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Generalising Raychaudhuri's equation[1]

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Abstract. Raychaudhuri's equation is fundamental for the analysis of behaviour of geodesic congruences. We describe two generalisations: the first to the case of the auto-parallel congruences of a linear connection with torsion, and the second to congruences of solutions of arbitrary second order ordinary differential equations on a manifold. These generalisations extend the efficacy of Raychaudhuri's equation to new and important domains.

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1. Introduction

Raychaudhuri's equation continues to play an important role in the singularity theory of geodesics and recently Jerie and Prince generalised the equation to the case of an arbitrary system of second order o.d.e.'s on a manifold. This generalisation opens the way to the study of, for example, the behaviour of congruences of charged particle orbits in various settings. In this paper we first of all show how to directly derive a Raychaudhuri's equation for the case of a manifold with a non-symmetric linear connection without recourse to the full generalisation. This equation shows the effect of both curvature and torsion on families of auto-parallel curves. Secondly we derive the generalisation of Jerie and Prince in a new and faster way and present the theorem which establishes the effectiveness of the generalisation in the analysis of congruences of solution curves of second order o.d.e.'s.

[1] This paper is in final form and no version of it will be submitted for publication elsewhere. M. Jerie wishes to acknowledge the support of an Australian Postgraduate Award.

2. The generalisation to the auto-parallel case

The setting is a manifold M of dimension n with a linear connection ∇ possessing torsion T. We are motivated by the case of a metric connection given in ([1]). Suppose that Z is the tangent field to a local congruence of curves on (M, ∇) with corresponding flow $\{\zeta_t\}$. If

$$\tau_t : T_x M \to T_{\zeta_t(x)} M$$

is the parallel transport map along $\{\zeta_t\}$ then the *shape map* $A_Z : T_x M \to T_x M$,

$$A_Z(\xi) := \frac{d}{dt}\bigg|_{t=0} (\tau_t^{-1} \circ \zeta_{t*})(\xi)$$

measures the deformation of tangent spaces by the flow. The following theorem shows that this map is precisely the difference $\nabla_Z - \mathcal{L}_Z$, cf. [6].

Theorem 2.1.

$$A_Z(\xi) = \nabla_\xi Z + T(Z_x, \xi),$$

i.e.,

$$A_Z = \nabla Z + T_Z \quad \text{or} \quad A_Z = \nabla_Z - \mathcal{L}_Z$$

($T_Z(X) := T(Z, X)$ and using $\nabla_Z X - \nabla_X Z - \mathcal{L}_Z X = T(Z, X)$).

Proof. Let X be the field obtained by Lie dragging ξ along the integral curve of Z through x. Then

$$A_Z(\xi) = \frac{d}{dt}\bigg|_{t=0} (\tau_t^{-1} X_{\zeta_t(x)}) = (\nabla_Z X)_x$$
$$= (\nabla_X Z)_x + T(Z, X)_x + (\mathcal{L}_Z X)_x = \nabla_\xi Z + T(Z_x, \xi)$$

where we have used $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ and $\mathcal{L}_Z X = 0$. □

The significance of trace(A_Z) will be demonstrated in a very general situation in the next section; to see the significance in the auto-parallel case suppose that Z is auto-parallel and let $\Omega^\perp \in \bigwedge^{n-1}(M)$ be a transverse volume form (that is, a characterising form for the distribution Sp$\{Z\}$) with $\nabla_Z \Omega^\perp = 0$ (it is not hard to establish the local existence of such a volume form). Then

Theorem 2.2.

$$\mathcal{L}_Z \Omega^\perp = \mathrm{tr}(A_Z)\Omega^\perp$$

Proof. A_Z induces a derivation on tensor fields which commutes with contractions, preserves type and vanishes on functions ([6]). We denote this action on $\Omega^\perp$ by $A_Z \Omega^\perp$ so that

$$A_Z \Omega^\perp = \nabla_Z \Omega^\perp - \mathcal{L}_Z \Omega^\perp = -\mathcal{L}_Z \Omega^\perp.$$

Now let $\{\theta^1, \ldots, \theta^{n-1}\}$ be a basis for the annihilating co–distribution of $\mathrm{Sp}\{Z\}$ at each point and let $\{X_1, \ldots, X_{n-1}\}$ be its dual. Then

$$\begin{aligned}
&(A_Z\Omega^{\perp})(X_1, \ldots, X_{n-1}) = \\
&\quad = -\Omega^{\perp}\big(A_Z(X_1), \ldots, X_{n-1}\big) - \cdots - \Omega^{\perp}\big(X_1, \ldots, A_Z(X_{n-1})\big) \\
&\quad = -\Big\{\theta^1\big(A_Z(X_1)\big) + \cdots + \theta^{n-1}\big(A_Z(X_{n-1})\big)\Big\}\Omega^{\perp}(X_1, \ldots, X_{n-1}) \\
&\quad = -\operatorname{tr}(A_Z)\Omega^{\perp}(X_1, \ldots, X_{n-1}).
\end{aligned}$$

This identification of tr (A_Z) follows since $A_Z(Z) = 0$. Since $Z \lrcorner\, \Omega^{\perp} = 0$ we have the result. $\square$

Clearly we want the evolution of $\operatorname{tr}(A_Z)$ along Z in a discussion of singularities (see Theorem 3.6 for details). This evolution equation is the trace of the corresponding equation for A_Z given in the next theorem.

Theorem 2.3. *Let Z be auto-parallel, then*

$$\mathcal{L}_Z A_Z = -A_Z^2 - R_Z + \mathcal{L}_Z T_Z + A_Z \circ T_Z$$

or

$$\mathcal{L}_Z(\nabla_Z) = -R_Z - A_Z \circ \nabla_Z,$$

where $R_Z(X) := R(X, Z)Z$.

Proof. It is a routine matter to show that $\mathcal{L}_Z A_Z = \nabla_Z A_Z$. Now $A_Z(X) = \nabla_X Z + T(Z, X)$ so that $\nabla_Z(A_Z(X)) = \nabla_Z\nabla_X Z + \nabla_Z(T(Z, X))$.

Hence

$$\begin{aligned}
(\nabla_Z A_Z)(X) &= -A_Z(\nabla_Z X) + \nabla_Z\nabla_X Z + \nabla_Z\big(T(Z, X)\big) \\
&= -A_Z\big(T(Z, X) + [Z, X] + \nabla_X Z\big) + \nabla_Z\nabla_X Z + \nabla_Z\big(T(Z, X)\big) \\
&= -A_Z\big(A_Z(X)\big) - A_Z\big([Z, X]\big) - (\nabla_X\nabla_Z Z - \nabla_Z\nabla_X Z - \nabla_{[X,Z]}Z) \\
&\quad - \nabla_{[X,Z]}Z + \nabla_Z\big(T(Z, X)\big) \\
&= -A_Z^2(X) - R(X, Z)Z + A_Z\big([X, Z]\big) - \nabla_{[X,Z]}Z + \nabla_Z\big(T(Z, X)\big) \\
&= -A_Z^2(X) - R(X, Z)Z + A_Z\big([X, Z]\big) - \nabla_{[X,Z]}Z + \nabla_Z T(Z, X) \\
&\quad + T(Z, \nabla_Z X) \\
&= -A_Z^2(X) - R(X, Z)Z + T\big(Z, \nabla_Z X - [Z, X]\big) + \nabla_Z T(Z, X) \\
&= -A_Z^2(X) - R(X, Z)Z + T\big(Z, A_Z(X)\big) + \nabla_Z T(Z, X).
\end{aligned}$$

The results that $(\nabla_Z T_Z)(X) = (\nabla_Z T)(Z, X)$ and $\mathcal{L}_Z T_Z(X) = \mathcal{L}_Z T(Z, X)$ are easy to establish and the fact that

$$\mathcal{L}_Z T_Z(X) = (\nabla_Z T_Z)(X) + T_Z\big(A_Z(X)\big) - A_Z\big(T_Z(X)\big)$$

then gives

$$\mathcal{L}_Z A_Z = \nabla_Z A_Z = -A_Z^2 - R_Z + \mathcal{L}_Z T_Z + A_Z \circ T_Z. \quad \square$$

When ∇ is metric (and Z is unit) taking the trace gives (see [1]) Raychaudhuri's equation ($\theta :=$ tr (A_Z))

$$Z(\theta) = -\mathrm{Ric}(Z, Z) - \mathrm{tr}(\omega^2) - \mathrm{tr}(\sigma^2) - \tfrac{1}{3}\theta^2.$$

θ, ω and σ are respectively the divergence, shear and vorticity of Z obtained through the decomposition of A_Z into the sum of a multiple of the identity θI, a trace free symmetric part ω and skew symmetric part σ. Symmetry and skew symmetry are defined with respect to the metric so that ω satisfies $g(\omega(X), Y) = g(\omega(Y), X)$ and σ satisfies the skew symmetric version. This equation has been used to great effect in general relativity (see, for example, [3, 7]).

In order to see how to generalise Raychaudhuri's equation to a situation where there is no linear connection or metric but simply a horizontal distribution, we present the following theorem which relates A_Z on (M, ∇) to the projection along the horizontal distribution onto the vertical. The connection produces the usual direct sum decomposition of $T_x M$ into vertical and horizontal subspaces with basis fields $V_a := \partial/\partial u^a$ and $H_a := \partial/\partial x^a - \{\Gamma^b_{ad} + T^b_{da}\} u^d\, \partial/\partial u^d$ respectively (TM has local adapted coordinates (x^a, u^a)). ξ^{h} and ξ^{v} are the vertical and horizontal lifts of $\xi \in T_x M$, P_V is the vertical projector along the horizontal and P_H the horizontal projector along the vertical.

Now assume that Z is auto-parallel with corresponding section $\sigma_Z : M \to TM$, then

$$\sigma_{Z*}(\xi) = \xi^{\mathsf{h}} + (\nabla_\xi Z + T(Z, \xi))^{\mathsf{v}}$$

and so

Theorem 2.4.

$$A_Z(\xi)^{\mathsf{v}} = P_V(\sigma_{Z*}\xi)$$

or

$$A_Z = \sigma_Z^* P_V$$

This result is the key to generalisation.

3. The generalisation to the SODE case

We present a review of the material in ([4, 5]). Our approach is to use the result of Theorem 2.4 as the definition of the map A_Z for congruences of solution curves of second order ordinary differential equations (SODE's) and to show that the trace of this map does indeed control congruence collapse. We emphasise again that in this setting there is no à priori metric or linear connection.

Crampin, Prince and Thompson ([2]) develop the basic differential geometry of SODE's and we give only a summary to establish the notation. In what follows we will be analysing a system of second order differential equations

$$(3.1)\qquad \ddot{x}^a = f^a(t, x, \dot{x})$$

on a manifold (the *configuration space*) M with local coordinates (x^a) and with associated bundles $\pi : \mathbb{R} \times M \to M$, $t : \mathbb{R} \times M \to \mathbb{R}$ and $\pi_1^0 : \mathbb{R} \times TM \to \mathbb{R} \times M$. ($\mathbb{R} \times M$ is the *extended configuration space* and the *evolution space* is $E := \mathbb{R} \times TM$.)

From (3.1) we construct on E (with local, adapted coordinates (t, x^a, u^a)) a second-order differential equation field (or SODE) :

$$(3.2)\qquad \Gamma = \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} + f^a \frac{\partial}{\partial u^a}$$

whose integral curves are the 1-jets of the solution curves of the given equations.

The vertical and contact structures of the bundle $\pi_1^0 : E \to \mathbb{R} \times M$ are combined in S, an intrinsic (1,1)-tensor field on E and known as the *vertical endomorphism*. In coordinates:

$$S = V_a \otimes \theta^a.$$

where $V_a := \partial/\partial u^a$ are the vertical basis fields and $\theta^a := \mathrm{d}x^a - u^a\,\mathrm{d}t$ are the local contact forms. From the first order deformation $\mathcal{L}_\Gamma S$ a *nonlinear connection* is constructed as follows: $\mathcal{L}_\Gamma S$ has eigenvalues 0, 1, -1 with corresponding eigenspaces spanned locally by Γ, the n *vertical* fields V_a and n *horizontal* fields

$$(3.3)\qquad H_a := \frac{\partial}{\partial x^a} - \Gamma_a^b \frac{\partial}{\partial u^a},$$

where $\Gamma_a^b := -\frac{1}{2}\partial f^a/\partial u^b$, respectively.

The vector fields $\{\Gamma, H_a, V_a\}$ form a local basis on E, with dual basis $\{\mathrm{d}t, \theta^a, \psi^a\}$ where

$$\psi^a := \mathrm{d}u^a - f^a\,\mathrm{d}t + \Gamma_b^a \theta^b.$$

The Γ_a^b form the components of the nonlinear connection thus induced by Γ. The resulting direct sum decomposition of $T(E)$ is $I = P_\Gamma + P_H + P_V$ where I is the identity type (1,1)-tensor field and P_Γ, P_H and P_V are the three projection operators given in coordinates by

$$P_\Gamma = \Gamma \otimes \mathrm{d}t\ , \quad P_H = H_a \otimes \theta^a, \quad P_V = V_a \otimes \psi^a.$$

The components of the Jacobi endomorphism, $\Phi := P_V \circ \mathcal{L}_\Gamma P_H$, a type (1,1)-tensor field on E can be calculated from

$$[\Gamma, H_a] = \Gamma_a^b H_b + \Phi_a^b V_b,$$

giving

$$\Phi = \Phi_a^b V_b \otimes \theta^a = \big(B_a^b - \Gamma_c^b \Gamma_a^c - \Gamma(\Gamma_a^b)\big) V_b \otimes \theta^a,$$

where $B^b_a := -(\partial f^b/\partial x^a)$. Other useful results:

$$[\Gamma, V_a] = -H_a + \Gamma^b_a V_b, \quad [H_a, H_b] = R^d_{ab} V_d, \tag{3.4}$$

this second fact is effectively the definition of the curvature, R, of the nonlinear connection Γ^b_a.

In ([2]) vertical and horizontal lifts to E of vector fields on $\mathbb{R}\times M$ are intrinsically defined; here it suffices to give their coordinate descriptions. If $X \in \mathfrak{X}(\mathbb{R} \times M)$ has representation $X = X^0\, \partial/\partial t + X^a\, \partial/\partial x^a$ then

$$X^{\mathsf{v}} = (X^a - u^a X^0)V_a$$

$$X^{\mathsf{h}} = (X^a - u^a X^0)H_a.$$

This means, for example, that for any vertical vector $\mu \in T_q(E)$ there exists a unique vector $\eta \in T_{\pi^0_1(q)}(\mathbb{R} \times M)$ with $\mathrm{d}t(\eta) = 0$ such that $\eta^{\mathsf{v}} = \mu$.

In order to arrive at a generalised Raychaudhuri equation for SODE's we need to introduce an arbitrary congruence of (graphs) of solution curves of (3.1). We follow ([4]): assume the existence of such a congruence with corresponding local tangent field $Z \in \mathfrak{X}(\mathbb{R} \times M)$. Then, for local functions Z^a on $\mathbb{R} \times M$, we can write

$$Z = \frac{\partial}{\partial t} + Z^a \frac{\partial}{\partial x^a}.$$

The relation between Z and (3.1) is given by

$$Z(Z^a) = f^a(t, x^b, Z^b).$$

Z defines a local section, σ_Z, of $\pi^0_1 : E \to \mathbb{R} \times M$ by

$$\sigma_Z(p) := (p, \pi_* Z_p).$$

We will use an overline to indicate the restriction in E to the image of the section. At the risk of a mild ambiguity we will also use an overline to denote the pull-back by the section, so that, for example, $\bar{\theta}^a := \mathrm{d}x^a - Z^a\, \mathrm{d}t$ denotes both the restriction and the pull-back of the contact forms. We will also use the symbol $\stackrel{*}{=}$ for section equality. Then the fact that Z is tangent to graphs of solution curves of (3.1) is expressed as $\overline{f}^a = Z(Z^a)$ (as already noted) and to $\Gamma \stackrel{*}{=} \sigma_{Z*}(Z)$.

We can still define a map A_Z, even though we don't have a linear connection on M, by using the result that $A_Z = \sigma^*_Z P_V$ in the geodesic and auto-parallel cases. We give a brief summary. Pull P_V back from E to $\mathbb{R} \times M$ using the section: let $\xi \in T_p(\mathbb{R} \times M)$, then $P_V(\sigma_{Z*}\xi)$ is vertical, and hence there is a unique vector $\eta \in T_p(\mathbb{R} \times M)$ such that $dt(\eta) = 0$ and $\eta^{\mathsf{v}} = P_V(\sigma_{Z*}\xi)$. We denote the linear map $\xi \mapsto \eta$ by $\sigma^*_Z P_V$, hence

$$\big(\sigma^*_Z P_V(\xi)\big)^{\mathsf{v}} = P_V(\sigma_{Z*}\xi) \quad \text{and} \quad dt(\sigma^*_Z P_V) = 0. \tag{3.5}$$

This can be done for any vertical (1,1)-tensor field B on E (vertical means that $P_V \circ B = B$) to give $\sigma^*_Z B$, see [4]. In particular, $\Phi := P_V \circ \mathcal{L}_\Gamma P_H$ is vertical and we will denote $\sigma^*_Z \Phi$ by $\overline{\Phi}$. In effect, $\overline{\Phi}(X)$ replaces $R_Z(X) := R(X, Z)Z$.

Definition 3.1. We define the type (1,1)-tensor field A_Z on $\mathbb{R} \times M$ associated with Z by

$$A_Z := \sigma_Z^* P_V. \tag{3.6}$$

In coordinates

$$A_Z = \left(\frac{\partial Z^a}{\partial x^b} + \bar{\Gamma}^a_b \right) \frac{\partial}{\partial x^a} \otimes \bar{\theta}^b. \tag{3.7}$$

In order to generalise Raychaudhuri's equation we need an analogue of ∇_Z. To obtain this we turn Theorem 2.1 on its head:

Definition 3.2. $\bar{\nabla}$ on $\mathbb{R} \times M$ is defined as follows:
(i) $\bar{\nabla}(f) := Z(f), \quad f \in C^\infty(\mathbb{R} \times M)$
(ii) $\bar{\nabla}(X) := [Z, X] + A_Z(X), \quad X \in \mathfrak{X}(\mathbb{R} \times M)$
(iii) $(\bar{\nabla}\omega)(X) := \bar{\nabla}(\omega(X)) - \omega(\bar{\nabla} X), \quad \omega \in \bigwedge(\mathbb{R} \times M)$
(iv) $\bar{\nabla}$ acts by Leibnitz rule on $\otimes$ and $\wedge$ and commutes with tensor contraction.

Lemma 3.3. $\bar{\nabla} A_Z = \mathcal{L}_Z A_Z$.

Proof. Use (ii) of definition 3.2 and the Leibnitz rule. □

Lemma 3.4.
(i) $\Phi = -P_V \circ \mathcal{L}_\Gamma P_V$
(ii) $\bar{\Phi} = -\bar{\nabla} \circ A_Z + A_Z \circ \mathcal{L}_Z$.

Proof. We prove (i) only:

$$\begin{aligned} P_V \circ \mathcal{L}_\Gamma P_V &= P_V \circ \mathcal{L}_\Gamma (I - P_\Gamma - P_H) \\ &= -P_V \circ \mathcal{L}_\Gamma P_H = -\Phi. \qquad \square \end{aligned}$$

Theorem 3.5.

$$\mathcal{L}_Z A_Z = A_Z^2 - \bar{\Phi} \tag{3.8}$$

Proof. Lemma 3.4 gives:

$$\begin{aligned} \bar{\Phi}(X) &= -\bar{\nabla}(A_Z(X)) + A_Z([Z, X]) \\ &= -\bar{\nabla} A_Z(X) - A_Z(\bar{\nabla} X) + A_Z([Z, X]) \\ &= -\mathcal{L}_Z A_Z(X) - A_Z(\bar{\nabla} X - [Z, X]) \quad \text{(Lemma 3.3)} \\ &= -\mathcal{L}_Z A_Z(X) - A_Z^2(X). \quad \square \end{aligned}$$

The generalised Raychaudhuri equation is just the trace of equation (3.8): if $\theta := \text{trace}(A_Z) = \partial Z^a / \partial x^a + \bar{\Gamma}^a_a$ then

$$Z(\theta) = -\text{trace}(A_Z^2 + \bar{\Phi}).$$

In order to establish that θ really is important in analysing congruence convergence and divergence consider an invariant Z-transverse volume form on $\mathbb{R} \times M$:

$$\Omega := \mu\bar{\theta}^1 \wedge \ldots \wedge \bar{\theta}^n, \ \bar{\nabla}\Omega = 0.$$

A simple calculation analogous to that in Theorem 2.2 establishes the following theorem proved in ([4]):

Theorem 3.6.

$$\mathcal{L}_Z\Omega = \theta\,\Omega$$

so that, in an obvious notation,

$$\Omega(s) = \Omega(0)\exp\left(\int_0^s \theta(s)\,ds\right).$$

The congruence collapses occurs when $\int_0^s \theta(s)\,ds$ *diverges. If* $I = [0, a)$ *and* $1/\theta$ *has no zero on* I *but a zero at* a*, then a sufficient condition for divergence is* $\theta < 0$, $(1/\theta)' > 0$ *on* I *and* $(1/\theta)'$ *exists at* a.

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