Differential Geometry and Its Applications
Proc. Conf., Opava (Czech Republic), August 27–31, 2001
Silesian University, Opava, 2001, 187–195

On the reduction of some systems of partial differential equations to first order systems with only one unknown function[1]

S. Jiménez, J. Muñoz and J. Rodríguez

Abstract. Inspired in part by some ideas of Lie, we define canonical correspondences between jet spaces which, when specialized to a system of partial differential equations, under certain conditions allow one to transform a given system into a first order one with only one unknown function. To illustrate the general theory, we describe the results that can be obtained for a class of second order systems, thus clarifying and completing some partial results of Lie.

Keywords. Jet, contact element, Lie correspondence, system of partial differential equations, involutive system, intermediate integral.

MS classification. 58A20, 35A30.

Introduction

In a long paper published in 1895, [6], S. Lie attempted to *reduce, as far as possible, the general theory of partial differential equations of arbitrary order to that of first order ones, thereby making its treatment amenable from the theory of groups* (page 327). He devotes the second chapter to such a reduction, making a detailed study of the systems of two second order equations with two independent variables and only one unknown function. However, his most important achievement is the idea for the reduction, which one can guess under statements and proofs, wrapped into the unavoidable imprecision caused by the state of the subject at that time. This idea consists in using some natural correspondences between jet spaces which

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.
The research of the first and third named authors was partially founded by Junta de Castilla y León under the contract SA30/00B.

apply submanifolds of a space into submanifolds of another one, and therefore systems of partial differential equations of one kind into systems of another kind.

The above correspondences between jet spaces appear by themselves when jets are considered as ideals of the ring of smooth functions of the total space (the manifold with the dependent and independent variables), as a result of the simple relation of inclusion between ideals. This presentation of the theory of jets was made in the PhD-thesis of J. Rodríguez [13] and is the natural continuation of the ideas of A. Weil [14], which did not arouse the same interest as those of Ehresmann. Ehresmann's point of view is not appropriate for the problem to be dealt with here, because when the manifold is fibred over different bases Lie's correspondences do not come out in a natural way.

1. Notations and preliminaries

In this section we recall some notions and results concerning Weil bundles and jet spaces which we shall use later on. A detailed discussion of the theory of jets used here can be found in [9, 11]. Finally, we recall some properties about jet spaces $\mathcal{J}^1_{n-1}(\mathcal{V})$ and first order systems with only one unknown function that we shall address along the paper. For the details, see [8, 7].

Let $\mathbb{R}^\ell_m$ be the quotient algebra of the ring of formal power series with real coefficients in m variables modulo the $(m+1)$-th power of its maximal ideal. Let $\mathcal{V}$ be an n-dimensional smooth manifold.

Definition 1.1. *An $\mathbb{R}^\ell_m$-point* or *(m,ℓ)-velocity* of $\mathcal{V}$ is an homomorphism of algebras $p^\ell_m\colon C^\infty(\mathcal{V}) \longrightarrow \mathbb{R}^\ell_m$ (see [5, 9, 11, 14]). A point p^ℓ_m is said to be regular if it is surjective.

The set of $\mathbb{R}^\ell_m$-points of $\mathcal{V}$ is a manifold, which we denote by $\mathcal{V}^\ell_m$, and the set $\check{\mathcal{V}}^\ell_m$ of regular $\mathbb{R}^\ell_m$-points is a dense open subset of $\mathcal{V}^\ell_m$.

Examples. (1) For $\ell = 0$ we have $\check{\mathcal{V}}^0_m = \mathcal{V}$.

(2) $\mathcal{V}^1_1 = T\mathcal{V}$.

(3) When $\ell = 1$, each point $p^1_m \in \check{\mathcal{V}}^1_m$ can be identified with the set formed by a point $p = p^0_m \in \mathcal{V}$ and m linearly independent derivations $D_{1\,p}, \dots, D_{m\,p} \in T_p\mathcal{V}$.

(4) $\check{\mathcal{V}}^1_n$ is the usual frame bundle on $\mathcal{V}$.

The kernel of each regular point p^ℓ_m is an ideal $\mathfrak{p}^\ell_m$ of the ring $C^\infty(\mathcal{V})$ called the jet of p^ℓ_m. The set of all these jets is a manifold, which will be denoted by $\mathcal{J}^\ell_m(\mathcal{V})$. Indeed, the projection $\check{\mathcal{V}}^\ell_m \longrightarrow \mathcal{J}^\ell_m(\mathcal{V})$ is a principal fibre bundle whose structure group is $\mathrm{Aut}(\mathbb{R}^\ell_m)$, see [9]. This is still true for an arbitrary Weil algebra A, see [1].

In particular, each jet $\mathfrak{p}^1_m$ is the ideal of $C^\infty(\mathcal{V})$ consisting of those functions vanishing at a given point $p \in \mathcal{V}$ which are annihilated by m given tangent vectors $(D_1)_p, \dots, (D_m)_p \in T_p\mathcal{V}$.

Let $\mathcal{X}$ be a locally closed m-dimensional submanifold of $\mathcal{V}$. For each point $p \in \mathcal{X}$, the quotient ring $C^\infty(\mathcal{X})/\mathfrak{m}_p^{\ell+1}$ (where $\mathfrak{m}_p$ is the ideal of the functions belonging to $C^\infty(\mathcal{X})$ which vanish at p) is isomorphic to $\mathbb{R}^1_m$. The kernel of the composition of the restriction morphism $C^\infty(\mathcal{V}) \longrightarrow C^\infty(\mathcal{X})$ with the quotient modulo $\mathfrak{m}_p^{\ell+1}$ is a jet $\mathfrak{p}_m^\ell \in \mathcal{J}_m^\ell(\mathcal{V})$; the set of all those jets is the prolongation $\mathcal{J}_m^\ell(\mathcal{X})$ of $\mathcal{X}$ to $\mathcal{J}_m^\ell(\mathcal{V})$. It is clear that $\mathcal{X} \approx \mathcal{J}_m^\ell(\mathcal{X})$ as manifolds.

There is a Pfaff system on $\mathcal{J}_m^\ell(\mathcal{V})$, the contact system, canonically determined by the following condition: the 1-forms belonging to it (contact forms) agree exactly with those that vanish on the prolongation $\mathcal{J}_m^\ell(\mathcal{X})$ of every locally closed m-dimensional submanifold $\mathcal{X}$ of $\mathcal{V}$ (see [11]).

This contact system will be denoted by $\Omega_m^\ell(\mathcal{V})$. Each point of $\mathcal{J}_m^\ell(\mathcal{V})$ has a neighbourhood endowed with local coordinates

$$(1) \qquad x_1, \dots, x_m, y_1, \dots, y_{n-m}, \dots, Y_{j,\alpha}, \dots,$$

where $\{x_1, \dots, x_m, y_1, \dots, y_{n-m}\}$ are local coordinates in $\mathcal{V}$, j runs from 1 to $n-m$ and $\alpha = (\alpha_1, \dots, \alpha_m)$ runs through the set of m-indexes such that $|\alpha| \le \ell$. Each jet $\mathfrak{p}_m^\ell$ belonging to this neighbourhood agrees with the ideal which contains $\mathfrak{m}_p^{\ell+1}$ (where $\mathfrak{m}_p$ is the maximal ideal at p, the base of $\mathfrak{p}_m^\ell$ in $\mathcal{V}$) and the $n-m$ functions

$$y_j - \sum_{|\alpha|\le\ell} Y_{j,\alpha}(\mathfrak{p}_m^\ell)\big(x - x(p)\big)^\alpha.$$

In this coordinated open subset, the contact system is spanned by the forms

$$\omega_{j,\alpha} = dY_{j,\alpha} - \sum_{i=1}^m Y_{j,\alpha+\varepsilon_i} dx_i, \qquad j = 1, \dots, n-m; \quad |\alpha| \le \ell - 1,$$

where $\alpha + \varepsilon_i$ is the m-index α, but replacing α_i by $\alpha_i + 1$, and $Y_{j,0} = y_j$.

Remark 1.2. In the jet spaces $\mathcal{J}_m^1(\mathcal{V})$, the corresponding contact system $\Omega_m^1(\mathcal{V})$ is precisely the Pfaff system which attaches to each $\mathfrak{p}_m^1 \in \mathcal{J}_m^1(\mathcal{V})$ the set of the differentials at p $(= \mathfrak{p}_m^0)$ of all functions $f \in \mathfrak{p}_m^1$ (considered as an ideal of $C^\infty(\mathcal{V})$). This follows immediately from the above expressions in local coordinates. Thus,

$$\big(\Omega_m^1(\mathcal{V})\big)_{\mathfrak{p}_m^1} = \mathfrak{p}_m^1/\mathfrak{m}_p^2.$$

This construction can be generalized to higher order jets.

When $m = n-1$, $\ell = 1$, each jet $\mathfrak{p}_{n-1}^1$ is the ideal of the smooth functions $f \in C^\infty(\mathcal{V})$ vanishing at $p = \mathfrak{p}_{n-1}^0$ that are annihilated by $n-1$ linearly independent tangent vectors $D_{1p}, \dots, D_{n-1p} \in T_p\mathcal{V}$; hence $\mathfrak{p}_{n-1}^1/\mathfrak{m}_p^2$ is a line of $T_p^*\mathcal{V}$. Accordingly, $\mathcal{J}_{n-1}^1(\mathcal{V})$ is the projectivized manifold of $T^*\mathcal{V} - \{0\}$.

It is well known that the manifold $T^*\mathcal{V}$ is endowed with a canonical 1-form θ: for each $\alpha_p \in T_p^*\mathcal{V}$, the value of θ at α_p is the lift to $T^*\mathcal{V}$ of the α_p itself via the projection $T^*\mathcal{V} \longrightarrow \mathcal{V}$; if we choose local coordinates $x_1, \dots, x_n$ in $\mathcal{V}$ and complete them to a coordinate system for $T^*\mathcal{V}$ with the "conjugated" ones, $p_1, \dots, p_n$, then $\theta = \sum_{i=1}^n p_i dx_i$.

The 2-form $d\theta$ endows $T^*\mathcal{V}$ with a symplectic structure; the lagrangian submanifolds of $T^*\mathcal{V}$ are the locally exact 1-forms (sections of $T^*\mathcal{V} \longrightarrow \mathcal{V}$) and the ones which can be obtained locally from them through canonical transformations. There is an unique tangent vector field H on $T^*\mathcal{V}$ such that $i_H d\theta = \theta$; this vector field satisfies the conditions $i_H\theta = 0$, $H^L\theta = \theta$. Its expression in local coordinates is

$$H = \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i}.$$

The orbits of H are the points in the section 0 and the lines in $T^*\mathcal{V}$ which, when considered as points of its projectivized space, are the points of $\mathcal{J}^1_{n-1}(\mathcal{V})$. The conditions $i_H\theta = 0, H^L\theta = \theta$ show that H belongs to the characteristic system of the Pfaff system spanned by θ and that such a system is (locally) projectable to the manifold of first integrals of H in $T^*\mathcal{V} - \{0\}$. However, this manifold agrees with $\mathcal{J}^1_{n-1}(\mathcal{V})$. Let us take local coordinates $x_1, \ldots, x_{n-1}, y$ in $\mathcal{V}$, these and $p_1, \ldots, p_{n-1}, q$ in $T^*\mathcal{V}$, and $x_1, \ldots, x_{n-1}, y, Y_1, \ldots, Y_{n-1}$ in $\mathcal{J}^1_{n-1}(\mathcal{V})$; then

$$\theta = p_1\, dx_1 + \cdots + p_{n-1}\, dx_{n-1} + q\, dy$$

and the contact system $\Omega^1_{n-1}(\mathcal{V})$ in $\mathcal{J}^1_{n-1}(\mathcal{V})$ is spanned by

$$\omega = dy - Y_1\, dx_1 - \cdots - Y_{n-1}\, dx_{n-1}.$$

The local equations of the projection $\pi : T^*\mathcal{V} \longrightarrow \mathcal{J}^1_{n-1}(\mathcal{V})$ are

$$Y_1 = -\frac{p_1}{q}, \ldots, Y_{n-1} = -\frac{p_{n-1}}{q},$$

and hence $\pi^*(\omega) = (1/q)\theta$. Thus, the projection of the Pfaff system spanned by θ to $\mathcal{J}^1_{n-1}(\mathcal{V})$ agrees with the contact system in this manifold.

A system of partial differential equations of order ℓ with m independent variables and $n - m$ unknown functions in $\mathcal{V}$ is a locally closed submanifold $\mathcal{R}^\ell_m$ of $\mathcal{J}^\ell_m(\mathcal{V})$ ($n = \dim\mathcal{V}$). A solution of $\mathcal{R}^\ell_m$ is an m-dimensional submanifold $\mathcal{X}$ of $\mathcal{V}$ such that $\mathcal{J}^\ell_m(\mathcal{X}) \subseteq \mathcal{R}^\ell_m$. Since $\mathcal{J}^\ell_m(\mathcal{X})$ is always a solution of the contact system $\Omega^\ell_m(\mathcal{V})$, the notion of solution can be generalized, as was done by Lie: a solution of $\mathcal{R}^\ell_m$ in Lie's sense is every m-dimensional submanifold $\widetilde{\mathcal{X}} \subseteq \mathcal{R}^\ell_m$, which is a solution of the contact system $\Omega^\ell_m(\mathcal{V})$ (specialized to $\mathcal{R}^\ell_m$).

In particular, when $\ell = 1$, $m = n - 1$, each $\mathcal{R}^1_{n-1} \subseteq \mathcal{J}^1_{n-1}(\mathcal{V})$ is a system of partial differential equations of first order (with an unknown function). The solutions of $\mathcal{R}^1_{n-1}$ in the Lie sense are the Legendre submanifolds of $\mathcal{J}^1_{n-1}(\mathcal{V})$ contained in $\mathcal{R}^1_{n-1}$. Among them (always with dimension $n-1$), there are the classical solutions: the hypersurfaces $\mathcal{X}_{n-1}$ of $\mathcal{V}$ whose manifolds of contact elements $\mathcal{J}^1_{n-1}(\mathcal{X}_{n-1})$ are contained in $\mathcal{R}^1_{n-1}$.

Passing from $\mathcal{J}^1_{n-1}(\mathcal{V})$ to the vector fibre bundle $T^*(\mathcal{V})$, the submanifolds $\mathcal{F} \subseteq T^*(\mathcal{V})$ are the first order systems of partial differential equations "which do not contain explicitly the unknown function". A solution of $\mathcal{F}$ is a lagrangian submanifold $\mathcal{X}_n \subseteq \mathcal{F}$ (in particular, a classical solution is an exact form dV that, when considered as a section of $T^*\mathcal{V} \longrightarrow \mathcal{V}$, takes its values in $\mathcal{F}$).

The homogeneous solutions, i.e., tangent to H, are precisely the inverse images of the Legendre manifolds contained in $\mathcal{J}^1_{n-1}(\mathcal{V})$.

2. Differential correspondences between jet spaces

In this section, $\mathcal{V}$ will be a fixed n-dimensional manifold and all jet spaces are referred to it; hence we can simplify the notations by omitting $\mathcal{V}$. Thus, $\mathcal{J}_m^\ell$ will denote $\mathcal{J}_m^\ell(\mathcal{V})$, Ω_m^ℓ will denote $\Omega_m^\ell(\mathcal{V})$, etc., when no confusion can arise.

Since each jet is an ideal of $C^\infty(\mathcal{V})$, the relation of inclusion between ideals produces canonical correspondences between jet spaces. At this point, it is essential to stop thinking of jets as "jets of cross–sections of a fibre bundle".

Focusing our attention on the case in which we are interested, we give the following:

Definition 2.1. Given the natural numbers $m \le r$, the *Lie correspondence* $\bigwedge_{m,r}(\mathcal{V})$ is the subset of the fibred product $\mathcal{J}_m^1 \times_{\mathcal{V}} \mathcal{J}_r^1$ consisting of the pairs of jets $(\mathfrak{p}_m^1, \mathfrak{p}_r^1)$ (with the same source $p = \mathfrak{p}_m^1 = \mathfrak{p}_r^0$), such that $\mathfrak{p}_m^1 \supseteq \mathfrak{p}_r^1$ (inclusion as ideals of $C^\infty(\mathcal{V})$).

Theorem 2.2 (Basic lemma). *The necessary and sufficient condition for a point* $(\mathfrak{p}_m^1, \mathfrak{p}_r^1) \in \mathcal{J}_m^1 \times_{\mathcal{V}} \mathcal{J}_r^1$ *to belong to* $\bigwedge_{m,r}(\mathcal{V})$ *is that the following inclusion of spaces of* 1*–forms should hold*

$$(\Omega_m^1)_{\mathfrak{p}_m^1} \supseteq (\Omega_r^1)_{\mathfrak{p}_r^1}.$$

Consequently, the equations deduced from the conditional inclusion $\Omega_m^1 \supseteq \Omega_r^1$ *are those of* $\bigwedge_{m,r}$ *as a submanifold of* $\mathcal{J}_m^1 \times_{\mathcal{V}} \mathcal{J}_r^1$.

Proof. This follows easily from Remark 1.2. □

Given a system of partial differential equations $\mathcal{R}_m^1 \subseteq \mathcal{J}_m^1$, its inverse image in $\bigwedge_{m,r}$ by the canonical projection $\bigwedge_{m,r} \longrightarrow \mathcal{J}_m^1$ will be denoted by $\mathcal{R}_{m,r}^1$.

We are mainly interested in the case $r = n-1$, because the submanifolds $\mathcal{F}_{n-1} \subseteq \mathcal{J}_{n-1}^1$ are first order systems with an unknown function, whose properties are well known.

A geometric interpretation of $\mathcal{R}_{m,n-1}^1$ can be deduced by thinking of each $\mathfrak{p}_m^1$ as an m-dimensional subspace of $T_p(\mathcal{V})$ ($p = \mathfrak{p}_m^0$). We can thus think of each $\mathfrak{p}_{n-1}^1$ contained (as an ideal) in $\mathfrak{p}_m^1$ as a hyperplane of $T_p(\mathcal{V})$ containing the m-dimensional subspace which is $\mathfrak{p}_m^1$. Thus, the fibre of $\mathcal{R}_{m,n-1}^1$ over $\mathfrak{p}_m^1$ is the subset of all hyperplanes of $T_p(\mathcal{V})$ passing through $\mathfrak{p}_m^1$.

Since $\mathcal{J}_{n-1}^1$ is the projectivized manifold of $T^*(\mathcal{V}) - \{0\}$, for the correspondences $\bigwedge_{m,n-1}$ we can replace the second factor in $\mathcal{J}_m^1 \times_{\mathcal{V}} \mathcal{J}_{n-1}^1$ by $T^*(\mathcal{V})$, which we will do in the sequel.

For each system of partial differential equations $\mathcal{R}_m^1 \subseteq \mathcal{J}_m^1$, we shall denote by $\mathcal{R}_{m,*}^1$ the set of all pairs $(\mathfrak{p}_m^1, \alpha_p) \subseteq \mathcal{R}_m^1 \times_{\mathcal{V}} T^*(\mathcal{V})$ such that $\alpha_p \in \mathfrak{p}_m^1/\mathfrak{m}_p^2$. The fibre of $\mathcal{R}_{m,*}^1$ over $\mathfrak{p}_m^1$ is the set of the differentials at p of all the functions of $\mathfrak{p}_m^1$, or, what is the same, $(\Omega_m^1)_{\mathfrak{p}_m^1}$ (see the previous section). The projectivized space of this fibre is the collection of hyperplanes of $T_p(\mathcal{V})$ passing through $\mathfrak{p}_m^1$.

From the basic lemma one has:

Proposition 2.3. *The equations of $\mathcal{R}^1_{m,*}$ inside $\mathcal{R}^1_m \times_{\mathcal{V}} T^*(\mathcal{V})$ are those deduced from the conditional inclusion $\theta \in \Omega^1_m$.*

Definition 2.4. The projection of $\mathcal{R}^1_{m,*}$ over $T^*(\mathcal{V})$ will be called the *first order system of partial differential equations associated with* $\mathcal{R}^1_m$ and we shall denote it by $(\mathcal{R}^1_m)^*$.

Remark 2.5. $(\mathcal{R}^1_m)^*$ may not be, a priori, a submanifold of $T^*(\mathcal{V})$. When one is working in the complex domain and $\mathcal{R}^1_m$ is an algebraic submanifold of $\mathcal{J}^1_m$, then $(\mathcal{R}^1_m)^*$ can be said to contain a dense subset which is a manifold.

Example. If $\mathcal{X}$ is an m-dimensional submanifold of $\mathcal{V}$, $\mathcal{R}^1_m = \mathcal{J}^1_m(\mathcal{X}) \subseteq \mathcal{J}^1_m(\mathcal{V})$ is a system of partial differential equations which has the solution $\mathcal{X}$ (and its open sets) only. Each point $\mathfrak{p}^1_m \in \mathcal{R}^1_m$ is the same as $T_p\mathcal{X}$ (where $p = \mathfrak{p}^0_m$ is the source of $\mathfrak{p}^1_m$). We can think of each $\mathfrak{p}^1_{n-1} \subseteq \mathfrak{p}^1_m$ as a hyperplane $H_p \subseteq T_p\mathcal{V}$ containing $T_p\mathcal{X}$. Thus, $\mathcal{R}^1_{m,n-1}$ is the collection of all contact elements H_ρ tangent to $\mathcal{X}$: it is the manifold of contact elements of $\mathcal{X}$ in the Lie terminology; it is a Legendre submanifold of $\mathcal{J}^1_{n-1}(\mathcal{V})$. Therefore, $(\mathcal{R}^1_m)^*$ is a lagrangian submanifold of $T^*(\mathcal{V})$.

From Definition 2.4 it follows easily that if $\mathcal{R}^1_m \subseteq \overline{\mathcal{R}}^1_m$ then $(\mathcal{R}^1_m)^* \subseteq (\overline{\mathcal{R}}^1_m)^*$. From this and the previous example we deduce the following:

Proposition 2.6 (Lie). *If $\mathcal{X}$ is a solution of the system of partial differential equations $\mathcal{R}^1_m$, $\mathcal{X}$ is also solution, in the generalized Lie sense, of the first order system $(\mathcal{R}^1_m)^*$, that is to say, $(\mathcal{J}^1_m(\mathcal{X}))^*$ is a lagrangian submanifold of $(\mathcal{R}^1_m)^*$.*

The systems of partial differential equations considered by Lie in [6] are those such that $(\mathcal{R}^1_m)^*$ parametrizes the correspondence $\mathcal{R}^1_{m,*}$. They fulfill the following:

Definition 2.7. A system $\mathcal{R}^1_m$ is a *Lie system* when the projection of the correspondence $\mathcal{R}^1_{m,*}$ over $(\mathcal{R}^1_m)^*$ is an isomorphism.

Remark 2.8. For each Lie system, each $\mathfrak{p}^1_{n-1}$ that is contained as an ideal in a $\mathfrak{p}^1_m \in \mathcal{R}^1_m$, is contained in only one of them. In dual terms: each contact element $H_p \in \mathcal{J}^1_{n-1}(\mathcal{V})$ which contains an m- dimensional manifold $P_p \subseteq T_p(\mathcal{V})$ determining a jet of $\mathcal{R}^1_m$ contains only one of them.

The composition of $(\mathcal{R}^1_m)^* \approx \mathcal{R}^1_{m,*}$ with the projection $\mathcal{R}^1_{m,*} \longrightarrow \mathcal{R}^1_m$ gives a parametrization of the system $\mathcal{R}^1_m$ by the first order system $(\mathcal{R}^1_m)^*$. The situation is as follows: a first order system $\mathcal{F} = (\mathcal{R}^1_m)^*$ and a smooth map $\lambda : \mathcal{F} \longrightarrow \mathcal{R}^1_m$ such that for each $\alpha_p \in \mathcal{F}$, $\alpha_p \in \lambda(\alpha_p)/\mathfrak{m}^2_p$.

The basic lemma gives:

Proposition 2.9. *Let $\mathcal{F} \subseteq T^*(\mathcal{V})$ be a first order system of partial differential equations, and let $\lambda : \mathcal{F} \longrightarrow \mathcal{J}^1_m(\mathcal{V})$ be a smooth map. The necessary and sufficient condition for $\alpha_p \in \lambda(\alpha_p)/\mathfrak{m}^2_p$ for all $\alpha_p \in \mathcal{F}$ is that $\bar{\theta} \in \lambda^*(\Omega^1_m)$, $\bar{\theta}$ being the specialization to $\mathcal{F}$ of the canonical* 1*-form of $T^*(\mathcal{V})$.*

3. Systems of two partial differential equations of second order with two independent variables and one unknown function

Systems of two partial differential equations of second order in two independent variables and only one unknown function are those the most studied in the classical literature; it is known that each involutive system of this kind is integrable by a generalization of Cauchy's characteristics method for a single first order partial differential equation because of the existence of characteristic vector fields (see for instance [4, 2]).

In the light of the discussion at the end of Section 1, a system of this kind can be defined as a 6-dimensional locally closed submanifold $\mathcal{R}_2^2$ of $\mathcal{J}_2^2(\mathcal{V})$, where $\mathcal{V}$ is a smooth manifold of dimension 3; furthermore, we assume that the projection $\mathcal{R}_2^2 \longrightarrow \mathcal{J}_2^1(\mathcal{V})$ has the highest rank $(= 5)$ at every point of $\mathcal{R}_2^2$. Inspired by the indications of Lie in [6, page 337], we wondered whether there are sufficient involutive distributions of rank 2 of vector fields tangent to $\mathcal{R}_2^2$ and annihilated by the specialization to it of the contact system of $\mathcal{J}_2^2(\mathcal{V})$ to obtain (locally) all the solutions of $\mathcal{R}_2^2$. The answer is affirmative and the results obtained can be summarized as follows:

Theorem 3.1. *The systems $\mathcal{R}_2^2$ of two partial differential equations of second order with two independent variables and one unknown function that are formally compatible when prolonged to third and fourth orders are those whose solutions are all solutions of involutive distributions of rank 2 of tangent vector fields annihilated by the contact system in $\mathcal{R}_2^2$. They can be divided into two and only two classes:*

1. *The prolongation $\mathcal{R}_2^3$ has dimension 6 and is isomorphic to $\mathcal{R}_2^2$ by the projection $\mathcal{R}_2^3 \longrightarrow \mathcal{R}_2^2$. There is only one involutive distribution of rank 2 of tangent vector fields on $\mathcal{R}_2^2$ annihilated by the contact system. The solutions of this distribution are all the solutions of $\mathcal{R}_2^2$. Thus, the general solution of $\mathcal{R}_2^2$ "depends on four arbitrary constants".*

2. *The prolongation $\mathcal{R}_2^3$ has dimension 7. $\mathcal{R}_2^3$ is formally compatible if and only if $\mathcal{R}_2^2$ is involutive in the sense of Spencer and Goldschmidt. The contact system on $\mathcal{R}_2^2$ is projectable by a tangent vector field D; there are infinite involutive distributions of rank 2 of vector fields tangent to $\mathcal{R}_2^2$ annihilated by the contact system, and each solution of $\mathcal{R}_2^2$ is a solution of one of these distributions, which are obtained by solving a partial differential equation of first order with six independent variables.*

As a consequence, *every solution of each system $\mathcal{R}_2^2$ formally compatible up to order 4 is obtained by solving ordinary differential equations.*

4. Relation with the Lie correspondence

The theory of Lie correspondences, as addressed here, can be applied to systems of partial differential equations with any number of independent variables and unknown functions, but always of first order, that is to say, submanifolds of a jet space $\mathcal{J}_m^1(\mathcal{W})$, with arbitrary $\dim \mathcal{W}$. Each system can be thought of in this way, due to the natural inclusions $\mathcal{J}_m^\ell(\mathcal{V}) \subseteq \mathcal{J}_m^1(\mathcal{J}_m^{\ell-1}(\mathcal{V}))$. In particular, the system $\mathcal{R}_2^2 \subseteq \mathcal{J}_2^2(\mathcal{V})$ can be considered as a submanifold of $\mathcal{J}_2^1(\mathcal{J}_2^1(\mathcal{V}))$, and in this case the base manifold for the Lie correspondence is $\mathcal{J}_2^1(\mathcal{V})$. Accordingly, $(\mathcal{R}_2^2)^*$ is a subset of $T^*\mathcal{J}_2^1(\mathcal{V})$.

We shall assume that $\mathcal{R}_2^2$ is a Lie system (Definition 2.7) and involutive. The situation is:

$$\begin{array}{ccc} T^*\mathcal{J}_2^1(\mathcal{V}) \supseteq (\mathcal{R}_2^2)^* & \xrightarrow{\lambda} & \mathcal{R}_2^2 \\ & \searrow \quad \swarrow & \\ & \mathcal{J}_2^1(\mathcal{V}). & \end{array}$$

and we denote by θ the canonical 1-form in $T^*\mathcal{J}_2^1(\mathcal{V})$ specialized to $(\mathcal{R}_2^2)^*$, and by $\Omega(\mathcal{R}_2^2)$ the contact system of $\mathcal{J}_2^1(\mathcal{J}_2^1(\mathcal{V}))$ specialized to $\mathcal{R}_2^2$.

Remark 4.1. Proposition 2.6 gives that each solution of $\mathcal{R}_2^2$ is a solution, in the generalized Lie sense, of the first order associated system $(\mathcal{R}_2^2)^*$.

Definition 4.2. Each solution of the system $(\mathcal{R}_2^2)^*$ will be called an *intermediate integral* of $\mathcal{R}_2^2$.

Therefore, an intermediate integral of $\mathcal{R}_2^2$ is a hypersurface of $\mathcal{J}_2^1(\mathcal{V})$; i.e., a first order partial differential equation. This definition does not agree with some of the classical ones, but is justified by our main result concerning this kind of system:

Theorem 4.3. *Let $\mathcal{R}_2^2$ be an involutive Lie system formally compatible up to fourth order; then, each solution of $\mathcal{R}_2^2$ is a solution of an intermediate integral; conversely, each intermediate integral has a complete integral formed by solutions of $\mathcal{R}_2^2$. Moreover, each solution of $\mathcal{R}_2^2$ is obtained as intersection of intermediate integrals.*

Remark 4.4. In this case, the program proposed by Lie can be carried out completely: the integration of an involutive Lie system $\mathcal{R}_2^2$ is reduced to that of its first order associated system.

The idea for the proof consists in using the fact that each solution of $\mathcal{R}_2^2$ can be obtained as a solution of an involutive distribution of vector fields tangent to $\mathcal{R}_2^2$ and annihilated by the contact system (see Theorem 3.1). Unfortunately, this is out of the scope of the present paper and will be published elsewhere.

References

[1] R.J. Alonso Blanco, Jet manifold associated to a Weil bundle, *Arch. Math. (Brno)* 36 (2000) 195–199.

[2] E. Cartan, Les systèmes de Pfaff a cinq variables et les équations aux dérivées partialles du second ordre, *Ann. Sci. École Nor. Sup.*, 3^e série 27 (1910) 153–206.

[3] H. Goldschmidt, Integrability criteria for systems of non-linear partial differential equations, *J. Differential Geom.* 1 (1967) 269–307.

[4] E. Goursat, *Leçons sur l'integration des équations aux dérivées partialles du second ordre à deux variables indépendantes,* Tom. II (Hermann, Paris, 1898).
[5] I. Kolář, P.W. Michor, and J. Slovák, *Natural Operations in Differential Geometry* (Springer, Berlin, 1993).
[6] S. Lie, Zur allgemeine Theorie der partiellen Differentialgleichungen beliebiger Ordnung, *Leipz. Ber.* (1895) (1) 53–128; (*Ges. Abh.*, Bd. 4, 320–384).
[7] V.V. Lychagin, Local classification of nonlinear first order partial differential equations, *Uspekhi Math. Nauk* 30 (1975) (in Russian); English translation *Russian Math. Surveys* 30 (1975) 105–176.
[8] J. Muñoz Díaz, *Ecuaciones Diferenciales I* (Universidad de Salamanca, Salamanca, 1982).
[9] J. Muñoz, F.J. Muriel, and J. Rodríguez, Weil bundles and jet spaces, *Czech. Math. J.* 50 (2000) (125) 721–748.
[10] J. Muñoz, F.J. Muriel, and J. Rodríguez, A remark on Goldschmidt's criterion on formal integrability, *J. Math. Anal. Appl.* 254 (2000) 275–290.
[11] J. Muñoz, F.J. Muriel, and J. Rodríguez, The contact system on the (m, ℓ)-jet spaces, *Arch. Math. (Brno)* 37 (2001) 291–300.
[12] J.F. Pommaret, *Systems of partial differential equations and Lie pseudogroups* (Gordon and Breach, New York, 1978).
[13] J. Rodríguez Lombardero, Sobre los espacios de jets y los fundamentos de la teoría de los sistemas de ecuaciones en derivadas parciales, PhD-thesis, Universidad de Salamanca, 1990.
[14] A. Weil, Théorie des points proches sur les variétés différentiables, in: *Colloque de Géometrie Différentielle* (Centre National de la Recherche Scientifique, Paris, 1953) 111–117.

S. Jiménez, J. Muñoz and J. Rodríguez
Departamento de Matemáticas, Univ. de Salamanca
Plaza de la Merced 1-4
E-37008 Salamanca
Spain
E-mail: sjv@usal.es, clint@usal.es, jrl@usal.es