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The characterization of biharmonic morphisms[1]

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Abstract. We present a characterization of biharmonic morphisms, i.e., maps between Riemannian manifolds which preserve (local) biharmonic functions, in a manner similar to the case of harmonic morphisms for harmonic functions. Of particular interest are maps which are at the same time harmonic and biharmonic morphisms.

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1. Harmonic maps

Consider two Riemannian manifolds (M^m, g) and (N^n, h), and a C^2-map $\phi : (M, g) \to (N, h)$ between them, then the energy of the map ϕ is:

$$E(\phi) = \frac{1}{2}\int_M |d\phi_x|^2\, v_g.$$

(or over any compact subset $K \subset M$).

A map is called *harmonic* if it is an extremum, in its homotopy class, of the energy functional E (or $E(K)$ for all compact subsets $K \subset M$).

The Euler–Lagrange equation corresponding to this functional is the *tension field*, a system of semi-linear second-order elliptic partial differential equations:

$$\tau(\phi) = \operatorname{trace} \nabla\, d\phi.$$

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In local coordinates, it takes the form:

$$\tau^\alpha(\phi) = g^{ij}\left(\frac{\partial^2\phi^\alpha}{\partial x^i \partial x^j} - {}^M\Gamma^k_{ij}\frac{\partial\phi^\alpha}{\partial x^k} + {}^N\Gamma^\alpha_{\beta\gamma}\frac{\partial\phi^\beta}{\partial x^i}\frac{\partial\phi^\gamma}{\partial x^j}\right),$$

where ${}^M\Gamma^k_{ij}$ and ${}^N\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the Levi-Civita connections on (M, g) and (N, h).

Harmonic maps generalize harmonic functions ($N = \mathbb{R}$) and geodesics ($M = \mathbb{S}^1$).

The first problem solved by Eells and Sampson was existence, more specifically within a given homotopy class. Using a heat equation method, they established:

Theorem 1.1 ([7]). *Let (M, g) be a compact manifold and (N, h) a compact manifold with non-positive Riemannian sectional curvature. Then any continuous map from M to N has a harmonic representant, of minimal energy, in its homotopy class. Or, in other words, any map can be continuously deformed into a map of minimal energy, perforce harmonic.*

Further evidence of the influence of curvature on harmonic maps is given by:

Theorem 1.2 ([18]). *Let M be a non-compact Riemannian manifold satisfying* $\mathrm{Ricci}^M \geq 0$ *and* $\mathrm{Riem}^N \leq 0$. *Then any harmonic map $\phi : M \to N$ of finite energy is constant.*

If ϕ is an immersion, then its mean curvature is, up to a constant, the trace of the second fundamental form $\nabla d\phi$. Therefore:

Theorem 1.3 ([7]). *A Riemannian immersion is harmonic if and only if it is minimal.*

We will see a sort of converse of this result in the section on harmonic morphisms. The framework of Hermitian geometry is particularly propitious to the study of harmonic maps, as was immediately observed by Eells and Sampson.

Theorem 1.4 ([7]). *If ϕ is a $\pm$holomorphic map between Kähler manifolds then it is harmonic.*

Taking specific Riemann surfaces yields greater rigidity, highlighting the limits of Theorem 1.1:

Theorem 1.5 ([8]). *Any harmonic non $\pm$holomorphic map from the two-torus to the two-sphere is null homotopic. In particular, there is no harmonic map of degree one from $\mathbb{T}^2$ to $\mathbb{S}^2$.*

2. Harmonic morphisms

Harmonic morphisms were introduced in potential theory by Constantinescu and Cornea in 1965 (cf. [5]) to generalize a well-known property of holomorphic maps.

Indeed, holomorphic maps between Riemann surfaces possess the characteristic property of pulling back (local) holomorphic functions onto holomorphic functions. Since holomorphic functions on a Riemann surface are harmonic, harmonic morphisms were defined as mappings between Riemannian manifolds which pull-back (local) harmonic functions onto (local) harmonic functions. More precisely:

Definition 2.1. Let $\phi : (M, g) \to (N, h)$ be a continuous mapping between Riemannian manifolds. Then ϕ is called a *harmonic morphism* if for any harmonic function $f : U \subset N \to \mathbb{R}$, its pull-back by ϕ, $f \circ \phi : \phi^{-1}(U) \subset M \to \mathbb{R}$ is harmonic as well.

An immediate consequence of this definition is that the composition of two harmonic morphisms is a harmonic morphism.

Moreover, since the Laplacian is (twice) the generator of the Brownian motion, this definition implies that a harmonic morphism maps a Brownian motion on M onto a Brownian motion on N. The image of a Brownian motion being a dense set, we conclude that, necessarily, if ϕ is a non-constant harmonic morphism from M to N, then $\dim M \geq \dim N$. This fact will be confirmed later on.

Definition 2.2. A C^1-map $\phi : (M^m, g) \to (N^n, h)$ is called *horizontally weakly conformal* if, at each point $x \in M$, either $d\phi_x \equiv 0$ or the linear map

$$d\phi_x|_{(\ker d\phi_x)^\perp} : (\ker d\phi_x)^\perp \to T_{\phi(x)}N$$

is a surjective and conformal map.

The conformal factor is usually called the *dilation* and denoted by $\lambda(x)$.

Call $V_x = \ker d\phi_x$ the vertical space and $H_x = (\ker d\phi_x)^\perp$ the horizontal space. In this case, horizontal weak conformality can be written as the condition:

$$h\big(d\phi(X), d\phi(Y)\big) = \lambda^2(x) g(X, Y), \qquad \forall X, Y \in H_x.$$

Using local coordinates $(x^i)_{i=1,\dots,m}$ and $(y^\alpha)_{\alpha=1,\dots,n}$ around x and $\phi(x)$, this reads as:

$$g^{ij}(x) \frac{\partial \phi^\alpha}{\partial x^i}(x) \frac{\partial \phi^\beta}{\partial x^j}(x) = \lambda^2(x) h^{\alpha\beta}\big(\phi(x)\big), \qquad \forall \alpha, \beta = 1, \dots, n.$$

This last equation makes more sense once one realizes that ϕ being horizontally weakly conformal is equivalent to $(d\phi)^*$ being conformal. Moreover, one can notice the relation between the energy density and the dilation of a horizontally weakly conformal map ϕ:

$$e_\phi = \frac{1}{2} |d\phi|^2 = \frac{n}{2} \lambda^2,$$

where n is the dimension of the target space.

It is clear that if $m < n$, then a horizontally weakly conformal map is constant.

Rather surprisingly, harmonic morphisms can be shown to be harmonic maps, this is the Fuglede–Ishihara Characterization:

Theorem 2.1 ([9, 13]). *A mapping between Riemannian manifolds is a harmonic morphism if and only if it is a horizontally weakly conformal harmonic map.*

Proof. The two original proofs use different methods. While ([9]) relies heavily on properties specific to the Laplacian, ([13]) offers a more direct, and more easily adaptable, procedure: write out the chain rule for $\Delta(f \circ \phi)$, at a given point, plug in harmonic functions with prescribed first and second derivatives at this point, deduce necessary and sufficient conditions on ϕ.

Let f be any C^2-function on N and ϕ a harmonic morphism from M^m to N^n, then:

$$ {}^{M}\Delta(f \circ \phi) = df\big(\tau(\phi)\big) + \mathrm{trace}\nabla\, df(d\phi, d\phi). \tag{1} $$

Let $q \in M$ and (x^i), $i = 1, \dots, m$, and (y^α), $\alpha = 1, \dots, n$, normal coordinates around q and $\phi(q) \in N$.

Choose a harmonic function f such that all its second derivatives vanish at the point $\phi(q)$ and

$$ \frac{\partial f}{\partial y^\alpha}\big(\phi(q)\big) = \delta_{\alpha\alpha_0}. $$

Then equation (1) reads:

$$ 0 = {}^{M}\Delta(f \circ \phi) = \tau^{\alpha_0}(\phi), $$

therefore ϕ is a harmonic map.

Next, take f harmonic such that all its first derivatives vanish at $\phi(q)$ and

$$ \frac{\partial^2 f}{\partial y^\alpha \partial y^\beta}\big(\phi(q)\big) = C_{\alpha\beta} \qquad (C_{\alpha\beta} = C_{\beta\alpha}), $$

with $C_{\alpha\beta} = 0$ if $\alpha \neq \beta$ and $\sum_\alpha C_{\alpha\alpha} = 0$ (this condition is forced by the harmonicity of f). Equation (1) becomes:

$$ 0 = {}^{M}\Delta(f \circ \phi) = \sum_\alpha C_{\alpha\alpha} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\alpha}{\partial x^j} g^{ij}. $$

Choosing different values for the $C_{\alpha\alpha}$'s shows that:

$$ g^{ij} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} = \lambda^2 h^{\alpha\beta}, $$

i.e., ϕ is horizontally weakly conformal.

The converse is easily deduced from (1). An important step here is the existence of such "test functions". □

Remark 2.1. It is well known that harmonic maps (or even harmonic functions) do not compose into harmonic maps, while, as was already observed, harmonic morphisms do. They form a closed sub-class of harmonic maps.

Remark 2.2. An often overlooked property of harmonic morphisms is their openness. Established by Fuglede with the help of the Harnack Inequality ([9]), it implies, in particular, that if the domain space M is compact and $\phi : M \to N$ is a non-constant harmonic morphism, then N has to be compact.

Remark 2.3. In 1996, Fuglede extended Theorem 2.1 to maps between semi--Riemannian manifolds, using Ishihara's method. Two steps proved crucial. First, establishing a new definition of horizontally weakly conformal, in this context of non-positive definite metrics. One has to add to Definition 2.2 the condition that if the vertical space $\ker d\phi_x$ is degenerate, then $(\ker d\phi_x)^\perp \subset \ker d\phi_x$. Next, since the Laplacian is no longer elliptic, the existence of harmonic test functions was, a priori, problematic. However such functions do exist, the proof of their availability relying, strangely enough on the semi-linearity of the operator. We refer the reader to ([10] and in particular Hörmander's appendix) for further details.

Though the cases $\dim M = \dim N$ are not trivial, they are of limited interest:

Theorem 2.2 ([9]). *A harmonic morphism between surfaces is a $\pm$holomorphic map (the converse being also true).*

A harmonic morphism between Riemannian manifolds of the same dimension (greater or equal to three) is a homothetic map, i.e., a conformal map of constant dilation.

An elementary computation shows that a holomorphic map into a Riemann surface is always horizontally weakly conformal, this yields:

Theorem 2.3. *A $\pm$holomorphic map from a Kähler manifold to a Riemann surface is a harmonic morphism.*

Conditions for the contrary have been sought, the closest seems to be:

Theorem 2.4 ([20]). *Let $\phi : M \to N$ be a non-constant harmonic morphism from an orientable Einstein four-manifold to a Riemann surface. Then ϕ is holomorphic with respect to some (integrable) Hermitian structure.*

The characterization gives us another:

Theorem 2.5. *A continuous map $\phi : (M, g) \to (N, h)$ is a harmonic morphism if and only if there exists a (unique and continuous) function λ such that:*

$$^{M}\Delta(f \circ \phi) = \lambda^2 (^{N}\Delta f) \circ \phi$$

for all $f \in C^2(U \subset N, \mathbb{R})$.

In fact this formula is also valid for maps and the tension field:

Theorem 2.6. *Let $\phi : (M, g) \to (N, h)$ be a harmonic morphism and $\psi : (N, h) \to (P, k)$ a harmonic map. Then $\psi \circ \phi : (M, g) \to (P, k)$ is a harmonic map.*

Though a lot has been written on harmonic morphisms (cf. [12]), a general existence theory is still lacking. However, for the lowest significant dimensions, the following classification was obtained by Baird and Wood (see [3, 4]):

Theorem 2.7. *Let $M = \mathbb{R}^3, \mathbb{S}^3$ or $\mathbb{H}^3$ with the standard metric, and a Riemann surface N. Then every non-constant harmonic morphism $\phi : M \to N$ has the form*

$$\phi = \sigma \circ \psi \circ i,$$

where $i : M \to M$ is an isometry, $\sigma : \widetilde{N} \to N$ a weakly conformal map between Riemann surfaces, while:

1. *If $M = \mathbb{R}^3$, then $\psi : \mathbb{R}^3 \to \mathbb{R}^2$ $(= \widetilde{N})$ is the projection of $\mathbb{R} \times \mathbb{R}^2$ onto $\mathbb{R}^2$.*
2. *If $M = \mathbb{S}^3$, then $\psi : \mathbb{S}^3 \to \mathbb{S}^2$ $(= \widetilde{N})$ is the Hopf fibration.*
3. *If $M = \mathbb{H}^3$, then $\psi : \mathbb{H}^3 \to \widetilde{N}$ is a certain explicit projection from $\mathbb{H}^3$ onto $\mathbb{R}^2$ or $\mathbb{H}^2$.*

More encouragements come from a recent striking result of Ababou–Baird–Brossard ([1]) on mappings between Euclidean spaces:

Theorem 2.8. *Let ϕ be a harmonic morphism from $\mathbb{R}^m \setminus K$, where K is a polar set, into $\mathbb{R}^n$ $(m \geq n \geq 3)$. Then ϕ is a polynomial map of degree less or equal than $(m-2)/(n-2)$.*

Employing a Bochner technique for harmonic morphisms, T. Mustafa has proved several non-existence results. First for compact domains:

Theorem 2.9 ([15]). *There exists no non-constant harmonic morphism from a compact Riemannian manifold of non-negative Ricci curvature to a compact Riemann surface of genus greater of equal than two.*

Then, in the non-compact case:

Theorem 2.10 ([16]). *Let $m \geq n$. There exists no non-constant harmonic morphism from $\mathbb{R}^m$ to $\mathbb{H}^n$.*

If ϕ from $\mathbb{H}^m$ to $\mathbb{R}^n$ is a harmonic morphism of dilation λ, then

$$\lambda^2 \leq \frac{m-1}{n-1}.$$

If $\phi : \mathbb{S}^m \to \mathbb{S}^n$ is a submersive harmonic morphism of dilation λ, then

$$\lambda^2 > \frac{m-1}{n-1}.$$

3. Biharmonic morphisms

Let $\phi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds. Define its tension field to be: $\tau(\phi) = \operatorname{trace} \nabla d\phi$, and, for a compact domain $K \subseteq$

M, the bienergy

$$E^2(\phi) = \frac{1}{2} \int_K \left|\tau(\phi)\right|^2 v_g.$$

Critical points of the functional E^2 are called *biharmonic maps*.

The Euler–Lagrange equation associated to E^2 is:

$$\tau^2(\phi) = -\Delta^\phi \tau(\phi) - \operatorname{trace}_g R^N\big(d\phi, \tau(\phi)\big)\, d\phi,$$

where

$$\Delta^\phi = -\operatorname{trace}_g \nabla^2 = -\operatorname{trace}_g(\nabla^\phi \nabla^\phi - \nabla^\phi_\nabla),$$

is the Laplacian on sections of $\phi^{-1}TN$, and R^N is the Riemannian curvature operator:

$$R^N(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.$$

Note that $\tau^2(\phi) = -J^\phi(\tau(\phi))$, where J^ϕ is the Jacobi operator associated to ϕ.

The definition of a harmonic morphism can be adapted to biharmonicity (cf. also [17]):

Definition 3.1. Let $\phi : (M, g) \to (N, h)$ be a continuous mapping between Riemannian manifolds. Then ϕ is called a *biharmonic morphism* if for any biharmonic function $f : U \subset N \to \mathbb{R}$, its pull-back by ϕ, $f \circ \phi : \phi^{-1}(U) \subset M \to \mathbb{R}$ is biharmonic as well.

4. Characterization

Proposition 4.1 ([6]). *Let (N^n, h) be a (smooth) Riemannian manifold and $q \in N$. There exists harmonic coordinates $(y^1, \ldots, y^n)$ on a neighborhood U of q, i.e., such that:*

$${}^N\Delta y^\alpha = -h^{\beta\gamma}\Gamma^\alpha_{\beta\gamma} = 0,$$

on U.

These coordinates are smooth and via linear transformations can be chosen so that $h^{\beta\gamma}(q) = \delta_{\beta\gamma}$.

Remark 4.1. 1. The existence of such coordinates relies on the local solvability, for each index, of the Laplacian with prescribed first derivatives.

2. In dimension 2, the usual isothermal coordinates are harmonic. Its higher dimensional counterpart is that holomorphic coordinates on Kähler manifolds are harmonic.

3. It is unreasonable, in dimension at least three, to expect coordinates, at the same time, harmonic and normal, except for the locally flat case.

The existence of local biharmonic test functions is given by:

Lemma 4.1. *Let $q \in N$ and symmetric coefficients $(C_\alpha, C_{\alpha\beta}, C_{\alpha\beta\gamma}, C_{\alpha\beta\gamma\delta})$ such that in local harmonic coordinates $(y^1, \dots, y^n)$ centered around q:*

$$h^{\alpha\beta}(q)h^{\gamma\delta}(q)C_{\alpha\beta\gamma\delta} + 2h^{\alpha\delta}(q)\frac{\partial h^{\beta\gamma}}{\partial y^\delta}(q)C_{\alpha\beta\gamma} + \Delta h^{\alpha\beta}(q)C_{\alpha\beta} = 0, \tag{2}$$

then there exist a neighborhood U of q and a smooth biharmonic function $f : U \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial y^\alpha}(q) = C_\alpha, \qquad \frac{\partial^2 f}{\partial y^\alpha \partial y^\beta}(q) = C_{\alpha\beta},$$

$$\frac{\partial^3 f}{\partial y^\alpha \partial y^\beta \partial y^\gamma}(q) = C_{\alpha\beta\gamma}, \qquad \frac{\partial^4 f}{\partial y^\alpha \partial y^\beta \partial y^\gamma \partial y^\delta}(q) = C_{\alpha\beta\gamma\delta}.$$

Proof. This is a straightforward application of a result of Alinhac and Gerard, proved with an implicit function theorem, on the local solvability of non-linear elliptic equations [2, Proposition 2.4]. □

Theorem 4.1. *Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds. Then ϕ is a biharmonic morphism if and only if it is a horizontally weakly conformal biharmonic map, of dilation λ, such that:*

$$\lambda^2 \tau(\phi) + d\phi \operatorname{grad} \lambda^2 = 0,$$

$$\left|\tau(\phi)\right|^4 - 2\Delta\lambda^2 \left|\tau(\phi)\right|^2 + 4\Delta\lambda^2 \operatorname{div} \left\langle d\phi\, \tau(\phi) t\right\rangle + n(\Delta\lambda^2)^2$$
$$+ 2\left\langle d\phi\, \tau(\phi)\right\rangle \nabla \left|\tau(\phi)\right|^2 + |S|^2 = 0,$$

where $S \in \bigodot^2 \phi^{-1}TN$ is the symmetrization of the g-trace of $d\phi \otimes \nabla^\phi \tau(\phi)$.

Proof. The proof follows the same procedure as for Theorem 2.1, though calculations are longer. □

The next result is also in ([17]):

Proposition 4.2. *A map $\phi : (M, g) \to (N, h)$ is a biharmonic morphism if and only if there exists a function $\lambda : M \to \mathbb{R}$ such that*

$$\Delta^2(f \circ \phi) = \lambda^4 \Delta^2(f) \circ \phi,$$

for all functions $f \in C^4(U \subset N, \mathbb{R})$.

Theorem 4.1 enables us to compare harmonic and biharmonic morphisms:

Proposition 4.3. *Let $\phi : (M, g) \to (N, h)$ be a non-constant map.*

If M is a compact manifold without boundary then ϕ is a biharmonic morphism if and only if ϕ is a harmonic morphism of constant dilation, hence a homothetic submersion with minimal fibers.

Besides N has to be compact as well.

Proof. An intermediate step in the proof of Theorem 4.1 shows that if ϕ is a biharmonic morphism then:

$$n\Delta(\lambda^2) + 2\,\mathrm{div}\,\big(d\phi\,\tau(\phi)\big) = \big|\tau(\phi)\big|^2. \tag{3}$$

Integrating this expression over M, gives:

$$0 = \int_M \big|\tau(\phi)\big|^2\, v_g,$$

by the Divergence Theorem. Therefore $\tau(\phi) = 0$, i.e., ϕ is harmonic, and equation (3) becomes: $\Delta\lambda^2 = 0$, but the only harmonic functions on a compact manifold are the constant ones. As a result the map ϕ must be a horizontally weakly conformal harmonic map, i.e., a harmonic morphism, with constant dilation.

Conversely, it is clear that such a map satisfies the conditions sufficient to be a biharmonic morphism.

Lastly, a harmonic morphism is an open map (see [9]), so $\phi(M)$ is both open and closed, thus $\phi(M) = N$, and N is compact. □

Proposition 4.4. *Let M be a complete manifold and $\phi : (M, g) \to (N, h)$ a biharmonic morphism. If $\phi \in W^{1,2}(M, N)$ and*

$$\int_M \big|\tau(\phi)\big|^2 < \infty,$$

then ϕ is a harmonic morphism of constant dilation.

Proof. If ϕ is a biharmonic morphism then it is horizontally weakly conformal, of dilation λ^2, and:

$$n\Delta(\lambda^2) + 2\,\mathrm{div}\big(d\phi\,\tau(\phi)\big) = \big|\tau(\phi)\big|^2. \tag{4}$$

As $\phi \in W^{1,2}(M, N)$

$$\int_M \nabla|d\phi|^2 = n\int_M \nabla\lambda^2 < \infty,$$

and

$$\int_M \big|d\phi\,\tau(\phi)\big| \le \left(\int_M |d\phi|^2\right)^{\frac{1}{2}} \left(\int_M \big|\tau(\phi)\big|^2\right)^{\frac{1}{2}} < \infty.$$

Recall Yau's generalised Stokes' theorem (see [21]):

Lemma 4.2. *Let M^m be a complete Riemannian manifold of dimension m. If ω is a smooth integrable $(m-1)$-form on M^m then there exists a sequence of domains B_i in M^m such that $M^m = \bigcup_i B_i$, $B_i \subset B_{i+1}$ and*

$$\lim_{i\longrightarrow+\infty} \int_{B_i} d\omega = 0.$$

Applying this lemma to equation (4) shows: $\lim_{i\longrightarrow+\infty} \int_{B_i} |\tau(\phi)|^2 = 0$, i.e.,

$\tau(\phi) \equiv 0$. This implies that $\Delta\lambda^2 = 0$ and, by [21], that either λ^2 is constant or

$$\int_M \lambda^{2p} = \frac{1}{n^p} \int_M |d\phi|^p = +\infty$$

for all $p > 1$, which would contradict the hypothesis. □

Proposition 4.5. *Let (M, g), (N, h) and (P, k) be Riemannian manifolds. Then a non-constant map $\phi : (M, g) \to (N, h)$ will be a biharmonic morphism if and only if there exists a continuous function $\lambda : M \to \mathbb{R}^+$ such that*

$$\tau^2(\psi \circ \phi) = \lambda^4 \tau^2(\psi) \circ \phi, \tag{5}$$

for any map $\psi : (N, h) \to (P, k)$.

Remark 4.2. Of course, Proposition 4.5 implies that biharmonic morphisms pull-back biharmonic maps onto biharmonic maps. Besides, if ϕ is a biharmonic morphism and a harmonic map, e.g., if λ is constant or ϕ is horizontally homothetic, then ϕ will also be a harmonic morphisms and satisfy:

$$\tau(\psi \circ \phi) = \lambda^2 \tau(\psi) \circ \phi.$$

In particular, this means that ϕ will pull-back non-harmonic biharmonic map onto non-harmonic biharmonic maps.

Example 4.1. Let $\mathbb{H}^m = (\mathbb{R}^{m-1} \times \mathbb{R}^+, (1/s^2)g_{\mathbb{R}^m})$ and consider the map $\pi : \mathbb{H}^m \to \mathbb{R}^{m-1}$, $(x_1, \ldots, x_{m-1}, s) \mapsto \alpha(x_1, \ldots, x_{m-1})$. This map is known to be horizontally homothetic ($d\phi(\nabla\lambda^2) = 0$) and to have totally geodesic fibers (cf. [11]), so it is a harmonic morphism, and will be a biharmonic morphism if and only if $\Delta\lambda^2 = 0$.

Since $\lambda^2 = \alpha^2 s^2$, $\Delta\lambda^2 = 0$ is equivalent to $\alpha^2 s^2 = s^{m-1}/(m-1)$, which happens if and only if $m = 3$ and $\alpha = \sqrt{2}/2$.

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