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Browder operations and heat kernel homology

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Abstract. We define a stochastic homology theory for the n-fold based loop space of a compact manifold related to the heat kernel diffeology and we define over it Browder operations.

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1. Introduction

Let us recall what is the n-fold loop space of a based manifold $(*, M)$. It is the space of maps from $[-1, 1]^n = I^n$ into M, which are continuous (or smooth), and which are equal to $*$ on the boundary of I^n. It carries an action of the little cube operad, as it is classically known in algebraic topology (See [12, 34, 35]). Its homology groups carry some operations, called Browder operations ([3, 7, 11, 12]), which can be presented following the operad formalism.

We are motivated in this communication by constructing Browder operations for stochastic homology groups for the stochastic based n-fold loop space. Operads are related to stochastic homology for the diffeology over the n-fold loop space given by all random fields ([29]). That is, we consider in [29] the stochastic homology groups associated to all Hoelder random n-fold based loops. It is a very big diffeology, and the stochastic cycles are homologous to random cycles, in the set of smooth n-fold loop space.

The only known examples for the moment of non-linear random fields are related to the theory of infinite-dimensional processes over infinite-dimensional manifolds. There are several approaches to these infinite-dimensional diffusion processes:

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– The construction of processes by using the theory of Dirichlet forms. The diffusion processes are constructed quasi-surely over a state-space having a measure. This produce the examples of diffusion processes on the loop space, defined quasi-surely ([2, 13, 23, 32]).

– An abstract theory of Brownian motion (or diffusion processes) over infinite-dimensional manifolds. We refer to [4, 5, 17].

– Our study is related to the procedure of Airault–Malliavin ([1]), who have constructed the Brownian motion over a loop group. The reader can see the work of Brzezniak–Elworthy [8], where a general presentation of diffusion processes over infinite-dimensional Banach manifold, which contains Airault–Malliavin equation, has been given. Inside this theory, Brzezniak–Léandre ([9, 10]) and Léandre ([28]) have constructed some random surfaces. It is a $(1+1)$-dimensional theory, because the time interval of the state space is one-dimensional.

Our motivation is related to a $(1 + n)$-dimensional theory: that is we consider some diffusion processes over the n-fold loop space, as it was done by Léandre in others geometrical contexts in [26, 27, 29, 31].

Diffeological Calculus was performed by Chen and Souriau ([14, 38]). A stochastic analogous was performed in one dimension for random loops in [18, 19, 20, 21, 22, 30, 33]. There is a big variety of stochastic diffeology for the loop space, because there is a suitable martingale theory in one dimension. For random tori, there is a poor stochastic diffeology studied in [26]. For the n-fold based loop space, there is the very rich diffeology of all random fields studied in [29].

We are motivated in this work by the heat kernel diffeology. We can define in the lines of [19] and [29] a stochastic homology theory related to it, as well as an action of the little cube operads over the stochastic homology groups. By using a stochastic analogous of the method of Cohen ([12]), we can define some Browder operations over the stochastic homology groups.

The reader interested by the relation between infinite-dimensional analysis and topology can see the two surveys of Léandre [23] and [25].

2. Heat kernel homology

Let (Ω, F_s, dP) be a filtered probability space, where the filtration F_s checks the habitual conditions. This probability space is fixed in the whole paper. Let $U \subseteq \mathbb{R}^m$ and $S \in I^n$.

Associated to $U \times I^n$, we consider an Hilbert space H from function from $U \times I^n$ into $\mathbb{R}$ satisfying the two following conditions:

(*) there exist a function $e_{u,S}(u', S')$ uniformly Hoelder such that

$$h(u, S) = \langle h, e_{u,S} \rangle \tag{2.1}$$

(**) $h(u, S) = 0$ on a neighborhood of the boundary of I^n, if h belongs to H.

It result from (*) that if $h \in H$, h is Hoelder. We consider a Brownian motion with values in H (if there exists one) $B_t(\cdot, \cdot)$. Its law is characterized by the correlation function, since it is a Gaussian process:

$$E\big[B_t(u,S)B_{t'}(u',S')\big] = t \wedge t' e_{u,S}(u',S') \tag{2.2}$$

(See [31] for the introduction of a similar object). $(t,u,S) \to B_t(u,S)$ is almost surely Hoelder, because $e_{\cdot,\cdot}(\cdot,\cdot)$ is Hoelder.

Let us consider $\widetilde{B}_t(\cdot,\cdot)$ the product of d independent copies of $B_t(\cdot,\cdot)$. We consider a finite sum $\widetilde{M}_t(\cdot,\cdot)$ of $\widetilde{B}^i_t(\cdot,\cdot)$ with different Hilbert spaces H^i. By the Kunita–Watanabe inequality , the right bracket of $\widetilde{M}_{\cdot}(u',S') - \widetilde{M}_{\cdot}(u,S)$ is smaller than a deterministic constant by $|u-u'|^\alpha + |S-S'|^\alpha$ for some convenient α. Let Δ^m be the canonical simplex of $\mathbb{R}^m$. Let U be a neighborhood of it. We imbedd the compact Riemannian manifold M into $\mathbb{R}^d$. Let Π be the orthogonal projection from $\mathbb{R}^d$ into $T_x(M)$. We consider the following stochastic differential equation (of Airault–Malliavin type, but depending on the parameter u):

$$dx_t(u,S) = \Pi(x_t(u,S))d_t\widetilde{M}_t(u,S) \tag{2.3}$$

where $x_0(u,S)$ is smooth in u, S and equal to $*$ in a neighborhood of the boundary of I^n. It is a family of Stratonovitch equations.

By an adaptation of [31, equation (2.8)] and by using the Burkholder–Davies–Gundy inequality, we deduce that:

$$\begin{aligned} E\Big[\big|x_t(u,S) - x_t(u',S')\big|^p\Big] &\le C\big(|u-u'|^{\alpha p/2} + |S-S'|^{\alpha p/2}\big) \\ &\quad + C\int_0^t E\Big[\big|x_s(u,S) - x_s(u',S')\big|^p\Big]ds \end{aligned} \tag{2.4}$$

where α is the Hoelder exponent of $e_{\cdot,\cdot}(\cdot,\cdot)$. We use Gronwall lemma and Kolmogorov lemma (See [36]) in order to deduce that the random field $(u,S) \to x_1(u,S)$ is almost surely $\alpha/2 - \epsilon$ Hoelder. Moreover, from condition (**), we deduce that $t \to B^i_t(u,S)$ is equal to 0 on a neighborhood of the boundary of I^n. Therefore, $x_1(u,S) = *$ on a neighborhood of the boundary of I^n.

Definition 2.1. An m-dimensional stochastic simplex associated to Δ^m is given by the following data:

(i) A finite family of Hilbert space H^i satisfying (*) and (**).

(ii) A finite family of Brownian motions $\widetilde{B}^i_t(\cdot,\cdot)$ associated to it as before, if it is possible to construct it from the probabilistic data.

(iii) A deterministic simplex in the n-fold based loop space $u \to \{S \to x_0(u,S)\}$ where x_0 is equal to $*$ on a neighborhood of the boundary of I^n.

(iv) The random simplex $u \to \{S \to x_1(u,S)\}$.

Briefly, we have two choices: the Brownian motions $\widetilde{B}^i_t(u,S)$ which allows to deform the deterministic simplex in the n-fold based loop space and the starting deterministic simplex in (2.3). These two choices depend on the considered stochastic simplex, and are not fixed in our study of heat kernel homology.

We can add and substract (oriented) stochastic simplices (not obligatory with the same perturbing family of Brownian motions $\widetilde{B}^i_t(u,S)$). Therefore we can define the random boundary of a stochastic simplex, by restricting the Brownian motion $\widetilde{B}^i_t(u, S)$ to the boundary of the simplex Δ^m. We get the notion of stochastic boundary ∂_{st} of an union of (oriented) stochastic simplices. As traditional,

$$\partial^2_{st} = 0.$$

By repeating the consideration of [18] and [29] (but here, without localization), we can define the stochastic homology groups:

Definition 2.2. $H_{*,st} = \mathrm{Ker}\, \partial_{st} / \mathrm{Im}\, \partial_{st}$.

Remark. In [18], we have computed explicitly the singular stochastic homology groups of the loop space related to the semi-martingale diffeology. With the present diffeology, it does not seem so easy to compute the stochastic homology groups. In particular, if $S \to x_1(\cdot, S)$ defines a random cycle, it is not a priori clear that $S \to x_0(\cdot, S)$ defines a (deterministic!) cycle in the n-fold based loop space.

3. Browder operations

Let us recall what is a topological operad: it is a family of topological spaces $C(k)$ endowed with an operation

$$C(k) \times C(j_1) \times \cdots \times C(j_k) \longrightarrow C(j_1 + \cdots + j_k),$$

which is compatible with the action of the symmetric group over $C(j_1) \times \cdots \times C(j_k)$. A classical example (see [12, 35]) is the little cube operad

$$C(k) : \amalg I^n \to I^n,$$

where we consider the disjoint union of k I^n and k maps $s_j(S) = a_j S + b_j$ such that $s_j(I^n)$ are disjoints. The operad operation is clear: we consider in the l box of $C(k)$, the j_l boxes which are deduced by translation and homotopy from the j_l boxes in I^n defined by $C(j_l)$. Let $s_. \in C(k)$ and $x_1(u_1, S), \ldots, x_1(u_k, S)$ $u_j \in \Delta^{m_j}$ be stochastic simplices. Since a product of simplices is an union of simplices, we deduce an union of stochastic simplices $(x_.(u_i, \widetilde{S}_i)$ where $\widetilde{S}_i$ belong to the i-th box defined by $C(k)$, after translating and rescaling the internal parameter time S. Since the boxes are disjoint, we get a union of stochastic simplices. If we tensorized $H_{*,st}$ by $\mathbb{R}$, this say that the stochastic homology groups are a n-algebra.

Let us consider $C(2)$. It is classically homotopic to the configuration space of I^n $I^n(2)$, see [12].

We deduce a map ψ from $H_*(I^n(2)) \times H_{*,st} \times H_{*,st}$ into $H_{*,st}$.

But since we consider fields which are equal to $*$ in a neighborhood of the boundary of I^n, we can consider $]-1, 1[^n$ such that the configuration space is homotopic to S^n, see [12, 15], such that we can replace $H_*(I^n(2))$ by $H_*(S^n)$.

Definition 3.1. Let σ_n be the generator of $H_n(S^n, Z) = Z$. Then $\psi(\sigma_n, \cdot, \cdot)$ is called the stochastic Browder operations associated to the n-fold loop space and applies $H_{*,st} \times H_{*,st}$ into $H_{*,st}$.

We refer to the work of Cohen ([12]) for the algebraic property of it.

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