Differential Geometry and Its Applications
Proc. Conf., Opava (Czech Republic), August 27–31, 2001
Silesian University, Opava, 2001, 197–201

Functional equations for metric geodesic arcs

Lubomír Klapka

Abstract. The functional equations which make possible to generalize the notion of geodesics from smooth Riemannian manifolds to arbitrary metric spaces are presented. Necessary and sufficient conditions for the set of all geodesics generated by some metric to be a set of generalized geodesics in the topological sense are derived. An example and a counterexample to such a situation are introduced.

Keywords. Functional definition of geodesics, metric geodesic arcs, topological geodesic arcs, Riemannian geometry, convex regions.

MS classification. 39B52, 39B70, 50C25, 51H20, 52A30, 53B20, 53B21, 53B40, 57M50, 57N16.

Let $\mathbb{R}$ be the set of real numbers, $[0,1] \subset \mathbb{R}$ be the closed interval, and M be a topological space. Any mapping $[0,1] \to M$ will be called an *arc* on M. For simplicity, in this paper, we deal only with geodesics which are arcs. Note that the other geodesics can be constructed from these ones (see, e.g., [1]).

Let M be a metric space and Δ be its metric. We shall have to do with arcs $x : [0,1] \to M$ satisfying

$$\Delta\big(x(\alpha), x(\beta)\big) = |\alpha - \beta| \Delta\big(x(0), x(1)\big). \tag{1}$$

for all $\alpha, \beta \in [0,1]$. Their meaning in Riemannian geometry is characterized by the following theorem.

Theorem 1. *Let $\Delta : M \times M \to \mathbb{R}$ be a Riemannian metric such that the corresponding metric tensor field is smooth, $U \subset M$ be a convex region. Then*

a) *every geodesic $x : [0,1] \to U$ of the Levi-Civita connection satisfies* (1),

b) *every $x : [0,1] \to M$ satisfying* (1) *is a geodesic of the Levi-Civita connection.*

This paper is in final form and no version of it will be submitted for publication elsewhere.

Proof. J.L. Synge, [2], showed that the smoothness of the metric tensor field implies the smoothness of the second power Δ^2 of the metric Δ and proved the local expressions

$$\text{(2)}\quad \begin{aligned} &\left.\frac{\partial^2\Delta^2(a,b)}{\partial b^i\partial b^j}\right|_{b=a} = 2\,g_{ij}(a), \quad \left.\frac{\partial^3\Delta^2(a,b)}{\partial a^i\partial b^j\partial b^k}\right|_{b=a} = -2\,g_{im}(a)\Gamma^m_{jk}(a),\\ &\left.\frac{\partial^3\Delta^2(a,b)}{\partial b^i\,\partial b^j\,\partial b^k}\right|_{b=a} = 2\left(g_{im}(a)\Gamma^m_{jk}(a)+g_{jm}(a)\Gamma^m_{ki}(a)+g_{km}(a)\Gamma^m_{ij}(a)\right),\end{aligned}$$

where g_{ij} and Γ^i_{jk} are components of metric tensor and Levi-Civita connection fields.

a) Since

$$\sqrt{g_{ij}\big(x(\tau)\big)\frac{dx^i(\tau)}{d\tau}\frac{dx^i(\tau)}{d\tau}} = \text{const}$$

and the points $x(0)$, $x(1)$ are joined by a unique segment of a geodesic which does not leave the region U, integrating in τ over $[0,1]$ we get const $= \Delta(x(0), x(1))$, and integrating from $\tau = \alpha$ to $\tau = \beta$ we get (1).

b) Consider the identity

$$\begin{aligned}\frac{\Delta^2(a,b)}{\tau}+\frac{\Delta^2(b,c)}{1-\tau}-\Delta^2(a,c) &= 2\Delta(a,c)\big(\Delta(a,b)+\Delta(b,c)-\Delta(a,c)\big)\\ &+\tau\left(\Delta(a,c)-\frac{\Delta(a,b)}{\tau}\right)^2+(1-\tau)\left(\Delta(a,c)-\frac{\Delta(b,c)}{1-\tau}\right)^2,\end{aligned}$$

where $\tau \in (0,1) \subset \mathbb{R}$. According to (1) and the triangle inequality, $a = x(\alpha)$, $b = x((1-\tau)\alpha+\tau\gamma)$, $c = x(\gamma)$ minimize the left-hand side. From the smoothness of Δ^2

$$\text{(3)}\qquad \frac{1}{\tau}\frac{\partial\Delta^2(a,b)}{\partial b^i} + \frac{1}{1-\tau}\frac{\partial\Delta^2(b,c)}{\partial b^i} = 0$$

at this point.

Consider $\tau_0 \in (0,1) \subset \mathbb{R}$. From (2)

$$\text{(4)}\qquad \operatorname{Det}\left(\frac{1}{\tau}\frac{\partial^2\Delta^2(a,b)}{\partial b^i\,\partial b^j} + \frac{1}{1-\tau}\frac{\partial^2\Delta^2(b,c)}{\partial b^i\,\partial b^j}\right) \neq 0.$$

for $a = b = c = x(\tau_0)$, $\tau = \tau_0$. Since the left-hand side is continuous, there exists an open ball $B \subset U$ of centre $x(\tau_0)$ and a closed interval $I \subset (0,1)$ of centre τ_0 such that (4) holds for each $a, b, c \in B$, $\tau \in I$. From (1) we see that x is continuous as well. Thus there exists a closed interval $[\alpha,\beta] \subset I$ such that $\alpha < \beta$, $\tau_0 \in [\alpha,\beta]$, $x([\alpha,\beta]) \subset B$. From (4) and the implicit function theorem, the mapping $\tau \to x((1-\tau)\alpha + \tau\gamma)$ defined by the equation (3) is smooth on some neighbourhood of the point $(\tau_0 - \alpha)/(\gamma - \alpha)$. Hence, the mapping x is smooth on some neighbourhood of the point τ_0. Since $\tau_0 \in (0, 1)$ is arbitrary, the mapping x is smooth on the whole interval $(0, 1)$. Substituting $a = x(\alpha)$, $b = x((1-\tau)\alpha + \tau\gamma)$, $c = x(\gamma)$ in (3), differentiating with respect to α and γ, putting $\gamma = \alpha$, and using (2), we get

$$2\, g_{ij}\big(x(\alpha)\big)\left(\frac{d^2 x^j(\alpha)}{d\alpha^2} + \Gamma^j_{kl}\big(x(\alpha)\big)\frac{dx^k(\alpha)}{d\alpha}\frac{dx^l(\alpha)}{d\alpha}\right) = 0,$$

hence $x|_{(0,1)}$ is a geodesic. According to the continuity, x is a geodesic as well. This completes the proof.

The functional equation (1) allows to generalize the notion of a geodesic arc to arbitrary metric spaces. Define the *metric geodesic arc* as a solution $x : [0, 1] \to M$ of this equation.

Now, we recall the definition of geodesic arcs on a general topological space, [1]. Let M be a set. Denote by $\operatorname{Arc} M$ the set of arcs on M. We say that a subset $U \subset M$ is *convex* with respect to $\operatorname{Arc} M$ if and only if for each two points $a, b \in U$ there exists a unique arc $x_{ab} \in \operatorname{Arc} M$ such that

$$x_{ab}(0) = a, \quad x_{ab}(1) = b, \quad x_{ab}([0, 1]) \subset U.$$

Furthermore, let M be equipped with the structure of a topological space. We say that M is *locally convex* with respect to $\operatorname{Arc} M$ if and only if there exists an open covering $\{U_i\}$ of M such that every intersection $U_i \cap U_j$ is convex with respect to $\operatorname{Arc} M$. We say that $\operatorname{Arc} M$ is a *set of topological geodesic arcs* if and only if M is locally convex with respect to $\operatorname{Arc} M$ and the mapping

$$[0, 1] \ni \tau \to x\big((1-\tau)\alpha + \tau\beta\big) \in M$$

belongs to $\operatorname{Arc} M$ for each $x \in \operatorname{Arc} M$, $\alpha \in [0, 1]$, $\beta \in [0, 1]$. Every element of a set of topological geodesic arcs is said to be a *topological geodesic arc*.

Furthermore, let M be equipped with the structure of a smooth manifold and the mapping

(5) $$f : \bigcup_i U_i \times U_i \times [0, 1] \ni (a, b, \tau) \to x_{ab}(\tau) \in M$$

is smooth. In such a case the notion of topological geodesic arc coincides with the usual notion of geodesic arc generated by a smooth linear connection, [1].

Theorem 2. *The set* $\operatorname{Arc} M$ *of all metric geodesic arcs of a metric*

$$\Delta : M \times M \to \mathbb{R}$$

is a set of topological geodesic arcs on M *if and only if* M *is locally convex with respect to* $\operatorname{Arc} M$.

Proof. Suppose $\operatorname{Arc} M$ is a set of topological geodesic arcs on M. Then, by definition of this notion, M is locally convex with respect to $\operatorname{Arc} M$.

Suppose M is locally convex with respect to the set $\operatorname{Arc} M$ of all metric geodesic arcs of the metric $\Delta : M \times M \to \mathbb{R}$ and $x \in \operatorname{Arc} M$, $\alpha, \beta, \gamma, \delta \in [0, 1]$. Then, from (1),

$$\Delta\Big(x\big((1-\gamma)\alpha+\gamma\beta\big), x\big((1-\delta)\alpha+\delta\beta\big)\Big) = |\gamma-\delta|\Delta\big(x(\alpha), x(\beta)\big),$$

the mapping $\tau \to x((1-\tau)\alpha+\beta)$ belongs to $\operatorname{Arc} M$, and $\operatorname{Arc} M$ is a set of topological geodesic arcs. This completes the proof.

Example (Riemannian metrics). After J.H.C. Whitehead, [3], if $\Delta : M \times M \to \mathbb{R}$ is a Riemannian metric with a smooth metric tensor field, then M is locally convex with respect to the set of all arcs which are geodesics of the Levi-Civita connection. Following the Theorem 1, M is locally convex with respect to the set $\operatorname{Arc} M$ of all metric geodesic arcs. Following the Theorem 2, $\operatorname{Arc} M$ is a set of topological geodesic arcs.

Counterexample (The metric L^1). Consider the metric

$$\Delta : \mathbb{R}^2 \times \mathbb{R}^2 \ni \big((a, b), (c, d)\big) \to \big(|a-c| + |b-d|\big) \in \mathbb{R}$$

and the open ball B_ε of centre (0,0) and radius $\varepsilon > 0$. There are two different metric geodesic arcs on B_ε

$$[0, 1] \ni \tau \to \big(\varepsilon \ln(1+\tau), \varepsilon\tau - \varepsilon \ln(1+\tau)\big) \in \mathbb{R}^2,$$

$$[0, 1] \ni \tau \to \big(\varepsilon\tau \ln 2, \varepsilon\tau(1-\ln 2)\big) \in \mathbb{R}^2,$$

which end points are the same. Since $\varepsilon > 0$ is arbitrary, $\mathbb{R}^2$ is not locally convex with respect to the set $\operatorname{Arc} \mathbb{R}^2$ of all metric geodesic arcs. Then, according to the Theorem 2, $\operatorname{Arc} \mathbb{R}^2$ is not a set of topological geodesic arcs.

The mapping (5) satisfies equations

$$f(a, b, 0) = a, \qquad f(a, b, 1) = b, \tag{6}$$

$$f\big(a, b, (1-\tau)\alpha+\tau\beta\big) = f\big(f(a, b, \alpha), f(a, b, \beta), \tau\big), \tag{7}$$

$$\Delta\big(f(a, b, \alpha), f(a, b, \beta)\big) = |\alpha-\beta|\Delta(a, b). \tag{8}$$

Relations (6) and (7) are functional equations for topological geodesic arcs, see [1]. Relations (6) and (8) are functional equations for metric geodesic arcs. The Theorem 2 asserts that if Δ is a metric and there exists a unique solution f of equations (6), (8), then f satisfies (7).

Acknowledgements

The research was supported by the Mathematical Institute of the Silesian University in Opava and by Grants MSM:J10/98:192400002, GAČR 201/00/0724. The author is grateful for this support.

References

[1] L. Klapka, The functional definition of generalized geodesics, *Aequationes Math.* 59 (2000) 201–213. Electronic version at http://arXiv.org/abs/dg-ga/ 9712007.
[2] J.L. Synge, A characteristic function in Riemannian space and its application to the solution of geodesic triangles, *Proc. London Math. Soc.* 32 (1931) 241–258.
[3] J.H.C. Whitehead, Convex regions in the geometry of paths, *Quart. J. Math.* 3 (1932) 33–42.

Lubomír Klapka
Silesian University in Opava
Mathematical Institute
Bezručovo nám. 13
746 01 Opava 1
Czech Republic
E-mail: Lubomir.Klapka@math.slu.cz