

Canonical gravity and gravitational energy¹

Katarzyna Grabowska and Jerzy Kijowski

Abstract. Hamiltonian evolution of gravitational field within a spatially compact world tube with non-vanishing boundary is described. It is shown that the standard A.D.M.-symplectic structure in the space of Cauchy data must be supplemented by an extra boundary term. Possible “initial value + boundary value” problems, compatible with this structure, are proposed. A possibility to define quasilocal energy as a Hamiltonian function of the resulting Hamiltonian system is analyzed.

Keywords. Canonical gravity, gravitational energy, A.D.M. symplectic structure, Hamiltonian formalism.

MS classification. 83C05, 83C40.

1. Introduction

There is a lot of ambiguities in the definition of gravitational energy. A textbook version of the Legendre transformation, which is often used to derive Hamiltonian formalism from the Lagrangian field theory, leads to a somewhat paradoxical result: gravitational energy vanishes *modulo boundary terms*. The same textbook version of the Canonical Field Theory (used, e.g., as a starting point for second quantization of Electrodynamics) is only “volume sensitive” but not “boundary sensitive”. This means that boundary phenomena are simply neglected. But here, in Gravity Theory, neglecting boundary terms means neglecting everything. Some authors improve this version of Canonical Gravity by imposing extra requirements on the energy functional in the asymptotically flat case (see, e.g., [1]).

In this way gravitational Hamiltonian is defined as “zero + boundary corrections”. These corrections are, however, often obtained not by a universal procedure, well defined for any field theory (e.g., electrodynamics), but *via ad hoc* improvements, which make no sense outside of Gravity Theory.

¹ This paper is in final form and no version of it will be submitted for publication elsewhere.

Another philosophy of such improvements is based on Lagrangian manipulations: as a remedy for the well-known diseases of the naive version of Canonical Gravity some authors propose adding boundary terms to the Lagrangian function of the theory (see [2]–[3], or [5]–[6]).

In the present paper we try to convince the reader that the Lagrangian manipulations are irrelevant for purposes of Canonical Gravity. Definitions of the gravitational energy must be based on a profound analysis of dynamical properties of Einstein equations. These properties cannot be improved by – even most sophisticated – boundary corrections of the gravitational action. As a starting point of our approach we take a version of Canonical Field Theory which is not only “volume sensitive” but also “boundary sensitive” (see [13], and [4]). In this approach boundary terms are fully legitimate: a Hamiltonian given by a boundary integral is not a paradox. When applied to gravity, this approach suggests definition of gravitational energy contained within a generic two-dimensional compact boundary S , as a quasi-local quantity (for the “free gravity” version of these results, see [8]–[9]; for a generalized version, when the interacting “gravity + matter fields” systems were analyzed, see [10]). Total energy is then obtained *via* a limiting procedure, when the surface S goes to infinity (spatial infinity for the A.D.M.-mass and null infinity for the Trautman–Bondi mass). In this paper we present for the first time an improved version of this result, where the vector field X generating dynamics is not necessarily time-like (as was always assumed in the previous versions of the theory). This way not only the *quasi*-local energy and static momentum may be defined, but also the *momentum* and the *angular momentum*.

We stress that the result presented here *does not* depend upon a choice of a variational principle, used for derivation of Einstein equations. As is generally known, there are different variational formulations of General Relativity. In the “purely metric” formulation, variation is taken with respect to the metric tensor $g_{\mu\nu}$. The corresponding Hilbert Lagrangian

$$L = \frac{1}{16\pi} \sqrt{|g|} R$$

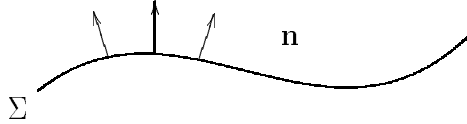
is of the second differential order. We may also use the non-invariant first order Einstein Lagrangian, obtained by subtracting a complete divergence from the Hilbert Lagrangian. Palatini proposed another (the so called “metric-affine”) formulation, where variation is taken with respect to both the metric and the connection $\Gamma_{\mu\nu}^{\lambda}$, treated *a priori* as independent quantities.

Finally, one of us (see [12]) proposed a “purely-affine” formulation, where the metric does not enter into the Lagrangian function and variation is taken with respect to the connection only. In this approach, metric tensor arises as a momentum canonically conjugate to the connection. Typically, the affine Lagrangian is of the form $L(\Gamma, \partial\Gamma) = c \cdot \sqrt{|\det R_{(\mu\nu)}|}$.

All the above variational formulations of General Relativity (and also everything, which may be obtained from them *via* boundary manipulations) lead to the same *volume part* of Canonical Gravity.

Its structure may be described as follows.

Given a space-like hypersurface $\Sigma \subset M$ (possibly *with* boundary) embedded in a general relativistic space-time M , denote by \mathbf{n} the unit, time oriented field, normal to Σ .



Consider the extrinsic curvature tensor

$$(1) \quad K_{mn} = (\nabla_m \partial_n | \mathbf{n}) = -(\partial_n | \nabla_m \mathbf{n}),$$

and its trace $K = K_{mn} \tilde{g}^{mn}$, where by \tilde{g}^{mn} we denote the three-dimensional inverse to the restriction g_{kl} to Σ of the metric $g_{\mu\nu}$ ($k, l = 1, 2, 3$; $\mu, \nu = 0, 1, 2, 3$). Take the Arnowitt–Deser–Misner momentum defined as follows:

$$(2) \quad P^{kl} = \sqrt{\det \tilde{g}} (K \tilde{g}^{kl} - K^{kl}).$$

The volume part ω_Σ of the symplectic form, defined in the space of Cauchy data (P^{kl}, g_{kl}) on Σ , the same for all the four variational formulations, equals

$$(3) \quad \omega_\Sigma = \frac{1}{16\pi} \int_\Sigma dP^{kl}(x) \wedge dg_{kl}(x).$$

This description of the phase space for gravity is correct only “in principle” because of the following problems:

- A boundary correction $\omega_{\partial\Sigma}$ to the symplectic form is necessary. We are going to derive it in the sequel and to show that the correct *total* symplectic structure is given by $\omega_\Sigma + \omega_{\partial\Sigma}$.
- There are Gauss–Codazzi constraints imposed on data (P^{kl}, g_{kl}) .
- There is an extra gauge invariance, dual to constraints. This implies that the “true” phase space of gravity is described by the quotient space of classes of data *modulo* gauge transformations.

In spite of these problems, formula (3) contains, as will be seen later, the only “variational” ingredient which is necessary for the construction of the satisfactory Canonical Gravity Theory.

2. Role of boundary integrals in the canonical field theory

To illustrate our approach, consider the symplectic formulation of mechanics. The phase space, parametrized by positions $q = (q^i)$ and momenta $p = (p_i)$, is equipped with the symplectic form

$$(4) \quad \omega = dp \wedge dq \quad (= dp_i \wedge dq^i),$$

A vector field

$$(5) \quad \mathcal{X} = \dot{p} \frac{\partial}{\partial p} + \dot{q} \frac{\partial}{\partial q},$$

is a Hamiltonian vector field generated by a Hamiltonian function $H = H(p, q)$ if and only if its components (\dot{p}, \dot{q}) fulfil the following equations:

$$(6) \quad \dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p},$$

or shortly: $-dH(p, q) = \dot{p} dq - \dot{q} dp$. This means, that for any vector

$$(7) \quad \mathcal{Y} = \delta p \frac{\partial}{\partial p} + \delta q \frac{\partial}{\partial q},$$

tangent to the phase space, the derivative of the Hamiltonian in the direction of \mathcal{Y} (i.e., the quantity $\delta H := \mathcal{Y}(H)$), satisfies the following equation

$$(8) \quad -\delta H(p, q) = \dot{p} \delta q - \dot{q} \delta p \quad (= \omega(\mathcal{X}, \mathcal{Y})).$$

Given dynamics of the system: $\dot{p} = \dot{p}(p, q)$ and $\dot{q} = \dot{q}(p, q)$, we may plug it into the right-hand side of the above equation and check, whether or not the resulting one-form is equal to a complete differential of a function on the phase space. In case of a positive answer, the dynamics is Hamiltonian and the function H may be reconstructed from (8) uniquely, up to an additive constant.

Consider now a simple example of a field theory: the scalar field satisfying Klein–Gordon equation

$$(9) \quad \ddot{\phi} - \Delta\phi + m^2\phi = 0,$$

or, equivalently,

$$(10) \quad \dot{\phi} = \pi, \quad \dot{\pi} = \Delta\phi - m^2\phi.$$

The phase space of Cauchy data $(\pi(x), \phi(x))$ on Σ is equipped with a symplectic form

$$(11) \quad \omega = \int_{\Sigma} d\pi(x) \wedge d\phi(x),$$

(no volume form ‘ d^3x ’ under the integral is needed because the momentum π is a scalar density and, hence, it is already a measure on Σ , see [4]). The object $\mathcal{X} = (\dot{\pi}, \dot{\phi})$ defines a vector in this phase space. Plugging the dynamics (10) into ω , we obtain for any other vector $\mathcal{Y} = (\delta\pi, \delta\phi)$:

$$\begin{aligned} \int_{\Sigma} \dot{\pi} \delta\phi - \dot{\phi} \delta\pi &= \int_{\Sigma} (\Delta\phi - m^2\phi) \delta\phi - \pi \delta\pi \\ &= -\delta \frac{1}{2} \int_{\Sigma} \{m^2\phi^2 + (\nabla\phi)^2 + \pi^2\} + \int_{\partial\Sigma} (\partial^\perp\phi \delta\phi), \end{aligned}$$

or, equivalently,

$$(12) \quad -\delta H = \int_{\Sigma} \dot{\pi} \delta\phi - \dot{\phi} \delta\pi + \int_{\partial\Sigma} p^{\perp} \delta\phi,$$

where $H = \frac{1}{2} \int_{\Sigma} \{m^2 \phi^2 + (\nabla\phi)^2 + \pi^2\}$, and we denote by $\pi = p^0$ the time component of the momentum $p^{\mu} := -g^{\mu\nu} \partial_{\nu} \phi$. Functional H could be interpreted as a Hamiltonian and the above formula as an infinite-dimensional analogue of the Hamiltonian formula (8), provided the surface term in (12) vanishes. This happens, e.g., when we limit ourselves to the space \mathcal{P}_D of Cauchy data fulfilling the Dirichlet boundary condition: $\phi|_{\partial\Sigma} = f$, where f is given *a priori*. Within this space we have $\delta\phi|_{\partial\Sigma} = 0$ and therefore

$$(13) \quad -\delta H_D = \int_{\Sigma} \dot{\pi} \delta\phi - \dot{\phi} \delta\pi,$$

or, equivalently,

$$(14) \quad \dot{\pi} = -\frac{\delta H_D}{\delta\phi}, \quad \dot{\phi} = \frac{\delta H_D}{\delta p},$$

where by H_D we denote the restriction of H to the phase space \mathcal{P}_D .

This method to translate field equations into the Hamiltonian language is not unique. Indeed, applying Legendre transformation to the boundary term of (12): $p^{\perp} \delta\phi = \delta(p^{\perp} \phi) - \phi \delta p^{\perp}$, we obtain

$$(15) \quad -\delta \tilde{H} = \int_{\Sigma} \dot{\pi} \delta\phi - \dot{\phi} \delta\pi - \int_{\partial\Sigma} \phi \delta p^{\perp},$$

where we have defined the following functional

$$(16) \quad \tilde{H} := H + \int_{\partial\Sigma} p^{\perp} \phi = \int_{\Sigma} \left\{ \frac{1}{2} m^2 \phi^2 + \frac{1}{2} (\nabla\phi)^2 + \frac{1}{2} \pi^2 - \partial_k (\phi \partial^k \phi) \right\}.$$

To derive Hamiltonian system from generating formula (15) let us choose the phase space \mathcal{P}_N of Cauchy data fulfilling Neuman boundary condition: $p^{\perp}|_{\partial\Sigma} = f$. Within this space we have $\delta p^{\perp}|_{\partial\Sigma} = 0$ and, therefore,

$$(17) \quad \delta H_N = \int_{\Sigma} \dot{\pi} \delta\phi - \dot{\phi} \delta\pi,$$

where by H_N we denote the restriction of the functional \tilde{H} to the phase space \mathcal{P}_N of Cauchy data satisfying Neuman boundary condition. We conclude that the field dynamics within Σ and its Hamiltonian is *not* defined, unless we specify field boundary conditions in an appropriate way (see again [4] for a more detailed analysis of this phenomenon).

The two (infinite-dimensional) Hamiltonian systems, described by equations (13) or (17), are equally legal. They describe field evolution in two different physical arrangements, described mathematically by two different functional spaces. Controlling boundary conditions means controlling the way the field in the in-

terior of Σ interacts with the rest of the World. Of course, there are infinitely many Hamiltonian systems which one may assign to a given field theory: one could control various combinations of Dirichlet and Neuman data over different pieces of $\partial\Sigma$. The question arises: Is there any criterion to choose one of them as a “fundamental control mode”, to consider its Hamiltonian as a “true field energy” and to call the corresponding boundary conditions as “adiabatic insulation” of Σ ?

Before we propose an answer to this question, we would like to stress that the existence of the variational principle, based on the Lagrangian function

$$(18) \quad L = -\frac{1}{2} \{g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) + m^2\phi^2\},$$

which is often used to derive equation (9) and to define canonical momentum

$$(19) \quad p^\mu := \frac{\partial L}{\partial(\partial_\mu\phi)},$$

is irrelevant for this purpose. Yes, generating formula (12) may be directly obtained *via* a standard Legendre transformation from the integral $\delta \int_\Sigma L dV$. Yes, controlling ϕ on $\partial\Sigma$ seems to be natural, because variation of L is taken with ϕ fixed on the boundary. These arguments are not, however, conclusive since there are many other variational formulations of the Klein–Gordon theory. In particular, take

$$(20) \quad \tilde{L} = \frac{1}{2} \left\{ g_{\mu\nu} p^\mu p^\nu + \frac{1}{m^2} (\partial_\mu p^\mu)^2 \right\}.$$

Variation of this Lagrangian function with respect to p^μ , together with definition of the conjugate momenta:

$$(21) \quad \Phi^v_\mu := \frac{\partial \tilde{L}}{\partial(\partial_\nu p^\mu)},$$

reproduces *exactly* the same theory. Indeed, (20) together with (21) imply that the momentum Φ^v_μ is proportional to Kronecker’s delta δ^v_μ . Denoting the proportionality coefficient by “ $-\phi$ ” we see that the “new” field theory derived from the new Lagrangian (together with the symplectic form in the space of Cauchy data) coincides with the old theory. The only difference is that applying the Legendre transformation to $\delta \int_\Sigma \tilde{L} dV$ we obtain now the Neuman formula (15) instead of the Dirichlet formula (12).

Hence, what was “Neuman” for one variational principle, can become “Dirichlet” for another one. We conclude that variational principles do not help us to find the appropriate form of the field energy among all possible formal definitions, related to all possible Legendre transformations which we may perform at the boundary $\partial\Sigma$. In fact, any generating formula of the type (15) or (12) gives rise to a corresponding variational principle. To be able to interpret such a formula as a generator of a Hamiltonian system, the corresponding mixed “initial value + boundary value” problem must be well posed, but this is the case for *many* different control modes.

In particular, both the Dirichlet and the Neuman problems for the Klein–Gordon theory satisfy this condition.

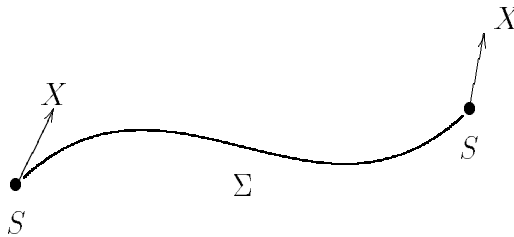
In our opinion, the only criterion which distinguishes H among all other candidates for the energy, is its positivity or, more precisely, the fact that it is bounded from below and convex. These are the fundamental physical properties which ensure stability of the physical system in question. Positivity is, therefore, the very reason to call H the field energy. Other functionals play role of a “free energy” and contain also a part of energy of the device used to control physically the boundary data (a “thermostat”).

3. Hamiltonian properties of Einstein equations

In this Section we present the so called “homogeneous generating formula” for Einstein equations, which may be used as a starting point for the construction of Canonical Relativity. We stress that the result presented here does not depend upon any “ideology”, which one might choose to formulate General Relativity theory or to derive its equations from any kind of a “least action principle”. The formula depends only upon intrinsic properties of Einstein equations.

Suppose that the following three objects have been chosen in a general relativistic spacetime M :

- 1) a two-dimensional, spacelike surface $S \subset M$,
- 2) a three-dimensional spacelike hypersurface $\Sigma \subset M$, such that $\partial\Sigma = S$,
- 3) a vector field X defined in a neighborhood of Σ :



Suppose, moreover, that a one-parameter family of solutions of Einstein equations $g_{\mu\nu} = g_{\mu\nu}(x; \sigma)$, defined in a neighborhood of Σ , has been chosen. Dragging these solutions along the vector field X we may construct a two-parameter family of solutions $g_{\mu\nu} = g_{\mu\nu}(x; \tau, \sigma)$, where τ is the parameter of the group of diffeomorphisms generated by X . Let $(P^{kl}(\tau, \sigma), g_{kl}(\tau, \sigma))$ denote the corresponding Cauchy data on Σ . Take the following two vectors in the space Cauchy data:

$$(22) \quad \mathcal{X} = \begin{cases} \dot{g} = \frac{\partial g}{\partial \tau} = \mathcal{L}_X g, \\ \dot{P} = \frac{\partial P}{\partial \tau} = \mathcal{L}_X P, \end{cases} \quad \mathcal{Y} = \begin{cases} \delta g = \frac{\partial g}{\partial \sigma}, \\ \delta P = \frac{\partial P}{\partial \sigma}, \end{cases}$$

and try to calculate their symplectic product

$$(23) \quad \omega_{\Sigma}(\mathcal{X}, \mathcal{Y}) = \frac{1}{16\pi} \int_{\Sigma} \dot{P}^{kl} \delta g_{kl} - \dot{g}_{kl} \delta P^{kl}.$$

Below, we give the result which is valid not only for pure gravity, but also for a broad class of matter fields ϕ interacting with gravity. We stress that, once we know the symplectic form ω_{Σ} , no variational principle is necessary to calculate (23), and only the field equations are needed. In fact, the proof given in [8, 9] used the “purely affine” variational principle whereas in [10], the same theorem was derived from the Hilbert “purely metric” Lagrangian. It is, however, an interesting and highly instructive exercise (which we leave to the reader) to derive the formula explicitly from Einstein equations and matter field equations.

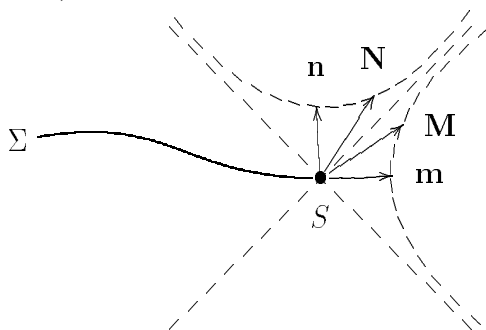
Theorem 1. *If $(g_{\mu\nu}(x; \tau, \sigma), \phi(x; \tau, \sigma))$ is a two-parameter family of solutions of the interacting system “Einstein equations + matter field equations”, if $X = \partial/\partial\tau$ (i.e., if $\mathcal{L}_{Xg} = \dot{g}$) and if $(P^{kl}(\tau, \sigma), g_{kl}(\tau, \sigma), \pi(\tau, \sigma), \phi(\tau, \sigma))$ are corresponding Cauchy data on Σ , then*

$$(24) \quad \begin{aligned} 0 = & \frac{1}{16\pi} \int_{\Sigma} \dot{P}^{kl} \delta g_{kl} - \dot{g}_{kl} \delta p^{kl} + \int_{\Sigma} \dot{\pi} \delta \phi - \dot{\phi} \delta \pi + \frac{1}{8\pi} \int_S \dot{\lambda} \delta \alpha - \dot{\alpha} \delta \lambda \\ & + \frac{1}{16\pi} \int_S \{2n\delta(\lambda k) + 2n_A \delta(\lambda \ell_A) + Q^{AB} \delta g_{AB}\} + \int_S p^{\perp} \delta \phi. \end{aligned}$$

In this formula the following notation has been used. If (x^A) , $A = 1, 2$; are coordinates on S , then

$$(25) \quad \lambda = \sqrt{\det g_{AB}} \, d^2x$$

is the volume form on S . The field X has been decomposed into the part tangent to S , which we denote by $X^{\parallel} = n^A \partial_A$, and the part X^{\perp} , orthogonal to S . We have, therefore, $X = X^{\perp} + X^{\parallel}$ and denote $n := \pm \sqrt{|(X^{\perp}|X^{\perp})|}$ where ‘+’ is taken if X is timelike and ‘-’ if X is spacelike. In the two-dimensional plane orthogonal to S (which may be identified with the two-dimensional Minkowski space) we use the following three normalized vectors: $\mathbf{N} := (1/n)X^{\perp}$, \mathbf{M} – orthogonal to \mathbf{N} and \mathbf{m} – tangent to Σ , directed outwards (we remind the reader that \mathbf{n} was the unit vector orthogonal to Σ).



By “ α ” we denote the “hyperbolic angle” between \mathbf{N} and \mathbf{n} , defined as follows:

$$(26) \quad \alpha = \begin{cases} \operatorname{arsinh}(\mathbf{N} | \mathbf{m}) & \text{for } X^\perp \text{ time-like,} \\ \operatorname{sgn}(\mathbf{N} | \mathbf{m}) \operatorname{arcosh}(\mathbf{N} | \mathbf{m}) & \text{for } X^\perp \text{ space-like.} \end{cases}$$

We also use the extrinsic geometry of S : the torsion covector

$$(27) \quad \ell_A := (\nabla_A \mathbf{N} | \mathbf{M}) = \frac{1}{n} (\nabla_A X^\perp | \mathbf{M}),$$

and the symmetric curvature tensor k in the direction of M

$$(28) \quad k_{AB} = k_{AB}(\mathbf{M}) := (\nabla_A \partial_B | \mathbf{M}).$$

This means, that for any pair (Y, Z) of vector fields tangent to S we have: $k(Y, Z) = (\nabla_Y Z | \mathbf{M})$. Finally, we consider also the “acceleration” scalar

$$(29) \quad s = (\nabla_X X | \mathbf{M}) = \mathcal{L}_X g(X, \mathbf{M}) - \frac{1}{2} \mathbf{M}(X | X).$$

Using the two-dimensional inverse $\tilde{\mathbf{g}}^{AB}$ to the metric g_{AB} on S we define the trace $k = \tilde{\mathbf{g}}^{AB} k_{AB}$ and the following tensor density:

$$(30) \quad Q^{AB} = \lambda \left\{ \left(\frac{s}{n} - 2n^c \ell_c - n^c n^d k_{cd} \right) \tilde{\mathbf{g}}^{AB} + n(k^{AB} - k \tilde{\mathbf{g}}^{AB}) \right\}.$$

This completes the list of geometric objects used in (24).

The “hamiltonian part” (first three terms) of (24) implies the following symplectic structure in the space of Cauchy data for the total “gravity + matter” system:

$$(31) \quad \omega = \frac{1}{16\pi} \int_\Sigma dP^{kl} \wedge dg_{kl} + \frac{1}{8\pi} \int_S d\lambda \wedge d\alpha + \int_\Sigma d\pi \wedge d\phi.$$

It contains not only the gravitational volume part (3) and the matter field part (11), but is supplemented by the gravitational surface part $\omega_{\partial\Sigma}$ (the second term on the right-hand side). This supplement is necessary for the gauge invariance of this symplectic structure.

Definition. Given S and X , by *gauge transformations* we mean those spacetime diffeomorphisms which do not move points of S and their trajectories under the group generated by X .

Theorem 2. *Symplectic structure (31) is invariant with respect to the above gauge transformations.*

The zero on the left-hand side of (24) *does not* mean that the Hamiltonian of the “gravity + matter” system vanishes. Indeed, this formula is analogous to the so called “homogeneous formulation” of mechanics of point particles. Consider spacetime coordinates (q^μ) , $\mu = 0, 1, 2, 3$; of a particle and the corresponding four momenta p_μ .

Hamiltonian mechanics (relativistic and also non-relativistic) may be formulated in terms of the following two equations:

$$(32) \quad 0 = \dot{p}_\mu \delta q^\mu - \dot{q}^\mu \delta p_\mu,$$

$$(33) \quad 0 = p_0 + H(p_k, q^k, q^0).$$

In this formulation, the parameter t along a trajectory is a pure gauge quantity and has no physical meaning. The theory is invariant with respect to reparameterizations of the trajectories. This is a consequence of the fact that the “control parameters” (p_μ, q^μ) in generating formula (32) are not free, but subject to constraint (33). To derive the standard (3+1)-formulation of mechanics we must fix a gauge putting, e.g.,

$$(34) \quad t \equiv q^0 \quad \implies \quad \dot{q}^0 = 1 \quad \text{and} \quad \delta q^0 = 0.$$

This implies:

$$(35) \quad 0 = \dot{p}_0 \delta q^0 - \dot{q}^0 \delta p_0 + \dot{p}_k \delta q^k - \dot{q}^k \delta p_k = \delta H + \dot{p}_k \delta q^k - \dot{q}^k \delta p_k,$$

or, simply

$$(36) \quad -\delta H(p_k, q^k, t) = \dot{p}_k \delta q^k - \dot{q}^k \delta p_k.$$

Similarly, boundary control parameters in equation (24) are subject to constraints, see [10]. As a consequence, these parameters do not imply the time lapse at the boundary. To derive a Hamiltonian dynamics from (24), we must choose boundary conditions in such a way, that they uniquely fix the time coordinate at the boundary. Here, different choices are possible. They must correspond to well posed boundary value problems for Einstein equations. We believe that the correct choice would be the one which leads to a positive – and, preferably, convex – energy functional. An analysis of the positivity theorem for the global gravitational energy given in [11, 7] is rather encouraging. At the moment, however, we do not know whether or not such a “good choice” is possible and we do not see any uniqueness in the choice of boundary conditions. We have, however, some conjectures which we present in the next Section (see also [10]).

Concluding this Section, we would like to stress that formula (24) is valid not only in the simplest case of a scalar field interacting with gravity, but also for a wide class of matter fields, including electromagnetism and gauge fields.

4. Examples of gravitational Hamiltonians

As an example of a possible choice of boundary conditions we may take the one obtained *via* the following Legendre transformation at the boundary:

$$(37) \quad \begin{aligned} n \delta(\lambda k) &= \delta(n \lambda k) - \lambda k \delta(n), \\ n^A \delta(\lambda \ell_A) &= \delta(\lambda n^A \ell_A) - \lambda \ell_A \delta(n^A). \end{aligned}$$

This enables us to rewrite (24) in the following way:

$$(38) \quad -\delta H = \frac{1}{16\pi} \int_{\Sigma} \dot{P}^{kl} \delta g_{kl} - \dot{g}_{kl} \delta P^{kl} + \int_{\Sigma} \dot{\pi} \delta \phi - \dot{\phi} \delta \pi + \frac{1}{8\pi} \int_S \dot{\lambda} \delta \alpha - \dot{\alpha} \delta \lambda \\ + \frac{1}{16\pi} \int_S -2\lambda k \delta(n) - 2\lambda \ell_A \delta n^A + Q^{AB} \delta g_{AB} + \int_S p^{\perp} \delta \phi.$$

with the Hamiltonian given by:

$$(39) \quad H = \frac{1}{16\pi} \int_S (2\lambda n^A \ell_A + 2\lambda nk) + E_0.$$

This choice consists, therefore, in controlling internal geometry of the three-dimensional world tube obtained from S by dragging it along X . In particular, putting $n = 1$ and $n^A = 0$ and controlling in formula (38) also the two-dimensional metric g_{AB} on the boundary (together with Dirichlet data $\phi|_S$ for the matter field), we may define the total energy of the “gravity + matter” system:

$$(40) \quad E = \frac{1}{8\pi} \int_S \lambda k + E_0,$$

where the additive constant E_0 (always free in the Hamiltonian formalism) may be fixed in such a way that $E = 0$ for the empty Minkowski spacetime. Although the above formula resembles the Brown–York proposal of defining the gravitational energy (see [2]–[3]), we stress that the latter contains the curvature κ of S , considered as a submanifold embedded in Σ (i.e., κ is taken with respect to the vector m and not M , as in (40) and, whence, is not gauge invariant). Putting $n = 0$ in (39), and taking vector n^A equal to the generator of a translations or rotations, we may define in a similar way the total momentum and the angular momentum of the system.

In our opinion, the above choice of boundary conditions is not the best one. Analyzing the linearization of the gravitational energy in an asymptotic region, i.e., when gravitational field on S is very weak, we came to conclusion that the correct energy control mode must be somehow related with the one obtained *via* the following Legendre transformation:

$$(41) \quad 2n\delta(\lambda k) = \delta(\lambda nk) - \lambda k^2 \delta\left(\frac{n}{k}\right) + nk\delta\lambda.$$

The last term vanishes when g_{AB} is controlled, since

$$(42) \quad \delta\lambda = \frac{1}{2} \lambda \tilde{g}^{AB} \delta g_{AB} = 0.$$

Hence, we may define a Hamiltonian related to the control the value of $b := k/n$. Fixing a “standard” value of b and keeping $n^A = 0$ we thus obtain for energy the following expression:

$$(43) \quad E = \frac{1}{16\pi} \int_S n\lambda k + E_0 = \frac{1}{16\pi} \int_S \lambda \frac{k^2}{b} + E_0.$$

As the “standard” b we take the extrinsic curvature k of the local, isometric embedding of S into the three-dimensional Euclidean space E^3 . If, e.g., g_{AB} is a sphere of radius r , then $b = -2/r$. The constant E_0 must be chosen in such a way that E vanishes for the Minkowski space. Hence, $E_0 = r/2$. This formula works especially well for the Schwarzschild solution, where it gives the correct value of mass on any Schwarzschild sphere (see [10] for a more detailed discussion).

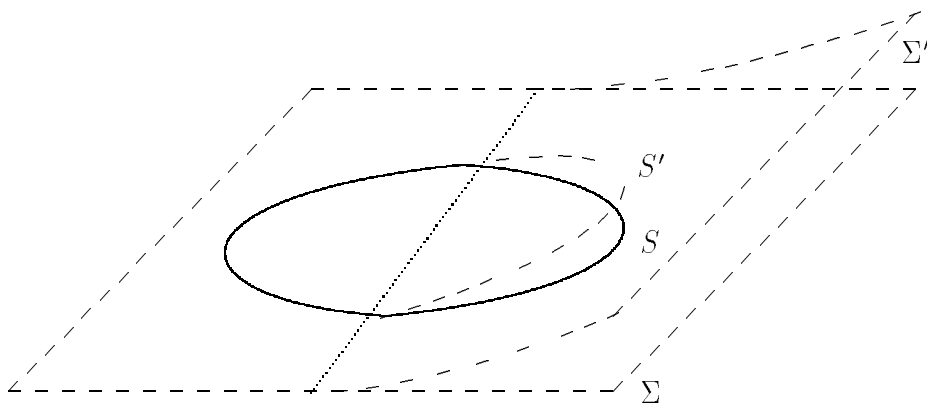
5. Rigid shells in General Relativity

In Special Relativity Theory, we do not expect any specific properties of a Hamiltonian assigned to a generic triplet (S, X, Σ) . Nice properties, which should be fulfilled by a “good energy functional”, are expected only when Σ is flat and X is orthogonal to Σ . In the general relativistic framework, the hypersurface Σ is no longer relevant, since it is a gauge quantity, but its flatness may be translated into the following property of S .

Definition. A two-dimensional, spacelike submanifold $S \subset M$, homeomorphic with the sphere S^2 , is called a *rigid shell* if there exist a non-vanishing vector field \mathbf{N} orthogonal to S such that the external curvature of S in direction of \mathbf{N} vanishes: $k_{AB}(\mathbf{N}) \equiv 0$. The manifold is called *weakly rigid* if there is a non-vanishing vector field \mathbf{N} orthogonal to S such that the traceless part of $k(\mathbf{N})$ vanishes:

$$k_{AB}(\mathbf{N}) - \frac{1}{2} g_{AB} \tilde{g}^{CD} k_{CD}(\mathbf{N}) \equiv 0.$$

In Minkowski space, every weakly rigid shell is also strongly rigid and must be embedded in a flat Euclidean hyperplane Σ . Below we illustrate the fact, that folding Σ in such a way that its internal geometry does not change, we may obtain a shell S' , whose internal geometry is isometric with internal geometry of a rigid shell S , but which is no longer rigid.



A rigid (or weakly rigid) shell defines automatically a reference frame: vector \mathbf{N} from the definition gives the time direction, whereas its orthonormal vector \mathbf{M} span

(together with vectors tangent to S) the local space directions. There is a conjecture that, in every asymptotically flat spacetime, there are “sufficiently many” weakly rigid shells, having a given internal geometry g_{AB} . If this conjecture is true, rigid spheres might be used to construct “good reference frames” in asymptotically flat regions of spacetime. Such frames would be unique up to an asymptotic Poincaré transformation. This way supertranslation ambiguities would be eliminated. Another conjecture says that a weakly rigid shell, which is a solution of the hamiltonian system (38) with control parameters constant in time and with vanishing matter field $\phi|_S$, must also be strongly rigid.

Analyzing properties of the quasilocal energy, defined in the previous Section, we came to yet another conjecture, that the positivity theorem might be satisfied for rigid shells, even if it is not universally valid. At the moment, we have no proof of this conjecture, but the work is already in progress.

References

- [1] A. Ashtekar, *Lectures on Non-Perturbative Canonical Gravity* (World Scientific, Singapore, 1991).
- [2] J.D. Brown, J.W. York, Quasilocal energy and conserved charges derived from the gravitational action, *Phys. Rev. D* 47 (1993) (4) 1407–1419.
- [3] J.D. Brown, S.R. Lau and J.W. York, Action and energy of the gravitational field, Preprint, ArXiv: gr-qc/0010024.
- [4] P. Chruściel, J. Jezierski and J. Kijowski, *Hamiltonian Field Theory in the Radiating Regime*, Lecture Notes in Physics 70 (Springer), to appear.
- [5] S.W. Hawking and G.T. Horowitz, The gravitational Hamiltonian, action, entropy and surface terms, *Class. Quantum Grav.* 13 (1996) 1487–1498.
- [6] S.W. Hawking and C.J. Hunter, The gravitational Hamiltonian in the presence of non-orthogonal boundaries, *Class. Quantum Grav.* 13 (1996) 2735–2752.
- [7] J. Jezierski and J. Kijowski, Positivity of total energy in general relativity, *Phys. Rev. D* 36 (1987) 1041–1044.
- [8] J. Kijowski, Asymptotic Degrees of Freedom and Gravitational Energy, in: *Proceedings of Journées Relativistes 1983, Torino*, ed. S. Benenti et al. (Pitagora Editrice Bologna, 1985) 205.
- [9] J. Kijowski, Unconstrained degrees of freedom of gravitational field and the positivity of gravitational energy, in: *Gravitation, Geometry and Relativistic Physics*, Lecture Notes in Physics 212 (Springer, Berlin, 1984) 40–50.
- [10] J. Kijowski, A simple derivation of canonical structure and quasi-local Hamiltonians in General Relativity, *Gen. Relativ. Gravit.* 29 (1997) 307–343.
- [11] J. Kijowski, On positivity of gravitational energy, in: *Proceedings of the IV Marcel Grossmann Meeting on General Relativity*, Rome 1985, ed. R. Ruffini (North-Holland, Amsterdam, 1986) 1681–1686.
- [12] J. Kijowski, On a new variational principle in general relativity and the energy of gravitational field, *GRG Journal* 9 (1978) 857–881.
- [13] J. Kijowski, W.M. Tulczyjew, *A Symplectic Framework for Field Theories*, Lecture Notes in Physics 107 (Springer, Berlin, 1979).

Katarzyna Grabowska
Institute for Mathematical Methods in Physics
Warsaw University
ul. Hoża 74
00-682 Warsaw
Poland
E-mail: konieczn@fuw.edu.pl

Jerzy Kijowski
Center for Theoretical Physics
Polish Academy of Sciences
al. Lotników 32/46
02-668 Warsaw
Poland
E-mail: kijowski@theta1.cft.edu.pl