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Examples of symplectic $Q^3 \times S^1$

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Abstract. We construct compatible symplectic structures on manifolds $Q^3 \times S^1$ for Waldhausen Q^3 and show it is possible iff Q^3 is a Stallings fibrations over S^1. We describe bordisms of Lagrangian T^2-bundles over $S^1 \times I$.

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Introduction

By Martinet's theorem ([10]) every closed orientable 3-dimensional manifold Q^3 is contact. Therefore by contactization procedure (see [1]) we get that the manifold $Q^3 \times I$ is symplectic. Moreover, the structure can be chosen so that the manifold has an immersion into $(\mathbb{R}^4, \omega_0)$. We look for a compactification of the form $M^4 = Q^3 \times S^1$. Not every manifold Q^3 has such a compact contactization. Among lens spaces only $S^2 \times S^1$ has this property. In what follows we consider only closed oriented manifolds Q^3.

To exhibit more examples consider *Stallings manifolds*. Such a manifold is by definition a 3-manifold fibered over the circle

$$Q^3 = P^2 \widetilde{\times}_\sigma S^1,$$

and (by symplectic reasons) we consider only orientation-preserving monodromy $\sigma : P^2 \to P^2$. To construct a symplectic structure Ω on $Q^3 \times S^1$ we note there exists a σ-invariant volume form τ on P^2. Let ψ be a coordinate on the base of the Stallings bundle $Q^3 \to S^1$. Denote by $\hat{\tau}$ any closed 2-form restricting to τ on the fibers P^2 and by $d\psi$ the lift of 1-form from S^1 to Q^3 (and later to $Q^3 \times S^1$). Then we define the symplectic structure by $\Omega = \hat{\tau} + d\psi \wedge d\varphi$. It turns out that no other examples are known in the meantime. Moreover Taubes ([15]) stated the following

This paper is in final form and no version of it will be submitted for publication elsewhere.

Conjecture T. *$Q^3 \times S^1$ is symplectic iff Q^3 is a Stallings.*

We won't discuss general case. We consider only special compatible symplectic structures on $Q^3 \times S^1$ and prove the hypothesis for them.

Let $\mathcal{I}$ be a (φ-independent) foliation of $Q_\varphi \simeq Q^3 \times \{\varphi\} \subset M^4$ with almost every leaf compact. We say a symplectic form ω on M^4 is *compatible* with the foliation $\mathcal{I}$ if Ker $\omega|_{Q_\varphi}$ belongs to $\mathcal{I}$. Consider at first the case of $\mathrm{rk}(\mathcal{I}) = 1$ (actually next statement holds also without compatibility assumption, see [4]).

Theorem 1 ([9]). *If $\mathcal{I}$ is a Seifert foliation of Q^3, then $M^4 = Q^3 \times S^1$ has an $\mathcal{I}$-compatible symplectic structure iff the base $Q/\mathcal{I}$ is an orbifold without reflecting circles and the Euler number $e(\mathcal{I}) = 0$.*

Now consider foliations of $\mathrm{rk}(\mathcal{I}) = 2$ with almost all leaves compact. We suppose that the rank of the leaves decreases by 1 at singularities and they are non--degenerate in Bott sense. This means that on (2-dimensional) transversal the singularities of the induced foliation are Morse-non-degenerate.

In this case $Q/\mathcal{I} = \Gamma$ is a graph. Moreover we suppose that a generic leaf L^2 is orientable and then compatibility yeilds $\chi(L^2) = 0$ and so implies $L^2 \simeq T^2$. Then the foliation (called *Liouville*) is well studied. The manifold Q is a (Waldhausen) graph-manifold, see [16]. Only such manifolds can occur as surfaces of constant energy of non-degenerate integrable Hamiltonian systems ([5]).

The vertices of the graph correspond to either (critical) circles $S^1 \simeq \gamma_i \subset Q^3$ or Klein bottles $K^2 \subset Q^3$ (note that there exists a double covering such that all Klein bottles disappear). We call γ_i critical cycles. Suppose the number of such cycles is positive (the case of no cycles is considered below).

Another way is to represent Q as a number of Seifert 3-manifolds glued along their toric boundaries. S^1-generator of each Seifert piece produces homology element $[\gamma_i] \in H_1(Q; \mathbb{R})$, a representative of which we still call critical cycle.

Theorem 2. *Let Q^3 be a Waldhausen manifold with non-homologous to zero critical cycles $[\gamma_i] \neq 0$. Then $M^4 = Q^3 \times S^1$ is symplectic.*

Moreover if $\mathcal{I}$ is a Liouville foliation on Q then symplectic structure can be chosen $\mathcal{I}$-compatible. For instance there is a Waldhausen but not Seifert Q^3 with symplectic $Q^3 \times S^1$.

However all symplectic manifolds constructed via this approach are Stallings:

Theorem 3. *A Waldhausen manifold Q^3 is a Stallings if $[\gamma_i] \neq 0$ for all critical cycles. A Stallings manifold Q^3 is a Waldhausen if it has a periodic monodromy map σ. The only exception is a Stallings manifold $Q^3 = T^2 \widetilde{\times}_A S^1$ with*

$$A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix},$$

$n \neq 0$ that is Seifert with fibers included into the Stallings-fibers.

The method of proof of Theorem 2 is as follows. At first we define a symplectic structure in $U_i \times S^1$ for neighborhoods U_i of critical elements. Each component of the remaining part is a product $T^2 \times (S^1 \times I)$ with Lagrangian T^2 for the symplectic structure defined near the boundary. Then we fill the rest of the component by a symplectic structure, the first factor T^2 still being Lagrangian.

This can be called bordism between the components of the boundary. In fact we use a criterion, from [7], that a Lagrangian tori bundle over an annulus is direct bordant to zero. Here we define *direct bordism* for two Lagrangian tori bundles $\pi_i : X_i^4 \to S^1 \times I_i$ to be a Lagrangian tori bundle $\bar{\pi} : M^4 \to S^1 \times I$ with $I = (a, b) \cup [b, c] \cup (c, d) = I_1 \cup \bar{I}_0 \cup I_2$ that restricts to π_i near the boundary of the base. It is not symmetric but can be made into equivalence relation as follows. We say π_1 is (up-down) bordant to π_2 if there is a sequence of Lagrangian tori bundles φ_k, $1 \leq k \leq N$, with $\varphi_1 = \pi_1$, $\varphi_N = \pi_N$ and for every $0 < i < N$ either φ_i is direct bordant to φ_{i+1} or φ_{i+1} is direct bordant to φ_i.

The first invariant of Lagrangian tori T^n bundle is a monodromy. Every such bundle determines a canonical integral affine structure on a manifold and hence a cycle $\beta \in H_1(B)$ on the base B^n of the bundle determines an element $\phi(\beta) \in G_n = \mathrm{GL}_n(\mathbb{Z}) \rightthreetimes \mathbb{R}^n$ upon identification $\mathbb{Z}^n \simeq H_1(T_x^n)$ for $T_x^n = \pi^{-1}(x)$.

When the base is 2-dimensional and closed, it is T^2 or K^2 and the invariants of Lagrangian tori bundles are monodromies together with an invariant of "twisting" ([11]). If there is a boundary, the base is either annulus or Möbius band with punctures (in the first case it has a submersion into $\mathbb{R}^2$); we can suppose no punctures in B^2 in our considerations and moreover restrict to only orientable case: $B = S^1 \times I$. Let us fix a generator $\beta \in H_1(B)$. Then there is exactly one monodromy $\phi = \phi(\beta)$.

In the case of topologically trivial bundle (we consider others in Section 2) the monodromy is $\phi(\vec{w}) = \vec{w} + \vec{v}_\phi$, $\vec{v}_\phi \in \mathbb{R}^2$. The generator β on the base determines a loop in Lagrangian tori-fibers. Its evaluation on the symplectic form gives a closed curve in $L_\phi = \mathbb{R}^2/\vec{v}_\phi\mathbb{Z}$ with Whitney number $\mathfrak{w} \in \mathbb{Z}$.

Theorem 4. *Two Lagrangian tori bundles π', π'' over annuli are (up-down) bordant iff $\vec{v}_{\phi'} = \vec{v}_{\phi''}$ for some choice of cycles $\gamma_i' \sim \gamma_i''$ and $\mathfrak{w}' = \mathfrak{w}''$.*

1. Seifert fibrations

If $\mathcal{I}$ is a rank 1 foliation of Q^3 with compact oriented fibers, then by Epstein theorem ([3]) $\mathcal{I}$ is a Seifert fibration π. We restrict to the original definition ([14]) with oriented fibers, since this is so for compatible symplectic $Q^3 \times S^1$.

We search for a nowhere zero closed 2-form on Q^3 with kernel along the fibers. Such a form induces an orientation of $P^2 = Q^3/\mathcal{I}$ and so the base P^2 of the Seifert fibration π should contain no reflecting circles ([13]).

If the Euler number $e(\mathcal{I}) = 0$, then there is a section $i : \widehat{P}^2 \to Q^3$ of π, where $p : \widehat{P}^2 \to P^2$ is the canonical covering of the orbifold P^2 ([13]). Let $\hat{\omega}$ be a volume form on $\widehat{P}^2$ invariant under the monodromy group of the covering p, i.e., $\hat{\omega} = p^*\omega'$, where ω' is a "smooth" form on the orbifold P^2 (described more precisely at singular points, [6]). Let $\omega = \pi^*\omega'$ be the lift to Q^3.

Define the map $\theta : Q^3 \to S^1 = \mathbb{R}(\mathrm{mod}\,1)$ as follows. Choose a vector field v generating S^1-action and let $\theta(i(\widehat{P}^2)) = 0$. Take $x_0 \in i(\widehat{P}^2)$ and $x \notin i(\widehat{P}^2)$ on a trajectory of v that connects x_0 to the next intersection point $\sigma(x_0)$ of the trajectory with the section. Denote by $\tau(x_0, x)$ the time of reaching x from x_0 along the field v. Then we set $\theta(x) = \tau(x_0, x)/\tau(x_0, \sigma(x_0))$. Now we denote by φ the coordinate on S^1-factor of $M^4 = Q^3 \times S^1$ and define the symplectic form by $\Omega = \omega + d\theta \wedge d\varphi$.

Consider now the case $e(\mathcal{I}) \neq 0$ and suppose there is a symplectic structure Ω on M^4. The compatibility implies $\Omega(H_2(Q)) = 0$. Therefore in the Künneth decomposition of de Rham cohomologies $H^* = H^*_{dR}$:

$$H^2(M) = \big(H^1(S^1) \otimes H^1(Q)\big) \oplus \big(H^0(S^1) \otimes H^2(Q)\big)$$
$$\ni [\Omega] = [\Omega]_1 + [\Omega]_2$$

we have: $[\Omega]_2 = 0$. Thus $[\Omega^2] = [\Omega]_1^2 \in H^2(S^1) \otimes H^2(Q) = 0$ and this contradiction proves Theorem 1.

2. Lagrangian tori bundles

Let $\pi : (X^4, \omega) \to S^1 \times I$ be a bundle over an annulus with compact orientable Lagrangian fibers (therefore tori $T^2_x = \pi^{-1}(x)$), which we consider at first topologically trivial. In this case the monodromy is a shift: $\phi(\vec{w}) = \vec{w} + \vec{v}_\phi$, $\vec{v}_\phi = (\phi_1, \phi_2)$. To define components ϕ_i let γ_1, γ_2 be some basis of cycles in $H_1(T^2_x) \simeq \mathbb{Z}^2$ and let β be a generator of $H_1(S^1 \times I)$. Then

$$\phi_i = \int\limits_{\gamma_i \times \beta} \omega, \quad i = 1, 2.$$

Consider the lattice $\mathbb{Z}^2 = H_1(T^2)$ in $\mathbb{R}^2 = H_1 \otimes \mathbb{R} \ni \vec{v}_\phi$. Denote $L_\phi = \mathbb{R}^2/\vec{v}_\phi\mathbb{Z}$. This is either the plane $\mathbb{R}^2$ or the cylinder $S^1 \times I$ subject to $\vec{v}_\phi = 0$ or not.

Fix a point x_0 on the base. Then the map

$$\Phi : S^1 \times I \to L_\phi, \quad \Phi(x) = \left(\int_{x_0}^{x} \oint_{\gamma_1} \omega, \int_{x_0}^{x} \oint_{\gamma_2} \omega\right),$$

is well-defined immersion of the annulus. We define $\mathfrak{w}$ as the Whitney number of this immersion, i.e., as the degree of the corresponding Gauss map.

Theorem 5 ([7]). *Lagrangian bundle $\pi : (X^4, \omega) \to S^1 \times I$ is direct bordant to zero, i.e., it can be extended to a Lagrangian bundle $\bar{\pi} : (M^4, \omega) \to D^2$ over a disk ($X^4 \subset M^4$) iff:*

1. $\vec{v}_\phi = 0$,
2. $\Phi : S^1 \times I \to \mathbb{R}^2$ *is the restriction to a neighborhood of boundary of an immersion of disk $\bar{\Phi} : D^2 \to \mathbb{R}^2$ (this is described by a criterion from* [12]).

Moreover this can be generalized to a statement that two Lagrangian tori bundles π' and π'' with $\vec{v}_\phi = 0$ are direct bordant if the corresponding maps Φ' and Φ'' are.

This means they are restrictions to neighborhoods of boundary components for some immersion of an annulus.

To get a modification denote $N^4 = D^2(r, \theta) \times S^1(\psi) \times S^1(\varphi)$ with polar coordinates on the disk and equip N with symplectic structure $r\,dr \wedge d\theta + d\psi \wedge d\varphi$. It has Lagrangian foliation $\{(r, \varphi) = \text{const}\}$, which correspond to the momentum map $I_1 = (r^2 + a^2)\cos\varphi$, $I_2 = (r^2 + a^2)\sin\varphi$. The image is called "hole".

Theorem 6 ([8]). *Lagrangian tori bundle $\pi : (X^4, \omega) \to S^1 \times I$ is direct bordant to the hole iff:*

1. *$\vec{v}_\phi$ is a non-zero vector with rational direction, i.e., for some choice of cycles γ_1, γ_2 we have $\vec{v}_\phi = (\phi_1, 0)$.*
2. *The corresponding map $\Phi : S^1 \times I \to L_\phi \simeq S^1 \times \mathbb{R}$ is direct bordant to the standard generator.*

Proof of Theorem 4. The Whitney number of an immersion $S^1 \to L_\phi$ is invariant under deformations in the class of immersions. It is easy to show that every curve with $\mathfrak{w} = 0$ is up-down bordant to zero (see figure 1). For arbitrary Whitney number there is (up-down) resolution of all self-intersections except $|\mathfrak{w}|$ standard ones. □

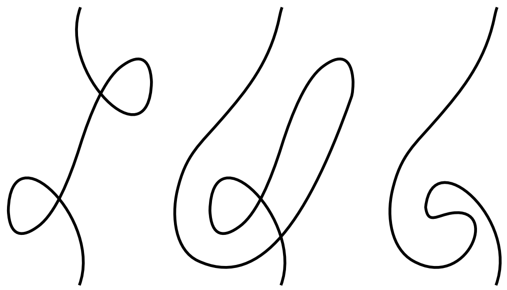

Fig. 1

General case

Arbitrary integer-affine transformation $\phi \in G_2$ has the form $\phi(\vec{w}) = A\vec{w} + \vec{v}$ with $A \in \mathrm{GL}_2(\mathbb{Z})$, $\vec{v} \in \mathbb{R}^2$. Since the base $S^1 \times I$ is orientable, the monodromy preserves orientation $A \in \mathrm{SL}_2(\mathbb{Z})$. Note that for eigenvalues different from 1 there is a choice of origin such that $\vec{v} = 0$.

The monodromy can be represented by means of integration along cylinders as follows. For a point $x \in S^1 \times I$ let loop β from x to x represent a generator of $H_1(S^1 \times I)$. Choose a basis of cycles γ_1, γ_2 in T_x^2. Then translating them along β we get two cylinders $\mathcal{C}_i^2(x)$. The monodromy map

$$M_{x_0} : \mathcal{O}(x_0) \to \mathcal{O}(x_0), \qquad M_{x_0}(x) = \left(\int_{\mathcal{C}_1^2(x)} \omega, \int_{\mathcal{C}_2^2(x)} \omega \right)$$

does not depend on a choice of cylinders and is integer-affine. So for eigenvalues $\lambda \neq 1$ we can find a point x_0, where the monodromy is linear $M_{x_0}(\vec{w}) = A\vec{w}$.

If A is semisimple let us direct the curve β along one of eigenvectors at x_0. Denote by $\mathcal{C}_i^2(x_0; x)$ a part of the cylinder $\mathcal{C}_i^2$ over a piece of β from x_0 to x. Then after one turn along β we get closed curve with respect to the map

$$\Phi(x) = \left(\int_{\mathcal{C}_1^2(x_0;x)} \omega, \int_{\mathcal{C}_2^2(x_0;x)} \omega \right).$$

Moreover it approaches x_0 with the same tangent if eigenvalues are positive and with the opposite one if they are negative. Thus the Whitney number $\mathfrak{w}$ is integer or half-integer respectively. The monodromy $A \in \mathrm{SL}_2(\mathbb{Z})$ defined by topological data and the number $\mathfrak{w} \in \frac{1}{2}\mathbb{Z}$ define the bordism class completely.

If eigenvalues of A are complex, the matrix is conjugated to rotation by $\pm 2\pi/k$, $k = 3, 4, 6$, and we define the Whitney number $\mathfrak{w} \in \mathbb{Z} \pm 1/k$. In the case of Jordan box, the monodromy is

$$\phi \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \vec{v}_\phi, \qquad \vec{v}_\phi = \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}.$$

In this case ϕ_1 is the invariant of Lagrangian bundle with integral formula as at the beginning of the section. So we get a loop in the torus $T^2 = \{w_1(\mathrm{mod}\,\phi_1), w_2(\mathrm{mod}\,n\phi_1)\}$ and we calculate its Whitney invariant $\mathfrak{w}$. This number together with ϕ_1 are complete invariants.

3. Liouville foliations

Let now $\mathcal{I}$ be a Liouville foliation. In this case $\Gamma = Q^3/\mathcal{I}$ is a graph with vertices called atoms. Each atom is the germ of neighborhoods of a connected graph in a surface $K \subset P^2$ with vertices of multiplicities $0, 2, 4$. Atom A has multiplicity 0 and it corresponds to the germ of central circle in trivially foliated solid torus. Atoms without stars are Morse foliated neighborhoods of graphs with multiplicity 4 vertices multiplied by S^1. For example singular leaf of foliation corresponding to atom B is pictured in figure 2. Multiplicity 2 vertices (stars) correspond to $(2, 1)$-Seifert fibers. Liouville foliation is classified by the graph Γ, list of atoms and some rational numbers ([2]).

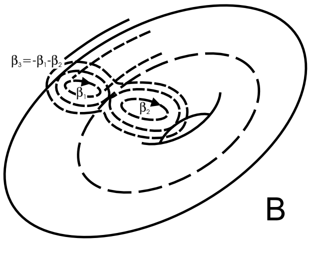

Fig. 2

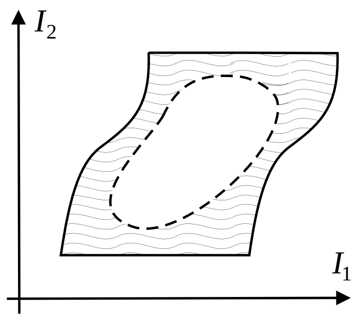

Fig. 3

Proof of Theorem 2. If $M^4 = Q^3 \times S^1(\varphi)$ has $\mathcal{I}$-compatible symplectic structure Ω we define element $F^\Omega \in H^1_{dR}(Q^3)$ by the formula $F^\Omega(\tau) = \langle \Omega, \tau \times S^1(\varphi) \rangle$. Since 1-parametric family of critical circles is a symplectic two-dimensional manifold, $F^\Omega(\gamma_i) \neq 0$ for all critical cycles.

Suppose now $[\gamma_i] \neq 0$. Then there is a rational class $F^\Omega \in H^1(Q^3; \mathbb{Q})$ with non-zero evaluation on every critical cycle γ_i.

Consider at first the case when all critical circles have orientable separatrix diagram and no leaf is a Klein bottle. We start by constructing symplectic form Ω on $U \times S^1(\varphi) \subset M^4$, where U is a neighborhood of critical leaves of $\mathcal{I}$. For each atom the neighborhood is $P^2 \times S^1(\theta)$ and the section P^2 can be chosen in such a way that F^Ω evaluates to zero on $H_1(P^2) \subset H_1(U)$. Then we set in U:

$$\Omega = \omega + d\theta \wedge d\varphi \tag{1}$$

for volume forms ω and $d\theta$ on P^2 and $S^1(\theta)$ correspondingly.

So we have defined Ω near singularities. The complement consists of edges $e_i \simeq T^2 \times I$ multiplied by $S^1(\varphi)$. Fix one $Q^3_{\varphi_0} = Q^3 \times \{\varphi_0\} \subset M^4$. For a neighborhood in M^4 of the edge $e_i \subset Q^3_{\varphi_0}$ we construct a symplectic form that extends Ω from a neighborhood of the boundaries. Define action coordinates, see [1], $(I_1^{(i)}, I_2^{(i)})$ on a subset of $e_i \times S^1 \subset M^4$, where Ω is already constructed:

$$I_1^{(i)} = \oint_{\sigma_i} \varkappa,\ I_2^{(i)} = \oint_{\alpha_i} \varkappa,\ d\varkappa = \Omega.$$

Here the cycle $\alpha_i \in H_1(T^2)$ is given by the restriction $F^\Omega = 0$ and σ_i is any complement cycle (for example a critical one). Thus the function $I_2^{(i)}$ is well defined, while $I_1^{(i)}$ is multiple-valued with respect to a turn around $S^1(\varphi) = M^4/Q^3$. If we cut $S^1(\varphi) \to [0,1]$ and double the edge e_i we get the momentum map

$$(I_1^{(i)}, I_2^{(i)}) : \partial(e_i \times S^1) \to \mathbb{R}^2$$

with the image pictured in figure 3. Since this image is an annulus bordant to zero, Theorem 5 implies that Ω can be extended to the whole M^4 so that all leaves of the Liouville foliation are Lagrangian.

Suppose now $\mathcal{I}$ contains critical circles with non-orientable separatrix diagram. A neighborhood U of such a fiber has Seifert fibration structure with singular fibers of the type (2,1) and base — a 2-dimensional manifold with boundary contracting to a graph, whose vertices have multiplicities 0 or 2 and 4 (atom). There is a double cover

$$\widehat{U} \xrightarrow{\mathbb{Z}_2} U$$

with trivial lifted fibration. The base of this new fibration doubly covers the original base

$$\widehat{P^2} \xrightarrow{\mathbb{Z}_2} P^2$$

and the monodromy (involution) preserves orientation and has fixed points at which the action linearizes to the reflection over a point. To define symplectic structure Ω on $U \times S^1$ we construct an involution-invariant symplectic form $\widehat{\Omega}$ on $\widehat{U} \times$

S^1. Restriction and projection of F^Ω produce an element $\widehat{F}^\Omega_U \in H^1(\widehat{U};\mathbb{R})$. We construct the form $\widehat{\Omega}$ on $\widehat{U} \times S^1$ using $\widehat{F}^\Omega_U$ as before. Standard $\mathbb{Z}_2$-averaging yields invariant $\widehat{\Omega}$ and hence a symplectic form Ω on $U \times S^1$ with Lagrangian foliation $\mathcal{I}|_{U\times\{pt\}}$. Now we use the product formula (1) and continue as in the orientable case.

Finally let us allow Klein bottle as a Liouville leaf. Homology group of its neighborhood $H_1(U;\mathbb{R}) = \mathbb{R}$ and the restriction F^Ω_U is non-zero. Let $\mathbb{Z}_2 : \widehat{U} \to U$ be the double cover such that the foliation with singular leaf K^2 lifts to a non-singular foliation by tori ([5, ch.4,§1.7]). The monodromy (involution) in $\widehat{U}$ is

$$(t,\theta,\psi) \mapsto (-t,\theta+\tfrac{1}{2},-\psi), \quad t \in (-1,1), \quad \theta,\psi \in S^1 = \mathbb{R}(\mathrm{mod}\,1).$$

The foliation is given by the formula $\mathcal{I} = \{t = \text{const}\}$ in these coordinates. We define involution-invariant symplectic form on $\widehat{U} \times S^1$ by

$$\widehat{\Omega} = c_1 \cdot dt \wedge d\psi + c_2 \cdot d\theta \wedge d\varphi,$$

and cohomological element F^Ω normalizes constants c_1, c_2. Thus we get a symplectic form Ω on $U \times S^1$ and this finishes the proof. □

Remark 7. Another approach for the Klein bottle part using Seifert sub-fibration $U \to D^2$ with two singular fibers of the type (2,1) is described in [6, Remark 3, fig. 1, 2].

4. Examples

1. Consider the Liouville foliation given by the molecule Γ of figure 4. Marks r_i shows how the distinguished cycles on the germs of letters intersect and n is the corresponding Euler number ([2]). We claim that $M^4 = Q^3 \times S^1$ has a compatible symplectic structure iff $r_1+r_2+r_3+n = 0$, in particular $-2 \le n \le 0$. Actually, the manifold if Seifert and the above condition means there exists a section, $e(\mathcal{I}) = 0$ (see [9] for details).

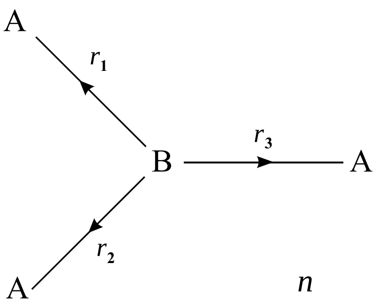

Fig. 4

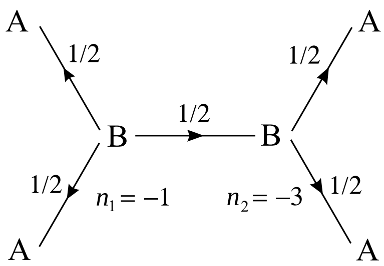

Fig. 5

2. Consider a Liouville foliation $\mathcal{I}$ of Q^3 given in figure 5. This Q^3 is obtained from 4 solid tori and 2 copies of the manifold $P^2 \times S^1$ (atom B), where $P^2 =$

$D^2 \setminus \bigcup_{i=1}^{2} D_i^2(\varepsilon)$ is a disk with two holes, by orientation-reversing diffeomorphisms of the boundary tori. We describe the diffeomorphisms precisely.

Numerate two left atoms A on the picture by 1 and 2 and two right ones by 3 and 4, numerate left and right atoms B by 1 and 2 correspondingly. Let us choose basis cycles (α_i, σ_i), $1 \le i \le 4$ on the boundary T^2 of atoms A_i: α_i is a simple cycle homologous to zero and σ_i additional. Similar choose cycles $(\gamma^{(j)}, \beta_i^{(j)})$, $1 \le i \le 3$ for the atoms B_j, $j = 1, 2$: γ being critical and the other additional (figure 2). We assume that the cycles

$$\beta_3^{(j)} = -\beta_1^{(j)} - \beta_2^{(j)}$$

correspond to the edge connecting B_1 and B_2. The gluing diffeomorphisms corresponding to arrows of figure 5 and the map $F^\Omega : H_1(Q^3; \mathbb{R}) \to \mathbb{R}$ have the formulae:

$$\begin{pmatrix} \alpha_i \\ \sigma_i \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \gamma^{(1)} \\ \beta_i^{(1)} \end{pmatrix}, \quad i = 1, 2;$$

$$\begin{pmatrix} \alpha_j \\ \sigma_j \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \gamma^{(2)} \\ \beta_j^{(2)} \end{pmatrix}, \quad j = 3, 4;$$

$$\begin{pmatrix} \gamma^{(2)} \\ \beta_3^{(2)} \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} \gamma^{(1)} \\ \beta_3^{(1)} \end{pmatrix}.$$

$$F^\Omega(\alpha_i) = 0, \ -F^\Omega(\beta_1^{(1)}) = -F^\Omega(\beta_2^{(1)}) = F^\Omega(\beta_1^{(2)}) = F^\Omega(\beta_2^{(2)}) = 1,$$

$$F^\Omega(\sigma_i) = -1, \ F^\Omega(\gamma^{(1)}) = F^\Omega(\gamma^{(2)}) = 2, \ F^\Omega(\beta_3^{(1)}) = 2 = -F^\Omega(\beta_3^{(2)}).$$

Thus, by Theorem 2, $M^4 = Q^3 \times S^1$ is symplectic. Since $H_1(Q^3) = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, the assumption Q^3 is Seifert implies that its base is S^2. So by the Waldhausen classification of automorphisms of graph-manifolds ([16]) Q^3 is not Seifert.

5. Waldhausen vs. Stallings

In this section we describe when a 3-manifold Q^3 is simultaneously Stallings and Waldhausen and prove Theorem 3. Let Q be a Stallings manifold. It is irreducible until $Q = S^2 \times S^1$. So consider a JSJ-decomposition of Q^3, i.e., decomposition induced by cuts along (canonical) essential 2-tori such that each piece (it has toric boundary) is either Seifert, or I-bundle or simple. According to the Thurston Geometrization Conjecture (proved for Haken manifolds) the last piece is hyperbolic.

If some of the pieces are simple, the manifold is not Waldhausen. Since we are interested in orientable Q^3 only, we exclude also I-bundles. So consider a Stallings manifold Q^3 with all the pieces in JSJ-decomposition being Seifert. Let regular Seifert fiber S^1 have projection to the Stallings base of degree d. Then the Seifert foliation can be deformed so that all regular circles intersect each Stallings fiber exactly d times (for $d = 0$ with one fiber excluded).

Now every non-compressible torus is isotopic to either fiber or to a transversal. If fiber is T^2, we get $Q^3 = T^2 \widetilde{\times}_A S^1$, where $A \in \mathrm{SL}_2(\mathbb{Z})$ and the Seifert condition implies $|\mathrm{tr}(A)| \leq 2$. In the case tr < 2 the matrix A is conjugated to rotation by the angle $\pm 2\pi/k$, $k = 3, 4, 6$. We can choose the Seifert structure with the degree $d \neq 0$ and then the claim is easily checked. $A = \pm\mathbf{1}$ is the trivial bundle and the case of Jordan box is the announced exception.

On the other hand if all JSJ-tori are transversal, they cut Q^3 to manifolds $F^2 \widetilde{\times} S^1$, each of which is Seifert with toric boundary. Suppose at first that all degrees $d \neq 0$, i.e., every Seifert circle is transversal to the fibers of the Stallings projection $\pi : Q^3 \to S^1(\theta)$. Then we construct 1-form $\pi^* d\theta$ and its class F^Ω is what is required for application of Theorem 2.

If there is a piece with $d = 0$, then F^2 is fibered by circles. Since it is oriented we get two possibilities: $F^2 = T^2$ or $S^1 \times I$. The first case is considered above while in the second the Stallings monodromy is periodic and so the Seifert foliation can be changed to one with $d \neq 0$.

Consider now the opposite statement of Theorem 3. Let Q^3 be Waldhausen such that for some element $F^\Omega \in H^1(Q)$ all evaluations on Seifert generators of Waldhausen pieces are non-zero. We can consider restrictions of F^Ω to these pieces and the compatibility of the evaluations on boundary tori (required to get a global class due to Mayer–Vietoris principle) constitute a system of linear equations with integer coefficients. If this system has a non-zero real solutions (as we assume in our hypothesis), it also have non-zero rational solutions. But existence of such a form leads to a class F^Ω that evaluates to non-zero integers on Seifert generators of Waldhausen pieces. This gives fibering over S^1 of these pieces that are compatible on the boundary tori and thus Q^3 is a Stallings.

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