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A K-theory for certain multisymplectic vector bundles[1]

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Abstract. A new topological K-theory is introduced. It is defined in terms of vector bundles equipped with a certain multisymplectic structure. Because of its connection with Hamiltonian field theories and some specific frameworks in topology and differential geometry, this K-theory promises to have several quite interesting applications. Here, however, only some rather basic properties will be developed.

Keywords. K-theory, multisymplectic structures, vector bundles.

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1. Introduction

The aim of this note is to introduce a new topological K-theory. It arose out of the study of vector bundles carrying a certain multisymplectic structure. The first important feature of this structure, which is locally the exterior product of a one form and a symplectic form on the kernel of this one form, is that it is in the underlying geometric structure for Hamiltonian field theories, [3]. Hence, the K-theory is a useful tool for studying global properties of Hamiltonian field theories. Yet, it has a substantially wider scope. This depends on the fact that this K-theory is actually a family of K-theories parametrised by the elements of the first cohomology group with coefficients in $\mathbb{Z}_2$. Hence, it suits well to deal with questions arising in the studies of families of bundles which are likewise parametrised. Such families arise, most notably, in the de Rham theory of forms valued in an arbitrary line bundle and as the families of Spin structures associated to a vector bundle. However, we will develop only the basic properties of the theory here.

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2. The special multisymplectic structure

Let V be a real vector space of odd dimension and W a subspaces of codimension one, i.e., there exists $u \in V$ such that $V = \mathbb{R} \cdot u \oplus W$ holds. Since W is even-dimensional, it can be equipped with a symplectic structure. By ω we denote the corresponding non-degenerate, alternating 2-form on W. Furthermore, define β to be a 1-form with $W = \ker \beta$. Then

$$\Omega := \beta \wedge \omega$$

is an alternating 3-form on V.

Definition 2.1. A multisymplectic vector space is a vector space equipped with an alternating k-form Ω, $k \geq 3$, which is non-degenerate in the following sense: the $(k-1)$-form $\iota_v \Omega$ vanishes for none $v \in V$.

Remark 2.2. For $k = 2$ the above definition is just the one of symplectic vector spaces.

Proposition 2.3. *(V, Ω) is a multisymplectic vector space.*

Proof. It has to be shown that $\iota_v \Omega \neq 0$ holds for all $v \in V$. Since $V = \mathbb{R} \cdot u \oplus W$, for every $v \in V$ we have $v = t \cdot u + w$ with $w \in W$ uniquely determined. Thus we get

$$\iota_v \Omega = \iota_{t \cdot u + w} \Omega = \iota_{t \cdot u} \Omega + \iota_w \Omega = t \cdot (\iota_u \beta) \wedge \omega - \beta \wedge (\iota_w \omega).$$

But this k-form is obviously not the zero-form. □

Definition 2.4. By $G(\Omega)$ we denote the group of linear automorphisms of V leaving Ω invariant, i.e., a linear automorphism ϕ is contained in $G(\Omega)$, if and only if

$$\Omega\big(\phi(v_1), \phi(v_2), \phi(v_3)\big) = \Omega\big((v_1), (v_2), (v_3)\big)$$

holds.

Proposition 2.5. *W is invariant under the action of $G(\Omega)$.*

Proof. First, we recall, that for a form θ and an one form α we have $\theta = \alpha \wedge \tilde{\theta}$, if and only if $\alpha \wedge \theta = 0$. By definition of Ω it is thus clear, that for a one form α with $\alpha \wedge \Omega = 0$, we must have $\alpha = t \cdot \beta$. Now let ϕ be an automorphism of Ω. Then we get

$$\begin{aligned} 0 &= \alpha \wedge \Omega\big(\phi(v_1), \ldots, \phi(v_4)\big) \\ &= \sum_{\pi} (-1)^{\pi} \alpha\big(\phi(v_{\pi(1)})\big) \Omega\big(\phi(v_{\pi(2)}), \ldots, \phi(v_{\pi(k+2)})\big) \\ &= \sum_{\pi} (-1)^{\pi} \alpha\big(\phi(v_{\pi(1)})\big) \Omega\big((v_{\pi(2)}), \ldots, (v_{\pi(k+2)})\big). \end{aligned}$$

Thus we have $(\alpha \circ \phi) \wedge \Omega = 0$. But, as stated above, this means that for any $\alpha = t \cdot \beta$ the 1-form $\alpha \circ \phi$ must again be of the type $r \cdot \beta$. For any $w \in W$ we get thus

$\alpha(\phi(w)) = r \cdot \beta(w) = 0$, since $W = \ker \beta$ by definition. If we had now $\phi(w) \notin W$ for some $w \in W$, we would get $\alpha(\phi(w)) \neq 0$, what gives a contradiction. Hence W has to be invariant under the action of $G(\Omega)$. $\square$

We are interested in the group $G(\Omega)$ since the condition for a vector bundle to be equipped with a multisymplectic structure of type Ω, i.e., each fibre is a multisymplectic vector space and the forms vary continuously with the fibres, is that the transition functions can be chosen to take values in $G(\Omega)$. However, it is well known that a vector bundle always can be equipped with a fibre metric, hence for questions concerning the topology of vector bundles it is sufficient to know the group

$$\widetilde{G}(\Omega) := G(\Omega) \cap O_{2m+1}.$$

Thus, we have to find a description of the group $\widetilde{G}(\Omega)$. For this we assume V to be equipped with a euclidean metric such that the decomposition $V = \mathbb{R} \cdot u \oplus W$ is orthogonal. Since every automorphism leaves W invariant, every $\tilde{\phi} \in \widetilde{G}(\Omega)$ leaves, since it preserves orthogonality, also $\mathbb{R} \cdot u$ invariant and hence splits into $\tilde{\phi}_u \oplus \tilde{\phi}_W$, with $\tilde{\phi}_u(u) = \pm u$ and $\tilde{\phi}_W \in O_{2m}(W)$. Thus, we have

$$\begin{aligned}\Omega(u, w_1, w_2) &= \Omega\big(\tilde{\phi}(u), \tilde{\phi}(w_1), \tilde{\phi}(w_2)\big)\\ &= \Omega\big(\tilde{\phi}_u(u), \tilde{\phi}_W(w_1), \tilde{\phi}_W(w_2)\big)\\ &= \pm\Omega\big(u, \tilde{\phi}_W(w_1), \tilde{\phi}_W(w_2)\big).\end{aligned}$$

Hence, we get

$$\Omega(u, w_1, w_2) = \pm\Omega\big(u, \tilde{\phi}_W(w_1), \tilde{\phi}_W(w_2)\big).$$

But since $\iota_u \Omega = \omega$ holds by the definition of Ω, it follows that

$$\omega\big(\tilde{\phi}_W(w_1), \tilde{\phi}_W(w_2)\big) = \mp\omega(w_1, w_2)$$

holds. Hence, every $\tilde{\phi}_W$ is up to a sign a symplectic automorphism of the symplectic vector space (W, ω). Since $\mathrm{Sp}_m(\mathbb{R}) \cap O_{2m} = U_m$, we get a short exact sequence

$$U_m \to \widetilde{G}(\Omega) \to \mathbb{Z}_2,$$

which splits. We can describe the action on $V = \mathbb{R} \cdot u \oplus W$ in the following way; U_m acts on W leaving ω invariant and acts trivial on $\mathbb{R} \cdot u$; $1 \in \mathbb{Z}_2$ is the identity on V and $-1 \in \mathbb{Z}_2$ acts as multiplication with -1 on $\mathbb{R} \cdot u$ and as complex conjugation compatible with the complex structure defined by the action of U_m on W.

With respect to this we will denote $\widetilde{G}(\Omega)$ by U_m^Ω and the restriction to W we will call U_m^T, the "twisted Unitary Group". Of course, U_m^Ω can be considered to be a special representation of U_m^T.

Remark 2.6. It is easy to see that using any embedding of U_p into U_{p+q} one can embed U_p^T into U_{p+q}^T by choosing the conjugation in U_{p+q}^T to be the image of the conjugation in U_p^T. However, $U_p^T \times U_q^T$ is not contained in U_{p+q}^T, because in the direct product the conjugation in one factor acts trivial on the other one. That this is not possible is seen most easily considering the symplectic structure on W: from the above reasoning it is clear that an element of U^T_{p+q} either leaves the symplectic form invariant or changes its sign, the direct product $U^T_p \times U^T_q$ however contains elements, especially the images of the conjugations, which change the sign only a subspace.

3. The topological K-theory of Ω-vector bundles

From the above discussion we can draw several conclusions on the particular structure of vector bundles carrying a form Ω. As already remarked, it is clear that their structure group can be reduced to U^T_m by choosing a Riemannian metric on the bundle. Also, it is clear by the invariance of W under the action of automorphisms that for a $(2m+1)$-dimensional Ω-bundle $\mathcal{V}$ we have a $2m$-dimensional subbundle $\mathcal{W}$ and a short exact sequence of vector bundles $\mathcal{W} \to \mathcal{V} \to \theta$, where θ is a line bundle. A quite remarkable consequence of the previous section is that neither $\mathcal{W}$ needs to carry a symplectic structure, nor needs to exist a globally non-vanishing one form on θ. This is, of course, due to the fact that U^Ω_m preserves each of them only up to a sign.

Obviously, the direct sum of two Ω-bundles cannot be again an Ω-bundle, simply because it has the wrong dimension, even instead of odd, and the wrong internal structure, two line subbundles instead of one. Even more, in view of the remark at the end of the previous section it is clear that the direct sum of two arbitrary U^T_*-bundles, i.e., vector bundles with structure group the twisted unitary group of some dimension, is in general not a U^T_*-bundle. On the other hand, in the case of a multisymplectic vector space (V, Ω) is it very well possible to extend W using the direct sum symplectic vector spaces. The following proposition shows now how this quite paradoxical situations is resolved.

Proposition 3.1. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two Ω-bundles over a space X, such that $\mathcal{V}_i = \theta \oplus \mathcal{W}_i$ for a fixed line bundle θ. Then*

$$\mathcal{V}_1 \oplus^\theta \mathcal{V}_2 := \theta \oplus \mathcal{W}_1 \oplus \mathcal{W}_2$$

is again an Ω-bundle

Proof. Let now denote $\oplus$ the direct sum of matrices. Every matrix in U^Ω_m is either of the type $1 \oplus A$ or $-1 \oplus AK$, where $A \in U_m$ and K is complex conjugation. Thus, for $A \in U_p$ and $B \in U_q$ we get

$$1 \oplus A \oplus B \in U^\Omega_{p+q}$$

and

$$-1 \oplus AK_p \oplus BK_q = -1 \oplus (A \oplus B)K_{p+q} \in U^\Omega_{p+q}.$$

Now we will apply this to the transition functions of the bundles $\mathcal{V}_i$. We can choose a covering U_ν and a set of local trivialisations over this covering such that for the transition functions $\phi^{\mathcal{V}_i}_{\nu\mu}$, $\phi^{\mathcal{W}_i}_{\nu\mu}$ and $\phi^{\theta}_{\nu\mu}$ of $\mathcal{V}_i$, $\mathcal{W}_i$ and θ the following holds:

i) $\phi^{\mathcal{V}_i}_{\nu\mu} \in U^{\Omega}_{p_i}$, $\phi^{\mathcal{W}_i}_{\nu\mu} \in U^{T}_{p_i}$ and $\phi^{\theta}_{\nu\mu} \in \{-1, 1\}$

ii) $\phi^{\mathcal{V}_i}_{\nu\mu} = \phi^{\theta}_{\nu\mu} \oplus \phi^{\mathcal{W}_i}_{\nu\mu}$

Thus, for $\phi^{\theta}_{\nu\mu} = 1$ we have $\phi^{\mathcal{W}_i}_{\nu\mu} = A^{\mathcal{W}_i}_{\nu\mu}$ with $A^{\mathcal{W}_i}_{\nu\mu} \in U_{p_i}$; and for $\phi^{\theta}_{\nu\mu} = -1$ we have $\phi^{\mathcal{W}_i}_{\nu\mu} = A^{\mathcal{W}_i}_{\nu\mu}K$ with $A^{\mathcal{W}_i}_{\nu\mu} \in U_{p_i}$. Since we can choose the transition functions of $\mathcal{V}_1 \oplus^{\theta} \mathcal{V}_2$ to be

$$\phi^{\mathcal{V}_1 \oplus^{\theta} \mathcal{V}_2}_{\nu\mu} = \phi^{\theta}_{\nu\mu} \oplus \phi^{\mathcal{W}_1}_{\nu\mu} \oplus \phi^{\mathcal{W}_2}_{\nu\mu},$$

it is clear by the above considerations about matrices that we can assume them to take values in $U^{\Omega}_{p_1+p_2}$. □

Clearly, $\oplus^{\theta}$ is commutative, associative and has a neutral element, namely θ, hence it is a semigroup, which we call $\Omega_{\theta}(X)$. Thus, there exists the corresponding Grothendieck group $K\Omega_{\theta}(X)$.

However, only with the above definition of the sum operation at hand it is hardly possible to derive any further information. To come to a better understanding of $K\Omega_{\theta}(X)$, we have to choose an alternative approach.

It is a well known fact (see, for example, [4, 5]) that every principal bundle with structure group G over a finite CW-complex can be constructed as the pull-back of a universal principal G-bundle. A principal G-bundle is universal if its total space is contractible, i.e., it is a principal bundle with fibre G, total space EG and base BG such that EG is a contractible free G-Space and $BG \simeq EG/G$.

Remark 3.2. BG is called a classifying space for principal G-bundles. Isomorphism classes of principal G-bundles over a finite CW-complex X are in one to one correspondence with homotopy classes of maps $X \to BG$. It is important to notice that BG and all constructions related to it are unique only up to homotopy equivalence.

Since every free G-space is also a free H-space for any closed subgroup $H \in G$, we see that the quotient EG/H a classifying space for principal H-bundles. Hence, BH is a bundle over BG with fibre G/H. Applying this to the exact sequence

$$U_m \to U^T_m \to \mathbb{Z}_2,$$

we see that BU_m is a double cover of BU^T_m. Now, for any principal U^T_m-bundle over X we have a classifying map $f : X \to BU^T_m$ and the following commutative diagram:

$$\begin{array}{ccc} X_{\theta} & \xrightarrow{f^{\theta}} & BU_m \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & BU^T_m. \end{array}$$

i.e., (X_{θ}, p, X) is the pull-back of the $\mathbb{Z}_2$-bundle (BU_m, π, BU^T_m) via the classifying map f. The map in the upper row is obtained from the identification of the fibre over x with the fibre over $f(x)$. Thus, every classifying map $f : X \to BU_m^T$ can be lifted to a $\mathbb{Z}_2$-equivariant map $f^\theta : X_\theta \to BU_m$ and every such $\mathbb{Z}_2$-equivariant map projects onto a classifying map of a principal U_m^T-bundle over X. However, it should be noticed that the lifting is not unique. This depends on the fact that the $\mathbb{Z}_2$-action defines a free involution on BU_m. Calling this involution τ, one sees immediately that if f^θ is a lifting of f, also $\tau \circ f^\theta$ is a lifting.

The connection with the K-theory constructed above is that X_θ can be identified with the "sphere" bundle of the line bundle θ or, equivalently, with the principal $\mathbb{Z}_2$-bundle associated to θ.

To see this, one has to take into account that every principal U_m^T-bundle over X has, via the one dimensional representation of U_m^T, a canonical associated line bundle. For the principal U_m^T-bundle associated to an Ω-bundle this is, of course, the one dimensional subbundle θ (this depends on the fact that U_m^Ω can be considered the direct sum of the standard representation and the one dimensional one). The principal $\mathbb{Z}_2$-bundle associated to this line bundle is thus the image of the principal U_m^T-bundle associated to $\mathcal{W}$ by the bundle map induced by the one dimensional representation. For the universal U_m^T-bundle over BU_m^T this $\mathbb{Z}_2$-bundle is obviously just (BU_m, π, BU_m^T). Hence, (X_θ, p, X) is the $\mathbb{Z}_2$-bundle corresponding to the principal U_m^T-bundle obtained from the classifying map f.

Using the above diagram one can give now, on the level of principal bundles, a new interpretation to the addition of Ω-bundles introduced in Proposition 3.1.

Given two maps $f_1 : X \to BU_m^T$ and $f_2 : X \to BU_n^T$, which define the same double covering X_θ of X, we can define the direct product

$$f_1^\theta \times f_2^\theta : X_\theta \to BU_m \times BU_n \to BU_{m+n}$$

for any two of their liftings. Now, if $f_1^\theta \times f_2^\theta$ is equivariant with respect to the usual $\mathbb{Z}_2$-action on BU_{m+n}, then it projects onto a map

$$f_1 \times^\theta f_2 : X \to BU_{m+n}^T.$$

$f_1 \times^\theta f_2$ is obviously independent of the chosen liftings. To show that the map $f_1^\theta \times f_2^\theta$ is equivariant the map it is necessary to show that $BU_m \times BU_n \to BU_{m+n}$ can be assumed to be equivariant.

For this reason we examine the free involution τ mentioned above more closely. This involution on BU_m can be described in the following way. Using the usual embedding $U_m \to O_{2m}$, we see that BU_m is a bundle over BO_{2m} with fibre O_{2m}/U_m. This fibre can be considered the space of complex structures on a given real vector space, i.e., the space of U_m-conjugacy classes of operators J with $J^2 = -1$. It has a obvious fixed point free involution, namely the map which assigns to each complex structure J its "complex conjugate" $-J$. Obviously, this involution extents to BU_m. That the map $BU_m \times BU_n \to BU_{m+n}$ can be considered to be equivariant with respect to $\mathbb{Z}_2$-action induced by the involution is clear. To see that this involution coincides in fact with τ one only has to recall from the previous section that U_m^T is generated by the unitary group and complex conjugation.

Hence, we have established an notion of product of principal U_m^T-bundle corresponding to the sum of Ω-bundles defined above. Infact we have achieved something more. Recall that M.F. Atiyah introduces in [2] the following notion of Real bundle over a Real space (other than Atiyah, we use the capital letter to rule out the obvious misunderstandings).

A Real space is any space with an involution τ and a Real bundle over a Real space Y is a complex vector bundle over Y for which

i) the total space E is also a Real space and the bundle projection π commutes with the involutions; i.e., $\pi \circ \tau_E = \tau_Y \circ \pi$,

ii) the map $E_y \mapsto E_{\tau_Y(y)}$ is complex antilinear.

Now, X_θ is obviously a Real space. Furthermore, since there is a canonical complex vector bundle associated to any principal U_m-bundle using the standard representation of the unitary group, it is clear that every lifting of $f : X \to BU_m^T$ defines a Real bundle over X_θ. On the other hand it is obvious that a complex vector bundle over X_θ can be equipped with a Real structure if and only if it admits a equivariant classifying map.

Hence, to state the relation with Atiyah's KR-theory (defined in [2]) it remains to understand "how many different liftings there are". Above we already mentioned that if f^θ is a lifting of f, $\tau \circ f^\theta$ is one, too. Hence, there are at least two. If $f \simeq f_1 \times^\theta f_2$, then this observation holds, of course, for each factor, i.e., we have already four. With this in mind our main result is immediately evident.

Theorem 3.3. *Let $[E]$ denote the KR-class of the real bundle E and denote by $\bar{E}$ its conjugate bundle. Then we have*

$$K\Omega_\theta(X) \simeq KR(X_\theta)/\mathcal{I}$$

as abelian groups, where $\mathcal{I}$ is the subgroup generated by $[E] - [\bar{E}]$.

Remark 3.4. It is quite interesting to examine case of θ being the trivial bundle, which we denote by ϵ. Then $\mathcal{W}_\epsilon$ itself can be equipped with a complex structure. Nevertheless, the above considerations still hold, since Ω-form on $\mathcal{V}_\epsilon$ cannot distinguish a complex structure from its conjugate on $\mathcal{W}_\epsilon$. In this case X_ϵ is just the trivial double covering of X Atiyah showed (in [2]) that $KR(X_\epsilon)$ is just ordinary complex K-theory $K(X)$ and thus we get $K\Omega_\epsilon(X) = K(X)/\mathcal{I}$, $\mathcal{I}$ defined as above.

Corollary 3.5. *For every Ω_θ-bundle $\mathcal{V}$ of finite type, there exists a Ω_θ-bundle $\mathcal{V}^\circ$, such that*

$$\mathcal{V} \oplus_\theta \mathcal{V}^\circ = \theta \oplus (\epsilon \oplus \theta)^n.$$

Proof. We pull the subbundle $\mathcal{W}$ back to X_θ. The pull-back $\mathcal{W}^C$ has a Real trivialisation for all possible complex structures,the quotient by the involution of the trivialisation gives us $\mathcal{W}^\circ$ and the quotient by the involution of a trivial Real bundle over X_θ is clearly of type $(\epsilon \oplus \theta)^n$. $\square$

Remark 3.6. Hence, it is possible to prove results analogous to the one stated at the beginning of the second chapter in ([1]). With the only difference that in $K\Omega_\theta$-theory instead of trivial bundles, one has bundles of type $(\epsilon \oplus \theta)^n$. Thus one gets "every class in $K\Omega_\theta(X)$ can be written as $\mathcal{V} - (\epsilon \oplus \theta)^n$" and "two bundles $\mathcal{V}_1, \mathcal{V}_2$ belong to the same class in $K\Omega_\theta(X)$ iff there exists a bundle $(\epsilon \oplus \theta)^n$, such that $\mathcal{V}_1 \oplus (\epsilon \oplus \theta)^n = \mathcal{V}_2 \oplus (\epsilon \oplus \theta)^n$".

Since the line bundles over X are in one to one correspondence with the elements of first cohomology group with $\mathbb{Z}_2$-coefficients, we can state the following definition of the topological K-theory of Ω-vector bundles.

Definition 3.7.

$$K\Omega(X) := \bigoplus_{\theta \in H^1(X,\mathbb{Z}_2)} K\Omega_\theta(X).$$

It is clear, that $K\Omega$ is a functor. Since every continuous map $f : X \to Y$ induces a cohomological homomorphism $f^* : H^1(Y, \mathbb{Z}_2) \to H^1(X, \mathbb{Z}_2)$ we get a commutative diagram

$$\begin{array}{ccc} X_{f^*(\theta)} & \longrightarrow & Y_\theta \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{\;f\;} & Y. \end{array}$$

So, the functoriality of $K\Omega$ follows from the functoriality of KR.

Remark 3.8. The motivation of the above definition is that there exists a finite dimensional projective space $\mathbb{R}P_k$, such that the line bundles over X are in one to one correspondence with the homotopy classes of maps $X \to \mathbb{R}P_k$. Because of this there exists a strong connection between $K\Omega(X)$ and $K\Omega_\theta(X \times \mathbb{R}P_k)$; here θ denotes the pull-back of the canonical line bundle over $\mathbb{R}P_k$ via the projection $p : X \times \mathbb{R}P_k \to \mathbb{R}P_k$. However, we will not develop this any further in the present paper.

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