{a}^{q}_{0}a^{0}_{0} - (\mathrm{i}/2)\, z_{;s}\, G^{0}_{pq}\bar{a}^{s}_{i}\bar{a}^{p}_{0}\bar{a}^{q}_{0}a\\
&\qquad - \tfrac{1}{2}\, zG^{0}_{pq}G^{0}_{rs}\, \bar{a}^{p}_{0}\bar{a}^{q}_{i}\bar{a}^{r}_{0}\bar{a}^{s}_{0}a^{0}_{0}a^{0}_{0}\big),\\
\bar{z}_{;i;0} &= e^{-\mathrm{i}\vartheta}\big(z_{;s;\mu}\bar{a}^{s}_{i}\bar{a}^{\mu}_{0} - \mathrm{i}z_{;\mu}G^{0}_{pq}\, \bar{a}^{\mu}_{0}\bar{a}^{p}_{i}\bar{a}^{q}_{0}a^{0}_{0} - (\mathrm{i}/2)z_{;s}G^{0}_{pq}\, \bar{a}^{s}_{i}\bar{a}^{p}_{0}\bar{a}^{q}_{0}a^{0}_{0}\\
&\qquad - \tfrac{1}{2}\, zG^{0}_{pq}G^{0}_{rs}\, \bar{a}^{p}_{0}\bar{a}^{q}_{0}\bar{a}^{r}_{i}\bar{a}^{s}_{0}a^{0}_{0}a^{0}_{0}\big),\\
\bar{z}_{;i;j} &= e^{-\mathrm{i}\vartheta}\big(z_{;r;s}\bar{a}^{r}_{i}\bar{a}^{s}_{j} - \mathrm{i}(z_{;r}G^{0}_{sp} + z_{;s}G^{0}_{rp})\bar{a}^{r}_{i}\bar{a}^{s}_{j}\bar{a}^{p}_{0}a^{0}_{0}\\
&\qquad - zG^{0}_{pq}G^{0}_{rs}\, \bar{a}^{p}_{i}\bar{a}^{q}_{0}\bar{a}^{r}_{j}\bar{a}^{s}_{0}a^{0}_{0}a^{0}_{0}\big).
\end{aligned}
$$

Theorem 4.20. *All covariant fibered morphisms over $\boldsymbol{E}$*

$$
(4.16)\qquad \mathfrak{q}_{\boldsymbol{Q}} : J_2\,\mathrm{Met}\,\boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_1\,\mathrm{Con}\,\boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_2\,\mathrm{Spec}\,\boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_1\,\mathrm{Con}\,\boldsymbol{Q}^{\uparrow} \underset{\boldsymbol{E}}{\times} J_2\boldsymbol{Q} \to \boldsymbol{Q}
$$

are given by

$$
\begin{aligned}
\mathfrak{q} = {} & \alpha f^0\big(\tfrac{1}{2} G_0^{ij}(z_{ij} - 2\mathrm{i}A_i z_j - \mathrm{i}A_{i,j} z - A_i A_j z) \\
& \qquad + \tfrac{1}{2} G_0^{ij} K_i{}^p{}_j (z_p - \mathrm{i}A_p z) + \mathrm{i}(z_0 - \mathrm{i}A_0 z) - (\mathrm{i}/2) K_0{}^p{}_p z\big) \\
& + \beta\big(-\mathrm{i} f^\circ z + f^0(z_0 - \mathrm{i}A_0 z) - f^i(z_i - \mathrm{i}A_i z)\big) \\
& + \gamma\big(-\mathrm{i} f^0_{,0} z - f^0_{,p} G_0^{ip}(z_i - \mathrm{i}A_i z)\big) \\
& + \delta(f^0_{,0} - f^0 K_0{}^p{}_p - G_0^{ij} K_i{}^p{}_j f^0_p - G_0^{ij} f^0_{i,j}) z \\
& + \epsilon(f^0_{,p} G_0^{ij} K_i{}^p{}_j + f^0_{,ij} G_0^{ij}) z + \phi f^0 G_0^{ij} R_{pi}{}^p{}_j z,
\end{aligned}
\tag{4.17}
$$

where $\alpha, \ldots, \phi$ are complex numbers.

Proof. Let us denote a $W^{2,2}_{(1,n)}$-equivariant function

$$F[J_1 \,\mathrm{Con}\, \boldsymbol{E}] \times F[J_2 \,\mathrm{spec}\, \boldsymbol{E}] \times F[J_1 \,\mathrm{Con}\, \boldsymbol{Q}^\uparrow] \times F[J_2 \boldsymbol{Q}] \to \mathbb{L}^{*n/2} \otimes \mathbb{C}$$

by $\mathfrak{q}$. Thus, $\mathfrak{q}$ is a function of the type

$$
\begin{aligned}
\mathfrak{q} = \mathfrak{q}(& G^0_{ij}, G^0_{ij,\lambda}, G^0_{ij,\lambda\mu}, K_\mu{}^i{}_\nu, K_\mu{}^i{}_{\nu,\rho}, f^\lambda, f^\lambda_{,\mu}, f^\lambda_{,\mu\nu}, f^\circ, f^\circ_{,\lambda}, f^\circ_{,\lambda\mu}, \\
& A_\lambda, A_{\lambda,\mu}, z, \bar{z}, z_\lambda, \bar{z}_\lambda, z_{\lambda\mu}, \bar{z}_{\lambda\mu}),
\end{aligned}
$$

where $\bar{z}\ \cdots$ denotes complex conjugation.

Now, according to the "orbit reduction theorem", [10], we can express $\mathfrak{q}$ as

$$\mathfrak{q} = \mathfrak{p} \circ (\rho_0, \nabla^{(2)}, d, \mathrm{Der}_2),$$

where $\mathfrak{p}$ is a $W^{1,0}_{(1,n)}$-equivariant function

$$
\begin{aligned}
& F[\mathrm{Met}\, \boldsymbol{E}] \times F[\mathrm{Curv}\, \boldsymbol{E}] \times F[\mathrm{Spec}^{(2)}\, \boldsymbol{E}] \times F[\Lambda^2 T^* \boldsymbol{E}] \times F[\mathrm{Der}_2\, \boldsymbol{E}] \\
& \qquad \to \mathbb{L}^{*n/2} \otimes \mathbb{C}
\end{aligned}
$$

i.e., a function of the type

$$
\begin{aligned}
\mathfrak{p} = \mathfrak{p}(& G^0_{ij}, R_{\lambda\mu}{}^i{}_\nu, f^\lambda, f^\lambda_{;\mu}, f^\lambda_{;\mu;\nu}, f^\circ, f^\circ_{;\lambda}, f^\circ_{;\lambda;\mu}, \Phi_{\lambda\mu}, z, \bar{z}, z_{;\lambda}, \bar{z}_{;\lambda}, \\
& z_{;\lambda;\mu}, \bar{z}_{;\lambda;\mu}).
\end{aligned}
$$

Moreover, in virtue of the "homogeneous function theorem", applied on a change of base in $\mathbb{L}$ and a change of gauge, $\mathfrak{p}$ is of the type

$$\mathfrak{p} = p\, z + p^\lambda\, z_{;\lambda} + p^{\lambda\mu}\, z_{;\lambda;\mu}, \tag{4.18}$$

where all coefficients are $G^1_{(1,n)}$-equivariant. Then, if we consider global solutions only, the homogeneous function theorem and linearity in f yield

$$p = \alpha_1 f^\circ + \alpha_2 f^0_{;0} + \alpha_3 f^i_{;i} + \alpha_4 G^{ij}_0 f^0_{;i;j} + \alpha_5 f^0 G^{ij}_0 R_{ki}{}^k{}_j;$$

$$p^0 = bf^0; \qquad p^i = c_1 f^i + c_2 f^0_{;p} G^{ip}_0;$$

$$p^{00} = p^{0i} = p^{i0} = 0; \qquad p^{ij} = d f^0 G^{ij}_0,$$

where α_i, b, c_j, d are complex numbers.

Now, considering the equivariancy of (4.18) with respect to the subgroup in $G^1_{(1,n)}$ given by elements $(\delta^\lambda_\mu, a^p_0)$, we get

$$\alpha_1 = \mathrm{i}c_1, \qquad \alpha_2 = -a_3 + \mathrm{i}c_2, \qquad b = -c_1 + 2\,\mathrm{i}d. \tag{4.19}$$

Then, (4.14) and (4.19), with $\alpha_3 = -\delta$, $\alpha_4 = \epsilon$, $\alpha_5 = \phi$, $c_1 = -\beta$, $c_2 = -\gamma$, $d = \frac{1}{2}\alpha$, yield the equivariant map which corresponds to (4.17). □

Corollary 4.21. *All second-order covariant pre-quantum operators depending on* G, K, $Ч^\uparrow$, Ψ *and linearly on a special quadratic function* f *are of the type*

$$\begin{aligned} \mathfrak{q}_Q(\Psi) = (&\alpha f'' \lrcorner\, \mathrm{S} + \beta L_{Y[f]} + \gamma L_{Y[f_Д]} + \delta\, \mathrm{div}_\upsilon X[f] \\ &+ \epsilon\, \mathrm{div}_\eta X[f_Д] + \phi f'' \lrcorner\, r[\varkappa])(\Psi), \end{aligned} \tag{4.20}$$

where $\alpha, \dots, \phi$ *are complex numbers.*

Corollary 4.22. *All second-order covariant pre-quantum operators associated with* $f \in \mathrm{quan}\,\boldsymbol{E}$, $f \in \mathrm{fine}\,\boldsymbol{E}$, $f \in \mathrm{aff}\,\boldsymbol{E}$, $f \in \mathrm{map}(\boldsymbol{E},\mathbb{R})$*, respectively, are of the type*

$$\begin{aligned} \mathfrak{q}_Q(\Psi) = \big(&\alpha f'' \lrcorner\, \mathrm{S} + \beta L_{Y[f]} + \gamma L_{Y[f_Д]} + \delta\, \mathrm{div}_\upsilon X[f] \\ &+ \phi f'' \lrcorner\, r[\varkappa]\big)(\Psi), \end{aligned}$$

$$\mathfrak{q}_Q(\Psi) = \big(\alpha f'' \lrcorner\, \mathrm{S} + \beta L_{Y[f]} + \delta\, \mathrm{div}_\upsilon X[f] + \phi f'' \lrcorner\, r[\varkappa]\big)(\Psi),$$

$$\mathfrak{q}_Q(\Psi) = \big(\beta L_{Y[f]} + \delta\, \mathrm{div}_\upsilon X[f]\big)(\Psi),$$

$$\mathfrak{q}_Q(\Psi) = \beta L_{f\mathbb{I}}(\Psi) = \beta f \Psi.$$

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