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Logarithm of differential forms and regularization of volume forms[1]

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Abstract. By using logarithm of derivation, logarithm of a differential form is defined. It allows to define fractional order differential forms on a flat space. Regularized volume form on a flat infinite-dimensional space is defined by using fractional order differential forms. Applying noncommutative connection, we show regularized volume form can be defined on a mapping space, if its string class vanishes.

Keywords. Logarithm of derivation, fractional order differential forms, spectral ζ function, regularized volume form.

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1. Introduction

In our study on regularization of differential operators on a Hilbert space H, infinite products of trigonometric functions appear as proper functions ([6]). It leads to the study of regularized infinite product of coordinate functions of H, see [7].

Precisely, to define regularization, we need to equip a positive Schatten class operator G such that $\zeta(G,s) = \mathrm{tr}(G^s)$ allows analytic continuation to $s = 0$ and holomorphic at $s = 0$. Such pairing is closely related to Connes' spectral triple ([8]). But our approach is more concrete and analytic. We assume $\zeta(G,s)$ has its first pole at $s = d$. We also set $\zeta(G,0) = \nu$ and call it the regularized dimension of H. The typical example of such pair is $\{L^2(X,E), G\}$, X a compact Riemannian manifold, E a symmetric (or Hermitian) vector bundle over X and $L^2(X,E)$ is the Hilbert space of L^2-sections of E, and G the Green operator of a nondegenerate selfadjoint elliptic differential operator acting on the sections of E ([11], see also [13] when X is noncompact).

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

The regularized infinite product $\vdots \prod x_n \vdots$ is defined to be the analytic continuation of $\prod x_n^{\mu_n^s}$ to $s = 0$, where μ_n is a proper value of G with the proper function e_n; $Ge_n = \mu_n e_n$. Strictly saying, one cannot define $\vdots \prod x_n \vdots$ if $x = \sum x_n e_n$ belongs to H, and we need to extend H as follows: Let $\|x\|_k = \|G^{-k}x\|$ be the k-th Sobolev norm of $x \in H$, and W^k the k-th Sobolev space constructed from H and $\|x\|_K$. We set $H^- = \bigcap_{k<0} W^k$ and define the spaces $H^-(\text{finite})$ and $H^-(0)$ by

$$H^-(\text{finite}) = \left\{ x = \sum x_n e_n \in H^- \,\middle|\, \lim_{n\to\infty} \mu_n^{-d/2} x_n \quad \text{exists} \right\}, \tag{1}$$

$$H^-(0) = \left\{ x = \sum x_n e_n \in H^- \,\middle|\, \lim_{n\to\infty} \mu_n^{-d/2} x_n = 0 \right\}. \tag{2}$$

By definition, we have $H^-(\text{finite}) = H^-(0) \oplus \mathbf{Ce}$, $\mathbf{e} = \sum \mu_n^{d/2} e_n$. Then $\vdots \prod x_n \vdots$ is defined for the elements of $H^-(\text{finite})$, ([7]). Since $\vdots \prod x_n \vdots$ is linear in each variable, we may consider $(\partial^N/\partial x_1 \cdots \partial x_N) \vdots \prod x_n \vdots = \vdots \prod_{n \geq N} x_n \vdots$, but we cannot compute $\lim_{N\to\infty} (\partial^N/\partial x_1 \cdots \partial x_N) \vdots \prod x_n \vdots$. To overcome this difficulty, we used fractional order derivation and define regularized infinite degree derivation $\vdots \partial^\infty / \prod \partial x_n \vdots$ by

$$\vdots \frac{\partial^\infty}{\prod \partial x_n} \vdots = \left(\prod \frac{1}{\Gamma(1+\mu_n^s)} \frac{\partial^{\mu_n^s}}{\partial x_n^{\mu_n^s}} \right)\Bigg|_{s=0}.$$

Then we have $\vdots \partial^\infty / \prod \partial x_n \vdots \vdots \prod x_n \vdots = 1$, see [7].

Motivated by these phenomenon, we introduce logarithm of differential forms by using logarithm of derivation, which is defined by

$$\log\left(\frac{\partial}{\partial x_n}\right) = \lim_{h\to+0} \frac{1}{h} \left(\frac{\partial^h}{\partial x_n^h} - \mathrm{I} \right).$$

There are several alternative definitions of $\log(\partial/\partial x_n)$. In this paper, we define $\log(\partial/\partial x_n)$ by using Borel transformation ([2]). Then we define logarithms of vector fields and differential forms $\log(dx_1), \log(dx_2), \ldots,$ by using logarithm of derivation. We also impose the commutation relation $[\log(dx_n), \log(dx_m)] = \pi i$, $n < m$. Fractional order differential form $d^a x_n$ is defined to be $\exp(a \log(dx_n))$, $0 \leq \Re a < 2$. *This definition differs from the definition in* [9] (cf. [11], see also [1]). Let $\omega(s)$ be $\sum \mu_n^s \log(dx_n)$. Then we define regularized volume form (regularized infinite degree wedge product) $\vdots \Lambda_{n=1}^{\infty,\to} dx_n \vdots$ by

$$\vdots \Lambda_{n=1}^{\infty,\to} dx_n \vdots = \mathrm{e}^{\zeta(G,s)(\zeta(G,s)-1)/2} \mathrm{e}^{\omega(s)}|_{s=0}. \tag{3}$$

Definitions of logarithm of differential forms and regularized volume form use flatness of H. If a Sobolev manifold $\mathcal{M}$ is parallelisable, then we can define regularized volume form of $\mathcal{M}$. If $\mathcal{M}$ is a mapping space $\mathrm{Map}(X, M)$, applying noncommutative connection ([3]), we can show that there is a subvariety $\mathcal{N}$ of $\mathrm{Map}(X, M)$ such that $\mathrm{Map}(X, M) \setminus \mathcal{N}$ is parallelisable ([4, 5]). Since regularized volume form generates a line bundle, regularized volume form of $\mathrm{Map}(X, M)$ is defined if this line bundle is extended to Map(X, M). Since string class gives the obstruction to this extension, Map(X, M) has the regularized volume form if the string class vanishes.

2. Logarithm of derivation I. One variable function

In this section, we recall the definition and property of $\log(d/dx)$.

We study $\log(d/dx)$ by using Borel transformation. For $f(x) = \sum c_n x^n$, its Borel transformation $\mathcal{B}[f] = \mathcal{B}[f(t)](x)$ is defined and has the following properties

$$\mathcal{B}[f] = \sum \frac{c_n}{n!} x^n = \frac{1}{2\pi i} \oint \frac{f(t)}{t} \mathrm{e}^{x/t}\, dt. \tag{4}$$

$$\frac{d}{dx}\mathcal{B}[f] = \mathcal{B}\left[\frac{f}{t}\right], \quad \mathcal{B}[f \cdot g] = \mathcal{B}[f] \sharp \mathcal{B}[g], \tag{5}$$

$$u \sharp v = \frac{d}{dx} \int_0^x u(x-t)v(t)dt. \tag{6}$$

In the rest, we use the notation

$$f(\sharp u) = \sum c_n u^{\sharp n}, \quad f(u) = \sum c_n u^n, \quad u^{\sharp n} = \overbrace{u \sharp \cdots \sharp u}^{n}.$$

For example, $\mathrm{e}^{\sharp u} = \sum (u^{\sharp n}/n!)$. The following lemma is proved in [1].

Lemma 1. *The following holds, where γ is the Euler constant.*

$$\mathrm{e}^{\sharp t \log x} = \frac{\mathrm{e}^{-\gamma t}}{\Gamma(1+t)} x^t. \tag{7}$$

If $\mathcal{B}[\log x]$ is defined, it must be $x^n/n! = \mathrm{e}^{\sharp n \log x}$. So by this lemma, we define $\mathcal{B}[\log x]$ by

$$\mathcal{B}[\log x] = \log x + \gamma.$$

We denote the algebra of finite exponential type entire functions with the variable $w = \log x$ by $\mathfrak{F}$. Since $w \sharp w^{n-1} = w^n - \mathrm{P}_{n-1}(w)$, where

$$\mathrm{P}_n(w) = \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+1)!}{k!} \zeta(n+2-k) w^k,$$

([2]), if $f(w) \in \mathfrak{F}$, then $f(\sharp w)$ acts on $\mathfrak{F}$ by the $\sharp$-product. We set $\mathbb{F} = \{f(\sharp w) | f \in \mathfrak{F}\}$.

Definition 1. We define fractional derivation and logarithm of derivation of $f \in \mathfrak{F}$ by

$$\frac{d^a f}{dx^a} = \mathrm{e}^{\sharp - a(w+\gamma)} \sharp f, \quad \log\left(\frac{d}{dx}\right) f = -(w+\gamma) \sharp f. \tag{8}$$

Example . We have

$$\frac{d^a}{dx^a}x^n = \frac{n!}{\Gamma(n+1-a)}x^{n-a},$$

$$\log\left(\frac{d}{dx}\right)x^n = -x^n\left(\log(x) + \left(\gamma - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\right)\right).$$

Note 1. Lemma 1 shows that $e^{\sharp - a(w+\gamma)}$ can be interpreted as a function, unless a is not a negative integer. We consider $e^{\sharp - n(w+\gamma)}$ to be the operator d^n/dx^n if n is a natural number. As for fractional derivation, d^a/dx^a is nonlocal unless a is an integer. We consider its initial to be 0 (cf. [9]).

Note 2. When $a < 0$, we consider $e^{\sharp - a(w+\gamma)}$ to be a fractional order indefinite integral operator. Similarly, we consider $(w + \gamma)\sharp f$ to be the logarithm of the indefinite integral operator

$$\int_0^x f(t)\,dt.$$

3. Logarithm of derivation II. Several variable function

Let w_n be $\log x_n$, where $x_1, x_2, \ldots,$ are coordinate functions of H^-(finite). Then the algebra of finite exponential type functions of $w_1, w_2, \ldots,$ is denoted by $\mathfrak{F}$. We also define the $\sharp$-product of w_n and w_m by

$$w_n \sharp w_m = w_n \cdot w_m + \frac{m-n}{|m-n|}\frac{\pi}{2}i, \quad n \neq m. \tag{9}$$

To define the action of $\mathbb{F}$ to $\mathfrak{F}$, we need some preliminaries. Let $S = \{p_1, \ldots, p_n\}$ be a set of natural numbers (may not be distinct form each other), T a subset of S and $\complement T$ the complement of T in S. Let $p_l \in T$, then we associate with p_l a natural number $\natural_S(p_l) = \natural(p_l)$ as follows;

$$\natural(p_l) = 1, \quad \text{if } p_l = \min.T,$$

$$\natural(p_l) = 2, \quad \text{if } p_l = \min.\complement\{p_j | \natural p_j = 1\} \cap T,$$

and so on. The sign $\mathrm{sgn}T = \mathrm{sgn}\{p_{j_1}, \ldots, p_{j_k}\}$ is defined by

$$\mathrm{sgn}\{p_{j_1}, \ldots, p_{j_k}\} = \mathrm{sgn}\begin{pmatrix} 1 & \ldots & k \\ \natural(p_{j_1}) & \ldots & \natural(p_{j_k}) \end{pmatrix} \tag{10}$$

Let $w_n \sharp' w_m$ be $w_n \sharp w_m$ if $n \neq m$ and w_n^2 if $n = m$. Then we define $w_{p_1}\sharp' \cdots \sharp' w_{p_n}$ to be

$$w_{p_1} \cdots w_{p_n} + \sum_{2m \leq m}\left(\frac{\pi i}{2}\right)^m \mathrm{sgn}\complement\{p_{j_1}, \ldots, p_{j_{n-2m}}\} w_{p_{j_1}} \cdots w_{p_{j_{n-2m}}}. \tag{11}$$

Definition 2. We define fractional partial derivation $\partial^a/\partial x_n^a$ and logarithm of partial derivation $\log(\partial/\partial x_n)$ by

$$\frac{\partial^a}{\partial x_n^a} f = \mathrm{e}^{\sharp -a(w_n+\gamma)}\sharp f, \qquad \log\left(\frac{\partial}{\partial x_n}\right) f = -(w_n+\gamma)\sharp f. \tag{12}$$

By (9), we have $[w_n, w_m] = \operatorname{sgn}\{n,m\}\pi i$. Hence we obtain

$$\mathrm{e}^{\sharp a(w_n+\gamma)}\sharp\mathrm{e}^{\sharp b(w_m+\gamma)} = \mathrm{e}^{\sharp(a(w_n+\gamma)+b(w_m+\gamma)+\operatorname{sgn}\{n,m\}ab\pi i/2)}. \tag{13}$$

By (13), we have

Proposition 1. (i) *Let $n \neq m$ and -1 be $\mathrm{e}^{\pi i}$ if $m > n$, and $\mathrm{e}^{-\pi i}$ if $m < n$. Then we have*

$$\mathrm{e}^{\sharp a(w_n+\gamma)}\sharp\mathrm{e}^{\sharp b(w_m+\gamma)} = (-1)^{ab}\mathrm{e}^{\sharp b(w_m+\gamma)}\sharp\mathrm{e}^{\sharp a(w_n+\gamma)}. \tag{14}$$

(ii) *Let -1 be the same as above, $a \neq 1$ and $\delta_{n,m}$ the Kronecker's delta. Then we have*

$$\mathrm{e}^{\sharp -a(w_n+\gamma)}\sharp\mathrm{e}^{\sharp a(w_m+\gamma)}\sharp 1 = (-1)^{-a^2}\frac{\sin(\pi a)}{\pi a}\left(\frac{x_m}{x_n}\right)^a, \tag{15}$$

$$\mathrm{e}^{\sharp -(w_n+\gamma)}\sharp\mathrm{e}^{\sharp(w_m+\gamma)}\sharp 1 = \mathrm{e}^{\sharp(w_m+\gamma)}\sharp\mathrm{e}^{\sharp -(w_n+\gamma)}\sharp 1 = \delta_{n,m}. \tag{16}$$

Note 3. Operators such as $\log(\partial/\partial x_1 + \partial/\partial x_2)$ are not contained in $\mathbb{F}$.

4. Regularized infinite product

Let $\Re s > d$. Then we may consider $\omega(s) = \sum \mu_n^s(w_n+\gamma)$ to be an element both of $\mathfrak{F}$ and $\mathbb{F}$. We mainly consider $\omega(s)$ to be an element of $\mathbb{F}$. But we use same notation when considered to be an element of $\mathfrak{F}$.

Lemma 2. *We have*

$$\mathrm{e}^{\sharp -\omega(s)} = \mathrm{e}^{(\zeta(G,s)(\zeta(G(s)-1)/2)\pi i}\mathrm{e}^{\sharp -\mu_1^s(w_1+\gamma)}\sharp\mathrm{e}^{\sharp -\mu_2^s(w_2+\gamma)}\sharp\cdots. \tag{17}$$

Proof. First we note, if $\sum c_n$ converges absolutely, then we have

$$\sum_{n=1}^{\infty}\sum_{m=n}^{\infty} c_n c_m = \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} c_n c_m, \qquad \sum_{n=1}^{m} c_n = C - \sum_{n=m+1}^{\infty},$$

where $C = \sum c_n$. Hence we have

$$\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} c_n c_m = \frac{C(C-1)}{2}.$$

Hence we obtain

$$\mathrm{e}^{\sharp c_1 w_1}\sharp\mathrm{e}^{\sharp c_2 w_2}\sharp\cdots = \mathrm{e}^{(C(C-1)\pi i)/2}\mathrm{e}^{\sharp\sum c_n w_n}.$$

Definition 3. Let $f \in \mathfrak{F}$. The regularized infinite product $\vdots \prod_{n=1}^{\infty,\rightarrow} \sharp e^{\sharp-(w_n+\gamma)} \vdots$ is defined to be the regularization of $e^{\sharp-(w_1+\gamma)}\sharp e^{\sharp-(w_2+\gamma)}\sharp\cdots$, by the formula

$$\vdots \prod_{n=1}^{\infty,\rightarrow} \sharp e^{\sharp-(w_n+\gamma)} \vdots \sharp f = e^{-\nu(\nu-1)\pi i/2} e^{\sharp-\omega(s)} \sharp f|_{s=0}. \tag{18}$$

However, $\vdots \prod_{n=1}^{\infty,\rightarrow} \sharp e^{\sharp-(w_n+\gamma)} \vdots$ does not act on $\mathfrak{F}$. But if f is a polynomial, we have

$$\vdots \prod_{n=1}^{\infty,\rightarrow} \sharp e^{\sharp-(w_n+\gamma)} \vdots \sharp f = \lim_{n\to\infty} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} f.$$

Since $\log\Gamma(1+\mu_n^s) = -\gamma\mu_n^s + O(\mu_n^{2s})$, we have

$$-\mu_n^s(w_n-\gamma) - \log\Gamma(1+\mu_n^s) = -\mu_n^s w_n + O(\mu_n^{2s}).$$

Hence $\Gamma(1+\mu_n^s)^{-1} e^{\sharp-\mu_n^s(w_n+\gamma)} = e^{\sharp-\mu_n^s w_n} + O(\mu_n^{2s})$. Therefore we may consider

$$\vdots \frac{\partial^\infty}{\prod \partial x_n} \vdots f = e^{\nu(\nu-1)\pi i/2} e^{\sharp-\sum \mu_n^s w_n} \sharp f|_{s=0}. \tag{19}$$

Note 4. Similarly, we may consider $e^{\nu(\nu-1)\pi i/2} e^{\sharp \sum \mu_n^s(w_n+\gamma)} \sharp f|_{s=0}$ to be the regularized infinite-dimensional indefinite integral $\int_{Q(x)} f(x) \vdots d^\infty x \vdots$, see [6].

Let $i_1 < i_2 < \cdots < i_m$ be an m-set of integers. Then we set

$$\omega_{1_1,i_2,\ldots,i_m}(s) = \sum_{n\notin\{i_1,\ldots,i_m\}} \mu_n^s(w_n+\gamma).$$

Starting $\omega_{i_1,\ldots,i_m}(s)$, regularized infinite product $\vdots \prod_{n\notin\{i_1,\ldots,i_m\}}^{\infty,\rightarrow} \sharp e^{\sharp-(w_n+\gamma)} \vdots$ is defined, and we have

$$\begin{aligned}\vdots \prod_{n\notin\{i_1,\ldots,i_m\}}^{\infty,\rightarrow} \sharp e^{\sharp-(w_n+\gamma)} \vdots &= (-1)^{(i_1-1)+\cdots+(i_m-1)} \\ &\quad e^{\sharp(w_{i_1}-\gamma)}\sharp\cdots\sharp e^{\sharp(w_{i_m}-\gamma)}\sharp \vdots \prod_{n=1}^{\infty,\rightarrow} \sharp e^{\sharp-(w_n+\gamma)} \vdots .\end{aligned} \tag{20}$$

5. Fractional degree differential forms

Let $\mathbb{F}_1$ be the submodule of $\mathbb{F}$ which consists of the homogeneous elements of degree 1. We set

$$\begin{aligned}\mathbb{F}_{1,-} &= \left\{\sum c_n w_n \in \mathbb{F}_1 \middle| \Re c_n \le 0\right\}, \\ \mathbb{F}_{1,+} &= \left\{\sum c_n w_n \in \mathbb{F}_1 \middle| \Re c_n \ge 0\right\}.\end{aligned} \tag{21}$$

The algebras generated by $\{e^{\sharp u} | u \in \mathbb{F}_{1,\pm}\}$ and 1 are denoted by $\mathrm{Exp}(\mathbb{F}_{1,\pm})$, respectively.

Let $\phi \in \mathrm{Exp}(\mathbb{F}_{1,-})$ and $\psi \in \mathrm{Exp}(\mathbb{F}_{1,+})$. Then $\phi\sharp\psi\sharp 1$ is a function of $x_1, x_2, \ldots, x_n = r_n e^{i\theta_n}$. The constant part $c_0(\phi\sharp\psi\sharp 1)$ of $\phi\sharp\psi\sharp 1$ is

$$\lim_{N\to\infty}\left(\lim_{n_1\to\infty,\ldots,n_N\to\infty}\frac{1}{2n_1\pi}\int_0^{2n_1\pi}d\theta_1\cdots\int\frac{1}{2n_N\pi}\int_0^{2n_N\pi}d\theta_N\phi\sharp\psi\sharp 1(\theta_1,\ldots)\right).$$

We denote $\mathrm{Exp}(\mathbb{F}_{1,\pm})_k$ the vector subspaces of $\mathrm{Exp}(\mathbb{F}_{1,\pm})$ generated by e^u, where $u = \sum_{j=1}^k c_{n_j} w_{n_j}$ such that no c_{n_j} vanishes.

Definition 4. Let $\phi \in \mathrm{Exp}(\mathbb{F}_{1,-})_k$ and $\psi \in \mathrm{Exp}(\mathbb{F}_{1,+})_k$. Then we define the pairing $\langle\phi, \psi\rangle$ by

$$\langle\phi, \psi\rangle = c_0(\phi\sharp\psi\sharp 1). \tag{22}$$

For example, by Proposition 1, we have

$$\langle e^{\sharp - a(w_n+\gamma)}, e^{\sharp b(w_m+\gamma)}\rangle = 0, \quad a \neq b, \text{ or } n \neq m, \tag{23}$$

$$\langle e^{\sharp - a(w_n+\gamma)}, e^{\sharp a(w_n+\gamma)}\rangle = -(-1)^{-(a-1)^2}\frac{\sin(\pi a)}{\pi a}, \; a \neq \mathbb{Z}. \tag{24}$$

Let $\mathbb{I}_+ = \mathbb{I}$ be the ideal generated by $e^{\sharp 2(w_n+\gamma)}$ in $\mathrm{Exp}(\mathbb{F}1, +)$. A similar ideal in $\mathrm{Exp}(\mathbb{F}_{1,-})$ is also denoted by $\mathbb{I}$.

Definition 5. The algebra of fractional differential forms $\widetilde{\Lambda} H$ is defined to be $\widetilde{\Lambda} H = \mathrm{Exp}(\mathbb{F}_{1,+})/\mathbb{I}$. Elements of $\widetilde{\Lambda} H$ are said to be fractional order differential forms.

Let $u^\flat$ be the class of $u \in \mathrm{Exp}(\mathbb{F}_{1,+})$ in $\widetilde{\Lambda} H$. Then the wedge product of $u^\flat$ and $v^\flat$ by $(u\sharp v)^\flat$. $d^a x_n$ means the class of $e^{\sharp a(w_n+\gamma)}$. By the definition of $\mathbb{I}$, we may assume $0 \leq \Re a < 2$. By (14), we have

$$d^a x_n \wedge d^b x_m = (-1)^{ab} d^b x_m \wedge d^a x_n, \quad n \neq m. \tag{25}$$

Here $-1 = e^{\pi i}$ if $n < m$ and $e^{-\pi i}$ if $n > m$. Note that *this multiplication rule is different from that of* [8].

Let $\psi = e^{\sharp u}$, where $u = \sum c_n w_n$. Then to set

$$u^\natural = \sum \widetilde{c_n} w_n, \quad \widetilde{c_n} \cong c_n, \quad \bmod 2, \quad 0 \leq \Re\widetilde{c_n} < 2,$$

we define the action $\psi^\flat * f$ of $\psi^\flat$ to $f \in \mathfrak{F}$ by

$$\psi^\flat * f = (e^{\sharp u})^\flat * f = e^{\sharp u^\natural}\sharp f. \tag{26}$$

For example, $\omega(s)^\natural = \omega(s)$ if $|s|$ is small. So

$$(e^{\sharp\omega(s)})\flat * f = e^{\sharp\omega(s)}\sharp f,$$

if $|s|$ is small.

Definition 6. We define regularized infinite wedge product $\vdots \bigwedge_{n=1}^{\infty,\rightarrow} dx_n \vdots$, i.e., $dx_1 \wedge dx_2 \wedge \cdots$, by

$$\vdots \bigwedge_{n=1}^{\infty,\rightarrow} dx_n \vdots * f = \mathrm{e}^{\nu(\nu-1)\pi i/2}(\mathrm{e}^{\sharp\omega(s)})^{\flat} * f|_{s=0}. \tag{27}$$

Note 5. Let $Q(x)$ be $\{\sum t_n e_n \mid 0 \le t_n \le x_n\}$. Then we may define the integral of $1 = 1\vdots \bigwedge_{n=1}^{\infty,\rightarrow} dx_n \vdots$ on $Q(x)$ by

$$\int_{Q(x)} 1\vdots \bigwedge_{n=1}^{\infty,\rightarrow} dx_n \vdots = \int_{Q(x)} 1\vdots d^{\infty}x \vdots. \tag{28}$$

This suggests if we can give a definition of infinite dimensional integral by using regularized infinite wedge product, then it provides a mathematical justification of the formula such as

$$\int \mathrm{e}^{(-2\pi\langle x, Dx\rangle)} \mathcal{D}x = \frac{1}{\sqrt{\det D}}.$$

Note 6. Similarly, by using $\omega_{i_1,\dots,i_m}(s)$, we can define regularized infinite wedge product $\vdots \bigwedge_{n\notin\{i_1,\dots,i_m\}}^{\infty,\rightarrow} dx_n \vdots$. We regard this form to be a regularized $(\infty - m)$-form (cf. [4, 5]).

6. Regularized volume form on a mapping space

Let X be a d-dimensional compact spin manifold, M an n-dimensional smooth manifold. Then the mapping space $\mathrm{Map}(X, M)$ is a Sobolev manifold modelled by $W^k(X) \otimes \mathbb{R}^n$. We fix the Sobolev metric of $W^k(X)$ by fixing a nondegenerate selfadjoint elliptic operator D, whose green operator is denoted by G. For simplicity, we denote the connected component of $\mathrm{Map}(X, M)$ consisting of 0-homotopic maps by the same notation $\mathrm{Map}(X, M)$.

It allows Clifford extension $\mathrm{Map}(X, M)_C$, which is modelled by $W^k(X, E) \otimes \mathbb{R}^n$,$W^k(X, E)$ is the k-th Sobolev space of spinor fields of X. In this case, we take for D the Dirac operator with a mass term on X tensored by the identity of $\mathbb{R}^n$, see [3]. But in the framework of this paper, we take for G the Green operator of D^2.

Let $\xi = \{g_{UV}\}$ be a G-bundle over M. Then defining g_{UV}^X by

$$g_{UV}^X(f)(x) = g_{UV}\big(f(x)\big),$$

we have a $\mathrm{Map}(X, \mathrm{G})$-bundle ξ^X over $\mathrm{Map}(X.M)$.

Especially, we have $\tau(\mathrm{Map}(X, M)) = \tau(M)^X$, where $\tau = \tau(M)$, etc., mean the tangent bundle of M, etc. Similar results hold for $\mathrm{Map}(X, M)_C$. In this case, $\mathrm{Map}(X, \mathrm{G})$ is contained in the restricted general linear group GL_p, where $p > d/2$ ([12]). In the rest, we also assume $\mathrm{Map}(X, M)_C$ *has a complex structure.*

Let $\mathfrak{G}$ be a Lie group with the Lie algebra $\mathfrak{g}$. For a Map$(X, \mathfrak{G})$-bundle $\tilde{\xi} = \{\tilde{g}_{UV}\}$, a collection of Map$(X, \mathfrak{g})$-valued functions $\{A_U\}$ such that

$$(D + A_U)\tilde{g}_{UV} = \tilde{g}_{UV}(D + A_V),$$

is said to be a connection of $\tilde{\xi}$ with respect to D ([4, 5]). In general, $D + A_U(x)$ degenerates at some x. If there exists a connection $\{A_U\}$ such that $D + A_U$ is nondegenerate at any point, then $\tilde{\xi}$ is trivial as a GL_p-bundle ([3]). Therefore, we take a connection $\{A_U\}$ of $\tau(\mathrm{Map}(X, M) = C)$, and set

$$Y = \Big\{ p \in \mathrm{Map}(X, M)_C \Big| \ker\big(D + A_U(p)\big) \neq \{0\} \Big\}. \tag{29}$$

$\tau(\mathrm{Map}(X, M)_C \setminus Y)$ is trivial.

Hence we can take H^-(finite) as the fibre of $\tau(\mathrm{Map}(X.M)_C \setminus Y)$. Here, $H = L^2(X, E) \otimes \mathbb{R}^n$. Then, a section of $(\mathrm{Map}(X, M)_C \setminus Y) \times \tilde{(}\Lambda)H$ is said to be a fractional order differential form on $\mathrm{Map}(X, M)_C \setminus Y$.

Note 7. Considering $\det(D+A_U(p))$ to be the analytic continuation of $\det(D+A_U(p)+m)$ to $m = 0$, see [3], we may regard Y to be the *o*-set of $\det(D + A_U)$.

By using the trivialization of $\tau(\mathrm{Map}(X, M)_C \setminus Y)$, we obtain a family of operators D_Y whose principal part is D, parametrized by $\mathrm{Map}(X, M)_C \setminus Y$. Let $\mu_1(p) \geq \mu_2(p) \geq \cdots > 0$ be the proper values of G_Y, the Green operator of D_Y^2, and let $e_1(p), e_2(p), \ldots,$ be their proper functions. We define $\omega(s)(p)$ to be $\sum \mu_n(p)^s w_n(p)$, where $w_n(p)$ is defined similar to w_n.

By assumption, $\lim_{x \longrightarrow Y} \omega(s)(p)$ diverges. Hence

$$\vdots \bigwedge_{n=1}^{\infty,\longrightarrow} dx_n(p) \vdots = \mathrm{e}^{\nu(p)(\nu(p)-1)/2} \mathrm{e}^{\omega(s)(p)}|_{s=0},$$

has singularities along Y. The counter term to this singularity is $(\det D_Y)^{-2d}$. However, $(\det D_Y)^{-2d} \vdots \bigwedge_{n=1}^{\infty,\longrightarrow} dx_n(p) \vdots$ may be discontinuous on $\mathrm{Map}(X, M)_C$. Because $\mathrm{e}^{\nu(p)(\nu(p)-1)\pi i/2}$ may be many valued on $\mathrm{Map}(X, M)_C$.

Summarizing these, we have (cf. [4, 5])

Theorem 1. *The regularized volume form*

$$d\vdots v \vdots = \vdots \bigwedge_{n=1}^{\infty,\longrightarrow} dx_n(p) \vdots \quad \text{on} \quad \mathrm{Map}(X, M)_C \setminus Y$$

defines a cross-section of the determinant bundle of $\mathrm{Map}(X, M)_C$. *So regularized volume form exists on* $\mathrm{Map}(X, M)_C$ *if its determinant bundle is trivial.*

Note 8. Since we used positive definite G as the *metric*, regularized volume form exists under more mild condition. In fact, starting from positive definite G, determinant bundle is trivial if $H^1(\mathrm{Map}(X, M), \mathbb{Z}) = 0$.

Note 9. By the same reason, $\vdots \bigwedge_{n \notin \{i_1, \ldots, i_m\}}^{\infty,\longrightarrow} dx_n(p) \vdots$ is defined on $\mathrm{Map}(X, M)_C$, if the determinant bundle of $\mathrm{Map}(X, M)$ is trivial.

We can regard this form to be the regularized $(\infty - m)$-form on $\mathrm{Map}(X, M)_C$. In fact, since G is trivial, $(\infty - m)$-forms on $\mathrm{Map}(X, M)$ are defined on $\mathrm{Map}(X, M)$ under more mild condition.

We conclude this paper to remark that although we can define regularized volume form and $(\infty - m)$-forms on a mapping space with vanishing string class, it is *not* known whether one can define fractional differential forms or logarithm of differential forms on a mapping space with nontrivial tangent bundle.

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