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On the covariant exterior differentiation on Weil bundles[1]

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Abstract. We study the prolongation of linear connections on a vector bundle E and of E-valued k-forms with respect to an arbitrary Weil functor T^A. We present a practical algorithm for these procedures that is heavily based on the multiplication in the Weil algebra A. Our main theoretical result is that the covariant exterior differential of E-valued k-forms in the sense of Koszul commutes with these prolongation procedures.

Keywords. Weil bundle, prolongation of exterior forms, linear connection, covariant exterior differential.

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It has been clarified recently that the Weil functors represent a unified technique for investigating a large class of geometric problems, see [7] for a survey. In [12], Slovák deduced the basic properties of the prolongations of general connections on arbitrary fibered manifolds with respect to a Weil functor T^A. The T^A-prolongations of tangent valued forms were studied in [1, 2, 4]. In particular, it was proved there that these prolongations preserve the Frölicher–Nijenhuis bracket.

The starting point of the present paper was the idea of an E-valued k-form for a vector bundle E and of its covariant exterior differential by Koszul, [8]. Our main theoretical result is Proposition 9 reading that the T^A-prolongations of linear connections and E-valued k-forms commute with the covariant exterior differentiation. However, we also intend to point out that the Weil algebras supply a very efficient practical machinery for the coordinate investigation of the above mentioned problems. Some basic ideas of these procedures were presented in ([4]) and we used

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this approach already in our research concerning the prolongations of second order connections to vertical Weil bundles, [3].

In Section 1 of the present paper we study the covariant derivative with respect to the T^A-prolongation of a linear connection on E. The prolongation formulae for tangent valued forms and E-valued forms are derived in Sections 2 and 3. In Section 4 we first remark that one proof of Proposition 9 can be based on the theoretical results from Sections 1 – 3 analogously to our proof of the fact that T^A preserves the Frölicher–Nijenhuis bracket in ([1]). However, we find it more conceptual to reduce the problem completely to the case of the Frölicher–Nijenhuis bracket by using an unpublished result by Marco Modugno, [10]. We acknowledge him for making us familiar with his manuscript.

All manifolds and maps are assumed to be infinitely differentiable and all manifolds are paracompact. Unless otherwise specified, we use the terminology and notation from ([7]).

1. On the prolongation of connections

For every manifold M and every Weil algebra A, we denote by $\varkappa_M : T^A TM \to TT^A M$ the canonical exchange diffeomorphism, [7]. For a tangent valued k-form ω on M,

$$\omega : TM \times_M \cdots \times_M TM \longrightarrow TM,$$

its prolongation $\mathcal{T}^A\omega$ is the tangent valued k-form on $T^A M$ defined by

$$\mathcal{T}^A\omega = \varkappa_M \circ T^A\omega \circ (\varkappa_M^{-1} \times \cdots \times \varkappa_M^{-1}), \tag{1}$$

see [1, 4, 11].

We shall start from the approach by Slovák, [12], who studied a (general) connection Γ on a fibered manifold $p : Y \to M$ in the form of the lifting map

$$\Gamma : Y \times_M TM \longrightarrow TY.$$

Composing Γ with $Tp : TY \to TM$ and the bundle projection $TY \to Y$, we obtain a tangent valued one-form $h_\Gamma : TY \to TY$, which is called the horizontal form of Γ. The vertical projection (or vertical form) $v_\Gamma : TY \to VY$ is characterized by

$$h_\Gamma + v_\Gamma = \mathrm{id}_{TY} . \tag{2}$$

Slovák introduced the induced connection $\mathcal{T}^A\Gamma$ on $T^A p : T^A Y \to T^A M$ in the lifting form, [12],

$$\mathcal{T}^A\Gamma = \varkappa_Y \circ T^A\Gamma \circ (\mathrm{id}_Y \times \varkappa_M^{-1}) : T^A Y \times_{T^A M} TT^A M \longrightarrow TT^A Y. \tag{3}$$

In [1], we pointed out that the horizontal form of $\mathcal{T}^A\Gamma$ is the prolongation of h_Γ, i.e., $h_{\mathcal{T}^A\Gamma} = \mathcal{T}^A(h_\Gamma)$. Then (2) implies

$$v_{\mathcal{T}^A\Gamma} = \mathcal{T}^A(v_\Gamma). \tag{4}$$

From now on we assume Γ is a linear connection on a vector bundle $p : E \to M$ unless otherwise specified. Denote by $\mathrm{pr}_2 : VE \to E$ the second projection determined by the canonical identification of the vertical tangent bundle

$$VE \approx E \times_M E.$$

For a section $s : M \to E$ and a vector field $X : M \to TM$, the covariant derivative $\nabla^\Gamma_X s : M \to E$ is the composition

(5) $$\nabla^\Gamma_X s = \mathrm{pr}_2 \circ v_\Gamma \circ Ts \circ X.$$

Clearly, $T^A p : T^A E \to T^A M$ is a vector bundle and $\mathcal{T}^A\Gamma$ is a linear connection, [12]. Consider the induced section $T^A s : T^A M \to T^A E$. The flow prolongation $\mathcal{T}^A X : T^A M \to TT^A M$ of the vector field X satisfies $\mathcal{T}^A X = \varkappa_M \circ T^A X$, [7]. Applying T^A to (5), one rededuces the following result by Slovák, [12],

(6) $$\nabla^{\mathcal{T}^A\Gamma}_{\mathcal{T}^A X} T^A s = T^A(\nabla^\Gamma_X s).$$

We recall that every $a \in A = T^A\mathbb{R}$ induces a tensor field of type (1,1) on $T^A M$

$$L(a) : TT^A M \longrightarrow TT^A M$$

as follows, [7]. Let $\mu : \mathbb{R} \times TM \to TM$ be the multiplication of tangent vectors by reals. Then we define

$$\mathcal{T}^A\mu = \varkappa_M \circ T^A\mu \circ (\mathrm{id}_A \times \varkappa_M^{-1}) : A \times TT^A M \longrightarrow TT^A M$$

and we set $L(a) = \mathcal{T}^A\mu(a, -)$. Similarly, if $m : \mathbb{R} \times E \to E$ is the multiplication of the vectors of E by reals, we construct $T^A m : A \times T^A E \to T^A E$. In this case, we shall use the following simple notation

(7) $$T^A m(a, X) = aX, \qquad a \in A,\ X \in T^A E.$$

Proposition 1. *For every section s of E, every vector field X on M and every $a, b \in A$, we have*

(8) $$\nabla^{\mathcal{T}^A\Gamma}_{L(a)\mathcal{T}^A X}(bT^A s) = (ab)\nabla^{\mathcal{T}^A\Gamma}_{\mathcal{T}^A X} T^A s.$$

Proof. It is well known that for all $k, l \in \mathbb{R}$ we have

(9) $$\nabla^\Gamma_{kX} ls = kl\nabla^\Gamma_X s.$$

If we apply T^A to (9) and use (5), we prove our claim. $\square$

2. The Weil algebra calculus

Using the ideas from [4], we show that the coordinate form of the above operations can be written in a concise form, if we use the fact that the addition or multiplication in $A = T^A\mathbb{R}$ is the prolongation of the addition or multiplication of reals. We have $A = \mathbb{R} \times N$, where N is the nilpotent part. Choose a basis $e^1, \ldots, e^k$

of the vector space N and write $e^0 = 1$. Hence every element of A can be written in the form

$$x_\alpha e^\alpha \quad \text{with summation with respect to } \alpha = 0, 1, \dots, k.$$

For a function $f : \mathbb{R}^m \to \mathbb{R}$, we have $T^A f : A^m \to A$ and we define $D_\alpha f : A^m \to \mathbb{R}$ by the decomposition

$$T^A f = (D_\alpha f) e^\alpha \tag{10}$$

with $D_0 f = f \circ q$, where $q : A^m \to \mathbb{R}^m$ is the canonical projection. If we write $y = f(x^1, \dots, x^m)$, then (10) can be also expressed in the form $y_\alpha e^\alpha = e^\alpha D_\alpha f$.

Given a map $f : \mathbb{R}^m \to \mathbb{R}^n$, $y^p = f^p(x)$, we have the following formula for $T^A f : A^m \to A^n$

$$y^p_\alpha e^\alpha = e^\alpha D_\alpha f^p.$$

If dx^i are the additional coordinates on $T\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m$, then we write analogously

$$T^A(dx^i) = dx^i_\alpha e^\alpha. \tag{11}$$

This formula includes the exchange map $\varkappa_{\mathbb{R}^m} : T^A T\mathbb{R}^m \to T T^A \mathbb{R}^m$.

Let us consider a vector field X on M with the coordinate expression $dx^i = X^i(x^1, \dots, x^m)$. Since the flow prolongation $\mathcal{T}^A X$ satisfies $\mathcal{T}^A X = \varkappa_{\mathbb{R}^m} \circ T^A X$, its coordinate form is

$$dx^i_\alpha = D_\alpha X^i. \tag{12}$$

Example 2. For the sake of simplicity, we shall discuss only the tangent functor T in the examples of the present paper, even though the power of our approach becomes more instructive in the case of the functor T^r_h of arbitrary (h, r)-velocities, [7]. Hence $A = \mathbb{D}$ and its elements are of the form $x_0 + x_1 e$, $x_0, x_1 \in \mathbb{R}$, $e^2 = 0$. For every local coordinates x^i on M, we denote by x^i_1 the additional coordinates on TM and we write $Tf = f + eDf$. Hence

$$Df = \frac{\partial f}{\partial x^1} x^1_1 + \dots + \frac{\partial f}{\partial x^m} x^m_1. \tag{13}$$

Then (12) yields the well-known coordinate formula for the flow prolongation $\mathcal{T} X$

$$dx^i = X^i, \quad dx^i_1 = DX^i. \tag{14}$$

Let ω be a tangent valued one-form on M with the coordinate expression

$$d\bar{x}^i = a^i_j(x)\, dx^j. \tag{15}$$

On the right-hand side we have the products of $a^i_j(x)$ with dx^j and the summation with respect to j. Hence the coordinate expression of $\mathcal{T}^A \omega$ is

$$T^A d\bar{x}^i = (T^A a^i_j)(T^A\, dx^j) \tag{16}$$

with the addition and multiplication in A on the right-hand side. In other words,

$$e^\alpha d\bar{x}^i_\alpha = (e^\beta D_\beta a^i_j)(e^\alpha\, dx^j_\alpha) = e^\alpha e^\beta (D_\beta a^i_j)\, dx^j_\alpha, \tag{17}$$

where the product $e^\alpha e^\beta$ is in A.

Example 3. In the case $A = \mathbb{D}$, (17) yields

$$d\bar{x}^i + e\, d\bar{x}^i_1 = (a^i_j + eDa^i_j)(dx^j + e\, dx^j_1).$$

Hence the coordinate expression of $\mathcal{T}\omega$ is

$$d\bar{x}^i = a^i_j\, dx^j, \quad d\bar{x}^i_1 = (Da^i_j)\, dx^j + a^i_j\, dx^j_1.$$

Consider a general connection Γ on an arbitrary fibered manifold $Y \to M$ in the form of the lifting map

$$d\bar{x}^i = dx^i, \quad dy^p = F^p_i(x, y)\, dx^i. \tag{18}$$

Then the lifting map $\mathcal{T}^A\Gamma : T^AY \times_{T^AM} TT^AM \to TT^AY$ has the coordinate expression

$$d\bar{x}^i_\alpha = dx^i_\alpha, \quad e^\alpha dy^p_\alpha = e^\alpha e^\beta (D_\beta F^p_i)\, dx^i_\alpha. \tag{19}$$

Example 4. For $A = \mathbb{D}$, the essential part of (19) has the form, cf. [6],

$$dy^p = F^p_i\, dx^i, \quad dy^p_1 = (DF^p_i)\, dx^i + F^p_i\, dx^i_1.$$

The coordinate expression of a linear connection Γ on the vector bundle E is

$$dy^p = \Gamma^p_{qi}(x) y^q\, dx^i. \tag{20}$$

Hence the equations of $\mathcal{T}^A\Gamma$ are

$$T^A(dy^p) = (T^A\Gamma^p_{qi})(T^Ay^q)(T^A\, dx^i), \tag{21}$$

or, equivalently,

$$e^\alpha\, dy^p_\alpha = e^\alpha e^\beta e^\gamma (D_\gamma \Gamma^p_{qi})\, y^q_\beta\, dx^i_\alpha. \tag{22}$$

Example 5. In the case $A = \mathbb{D}$, (22) has the form (20) and

$$dy^p_1 = (D\Gamma^p_{qi})\, y^q\, dx^i + \Gamma^p_{qi} y^q_1\, dx^i + \Gamma^p_{qi} y^q\, dx^i_1.$$

For a section s of E, (5) implies the well-known coordinate expression of $\nabla^\Gamma_X s$

$$\frac{\partial s^p}{\partial x^i} X^i - \Gamma^p_{qi} s^q X^i. \tag{23}$$

Then the coordinate form of $T^A(\nabla^\Gamma_X s)$ is

$$T^A\left(\frac{\partial s^p}{\partial x^i} X^i\right) - (T^A\Gamma^p_{qi})(T^As^q)\,(T^AX^i). \tag{24}$$

The expression $(\partial s^p/\partial x^i)X^i$ can be interpreted as the derivative Xs^p of s^p with respect to the vector field X. Lemma 3 from ([1]) reads

$$T^A(Xs^p) = \mathcal{T}^A X(T^A s^p). \tag{25}$$

If we plug this onto (24), we obtain the coordinate expression of $\nabla^{\mathcal{T}^A\Gamma}_{\mathcal{T}^A X} T^A s$. This is a coordinate proof of (6). Moreover, since the coordinate form of $L(a)\mathcal{T}^A X$ is $aT^A X^i$ and the coordinate form of $bT^A s$ is $bT^A s^p$, (24) yields a coordinate proof of Proposition 1.

3. Prolongation of E-valued k-forms

Koszul introduced an E-valued k-form ω as a linear base preserving morphism $\Lambda^k TM \to E$, [8]. We define the prolongation $\mathcal{T}^A\omega$ as a $T^A E$-valued k-form determined by the following commutative diagram

$$\begin{array}{ccc} T^A TM \times_{T^A M} \cdots \times_{T^A M} T^A TM & \xrightarrow{\;T^A\omega\;} & T^A E \\ \varkappa_M \downarrow \qquad\qquad\qquad \downarrow \varkappa_M & & \| \\ TT^A M \times_{T^A M} \cdots \times_{T^A M} TT^A M & \xrightarrow{\;\mathcal{T}^A\omega\;} & T^A E. \end{array} \tag{26}$$

If $X_1, \dots, X_k$ are vector fields on M, then $\omega(X_1, \dots, X_k)$ is a section of E. Using (26), we obtain immediately

$$\mathcal{T}^A\omega(\mathcal{T}^A X_1, \dots, \mathcal{T}^A X_k) = T^A\big(\omega(X_1, \dots, X_k)\big). \tag{27}$$

This formula can be generalized as follows.

Proposition 6. *For every $a_1, \dots, a_k \in A$ we have*

$$\mathcal{T}^A\omega\big(L(a_1)\mathcal{T}^A X_1, \dots, L(a_k)\mathcal{T}^A X_k\big) = a_1 \cdots a_k T^A\omega(X_1, \dots, X_k). \tag{28}$$

Proof. The multilinearity of ω implies

$$\omega(t_1 X_1, \dots, t_k X_k) = t_1 \cdots t_k \omega(X_1, \dots, X_k)$$

for every $t_1, \dots, t_k \in \mathbb{R}$. Applying T^A, we prove our claim. □

To deduce the coordinate expression of $\mathcal{T}^A\omega$, we start with the case of a tensor field φ of type $(1, k)$ on M. The prolongation $\mathcal{T}^A\varphi$ is a tensor field of type $(1, k)$ on $T^A M$ defined by the same formula (1) as in the case of tangent valued k-forms. Let

$$d\bar{x}^i = a^i_{j_1\cdots j_k}(x)\, dx^{j_1} \otimes \cdots \otimes dx^{j_k} \tag{29}$$

be the coordinate expression of φ. If we rewrite (29) in the form of a multilinear map, we have

$$w^i = a^i_{j_1\cdots j_k}(x) y^{j_1} \cdots z^{j_k}. \tag{30}$$

Applying T^A, we obtain

$$T^A w^i = (T^A a^i_{j_1\cdots j_k})(T^A y^{j_1}) \cdots (T^A z^{j_k}). \tag{31}$$

Going back to the tensor notation, we replace each $y^{j_1}_{\alpha_1} \cdots z^{j_k}_{\alpha_k}$ by $dx^{j_1}_{\alpha_1} \otimes \cdots \otimes dx^{j_k}_{\alpha_k}$. Hence (31) can be rewritten as

$$e^\alpha\, d\bar{x}^i_\alpha = e^\beta e^{\alpha_1} \cdots e^{\alpha_k} D_\beta a^i_{j_1\cdots j_k}\, dx^{j_1}_{\alpha_1} \otimes \cdots \otimes dx^{j_k}_{\alpha_k}. \tag{32}$$

Consider now the case of a tangent valued k-form ω,

$$d\bar{x}^i = a^i_{j_1\cdots j_k}\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}. \tag{33}$$

We have $dx^{i_1} \wedge \cdots \wedge dx^{j_k} = \mathrm{Alt}(dx^{i_1} \otimes \cdots \otimes dx^{i_k})$. Since the multiplication in A is commutative,

$$\mathrm{Alt}(e^{\alpha_1} \cdots e^{\alpha_k}\, dx^{i_1}_{\alpha_1} \otimes \cdots \otimes dx^{i_k}_{\alpha_k}) = e^{\alpha_1} \cdots e^{\alpha_k}\, dx^{i_1}_{\alpha_1} \wedge \cdots \wedge dx^{i_k}_{\alpha_k}.$$

Hence the coordinate expression of $\mathcal{T}^A\omega$ is

$$e^\alpha d\bar{x}^i_\alpha = e^\beta e^{\alpha_1} \cdots e^{\alpha_k} D_\beta a^i_{j_1\cdots j_k}\, dx^{j_1}_{\alpha_1} \wedge \cdots \wedge dx^{j_k}_{\alpha_k}. \tag{34}$$

Example 7. In the case $A = \mathbb{D}$ and $k = 2$, we have

$$d\bar{x}^i + e\, d\bar{x}^i_1 = [a^i_{jk} + eDa^i_{jk}](dx^j + e\, dx^j_1) \wedge (dx^k + e\, dx^k_1).$$

This yields

$$d\bar{x}^i_1 = (Da^i_{jk})\, dx^j \wedge dx^k + a^i_{jk}(dx^j \wedge dx^k_1 + dx^j_1 \wedge dx^k). \tag{35}$$

If we assume $a^i_{jk} = -a^i_{kj}$, then the last term in (35) is equal to $a^i_{kj}\, dx^k \wedge dx^j_1$. Hence we have

$$d\bar{x}^i_1 = (Da^i_{jk}) dx^j \wedge dx^k + 2a^i_{jk} dx^j \wedge dx^k_1. \tag{36}$$

Example 8. In [1], we have deduced that T^A preserves the Frölicher–Nijenhuis bracket of tangent valued forms. Since the curvature of an arbitrary connection Γ can be expressed in terms of the Frölicher–Nijenhuis bracket, our procedures give an algorithm for evaluating the curvature $C\mathcal{T}^A\Gamma$ of $\mathcal{T}^A\Gamma$ from the curvature $C\Gamma$ of Γ. Let

$$dy^p = C^p_{ij}\, dx^i \wedge dx^j, \quad C^p_{ij} = -C^p_{ji}, \tag{37}$$

be the coordinate expression of $C\Gamma$. In the case $A = \mathbb{D}$, (36) implies the following coordinate formula for $C\mathcal{T}\Gamma$

$$dy^p_1 = (DC^p_{ij})\, dx^i \wedge dx^j + 2C^p_{ij}\, dx^i \wedge dx^j_1. \tag{38}$$

The coordinate procedure for evaluating the prolongation of an E-valued k-forms ω is of the same character. Let

$$y^p = a^p_{i_1\cdots i_k}(x)\, dx^i \wedge \cdots \wedge dx^{i_k} \tag{39}$$

be the coordinate expression of ω. Then the coordinate form of $\mathcal{T}^A\omega$ is

$$e^\alpha y^p_\alpha = e^\beta e^{\alpha_1} \cdots e^{\alpha_k} D_\beta a^p_{j_1\cdots j_k}\, dx^{j_1}_{\alpha_1} \wedge \cdots \wedge dx^{j_k}_{\alpha_k}. \tag{40}$$

4. The covariant exterior differential

According to Koszul, [8], the covariant exterior differential $d^\Gamma\omega$ of an E-valued k-form ω with respect to a linear connection Γ on E is an E-valued $(k+1)$-form defined by

$$\begin{aligned} d^\Gamma\omega(X_1, \ldots, X_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}\nabla^\Gamma_{X_i}\omega(X_1, \ldots, \widehat{X}_i, \ldots, X_{k+1}) \\ &\quad + \sum_{i<j}(-1)^{i+j}\omega\big([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1}\big). \end{aligned} \tag{41}$$

Then $d^{\mathcal{T}^A\Gamma}(\mathcal{T}^A\omega)$ is a T^AE-valued $(k+1)$-form.

Proposition 9. *We have*

$$d^{\mathcal{T}^A\Gamma}(\mathcal{T}^A\omega) = \mathcal{T}^A(d^\Gamma\omega).$$

First of all we remark there is a straightforward proof of Proposition 9 by using (8) and (28) analogously to the proof of ([1, Proposition 1]). However, we prefer a direct way of reducing Proposition 9 to the case of the Frölicher–Nijenhuis bracket. This is based on a manuscript by Modugno, [10], and on the fact that T^A preserves the Frölicher–Nijenhuis bracket, [1].

A vertical valued horizontal k-form on a fibered manifold $p : Y \to M$ can be defined as a linear base preserving morphism of the pull-back $p^* \bigwedge^k TM$ into VY. This is a special tangent valued k-form on Y. An E-valued k-form ω can be extended into a vertical valued horizontal k-form $\widetilde{\omega}$ on E by using the translations into the individual fibers of E. Modugno deduced, [10], that $d^\Gamma\omega$ is characterized by the following property

$$\widetilde{d^\Gamma\omega} = [h_\Gamma, \widetilde{\omega}] \tag{42}$$

with the Frölicher–Nijenhuis bracket on the right-hand side.

Hence the main tool for proving Proposition 9 is the following formula, where $\mathcal{T}^A$ on the left-hand side denotes the prolongation of a tangent valued k-form and $\mathcal{T}^A$ on the right-hand side means the prolongation of an E-valued k-form.

Lemma 10. *We have* $\mathcal{T}^A(\widetilde{\omega}) = \widetilde{\mathcal{T}^A\omega}$.

Proof. A vertical valued horizontal k-form ψ on E can be interpreted as a morphism

$$\varrho(\psi) : TM \times_M \cdots \times_M TM \times_M E \to E \tag{43}$$

with values in the second factor of $VE = E \times_M E$. Clearly, ψ is of the form $\psi = \widetilde{\omega}$, if and only if

$$\varrho(\psi) = \varrho(\psi) \circ \big(\mathrm{Id}_M \times (O_E \circ p)\big), \tag{44}$$

where Id_M means the identity of $TT^AM \times_{T^AM} \cdots \times_{T^AM} TT^AM$, $p : E \to M$ is the bundle projection and $O_E : M \to E$ is the zero section. In this case, $\omega = \varrho(\psi) \circ (\mathrm{Id}_M \times O_E)$. The morphism $\varrho(\mathcal{T}^A\psi)$ makes the following diagram commutative

$$\begin{array}{ccccccc} T^ATM \times_{T^AM} \cdots \times_{T^AM} & T^ATM \times_{T^AM} & T^AE & \xrightarrow{T^A\varrho(\psi)} & T^AE \\ \varkappa_M \big\downarrow & \big\downarrow \varkappa_M & \big\| & & \big\| \\ TT^AM \times_{T^AM} \cdots \times_{T^AM} & TT^AM \times_{T^AM} & T^AE & \xrightarrow{\varrho(\mathcal{T}^A\psi)} & T^AE. \end{array} \tag{45}$$

Applying T^A to (44), we obtain

$$T^A\varrho(\psi) = T^A\varrho(\psi) \circ \big(T^A(\mathrm{Id}_M) \times (O_{T^AE} \circ T^Ap)\big).$$

Taking into account (45), we deduce

$$\varrho(\mathcal{T}^A\psi) = \varrho(\mathcal{T}^A\psi) \circ \big(\mathrm{Id}_{T^AM} \times (O_{T^AE} \circ T^Ap)\big).$$

Furthermore, $\varrho(\mathcal{T}^A\psi) \circ (\mathrm{Id}_{T^AM} \circ O_{T^AE}) = \mathcal{T}^A\omega$. □

Using this lemma, (42) and the fact that $\mathcal{T}^A$ preserves the Frölicher–Nijenhuis bracket, we obtain

$$\begin{aligned} \big(\mathcal{T}^A(d^\Gamma\omega)\big)^{\sim} &= \mathcal{T}^A(\widetilde{d^\Gamma\omega}) = \mathcal{T}^A[h_\Gamma, \tilde{\omega}] = [\mathcal{T}^Ah_\Gamma, \mathcal{T}^A\tilde{\omega}] \\ &= [h_{\mathcal{T}^A\Gamma}, \widetilde{\mathcal{T}^A\omega}]. \end{aligned}$$

This proves Proposition 9.

We remark that the coordinate formula for $d^{\mathcal{T}^A\Gamma}\mathcal{T}^A\omega$ can be derived from the coordinate formula for $d^\Gamma\omega$ by means of the algorithms of Sections 2 and 3.

References

[1] A. Cabras and I. Kolář, Prolongation of tangent valued forms to Weil bundles, *Archivum Math. (Brno)* 31 (1995) 139–145.

[2] A. Cabras and I. Kolář, Prolongation of projectable tangent valued forms, *Rend. Circ. Mat. Palermo*, to appear.

[3] A. Cabras and I. Kolář, Prolongation of second order connections to vertical Weil bundles, *Archivum Math. (Brno)* 37 (2001) 333–347.

[4] J. Gancarzewicz, W. Mikulski and Z. Pogoda, Lifts of some tensor fields and connections to product preserving bundles, *Nagoya Math. J.* 135 (1994) 1–41.
[5] J. Gancarzewicz, W. Mikulski and Z. Pogoda, Product preserving functors and vector bundles, IMUJ Preprint 1997/24, Cracow.
[6] I. Kolář, On generalized connections, *Beiträge zur Algebra und Geometrie* 11 (1981) 29–34.
[7] I. Kolář, P.W. Michor and J. Slovák, *Natural Operations in Differential Geometry* (Springer, Berlin, 1993).
[8] J.L. Koszul, *Lectures on Fiber Bundles and Differential Geometry* (Tata Inst., Bombay, 1960).
[9] L. Mangiarotti and M. Modugno, Graded Lie algebras and connections on a fibered space, *J. Math. Pures et Appl.* 63 (1984) 111–120.
[10] M. Modugno, Differential calculus on fibered manifolds, manuscript.
[11] A. Morimoto, Prolongations of connections to bundles of infinitely near points, *J. Diff. Geometry* 11 (1976) 479–498.
[12] J. Slovák, Prolongations of connections and sprays with respect to Weil functors, *Suppl. Rend. Circ. Mat. Palermo*, Serie II. 14 (1987) 143–155.

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