Differential Geometry and Its Applications
Proc. Conf., Opava (Czech Republic), August 27–31, 2001
Silesian University, Opava, 2001, 23–26

When is a diffeomorphism of a hyperbolic space isotopic to the identity?

Midori Goto

Abstract. We prove that a diffeomorphism of a compact oriented hyperbolic space is isotopic to the identity if the displacement distance and the covariant derivative of its associated vector field are sufficiently restricted.

Keywords. Hyperbolic space, diffeomorphism, isotopy, Jacobi field.

MS classification. 53C20, 57N37.

1. Introduction

Let M be a complete connected Riemannian manifold of dimension n. We denote by $d(x, y)$ the distance between two points x, y of M, and by $i(x)$ the injectivity radius at x. If $d(x, y) < i(x)$, then there exists a unique geodesic connecting x and y. Hence, if f is a diffeomorphism of M that satisfies the condition

$$(1) \qquad d\big(x,\ f(x)\big) < i(x) \qquad \text{for all } x \in M,$$

then f is smoothly homotopic to the identity. Given such an f, it is not clear whether two geodesics connecting x to $f(x)$, and y to $f(y)$ intersect for two points x, y of M. If no two geodesics connecting a point of M to its image by f intersect, then we could find an isotopy between f and the identity. In this paper we shall investigate when diffeomorphisms of M satisfying the condition (1) is smoothly isotopic to the identity. We will have the following theorem:

Theorem 1. *Let M be a compact, connected and oriented Riemannian manifold with constant curvature -1, and f an orientation-preserving diffeomorphism that satisfies the condition* (1). *Set $W(x) = \exp_x^{-1} f(x)$. If $\max |\nabla_v W| \le 1$ holds for any $x \in M$ and any unit tangent vector v at x, then f is smoothly isotopic to the identity.*

This paper is in final form and no version of it will be submitted for publication elsewhere.

As an immediate consequence, we have

Corollary 1. *Let M be as in the Theorem* 1. *Then the group of orientation--preserving diffeomorphisms of M is locally contractible.*

Remark 1. It is natural that the group of orientation-preserving diffeomorphisms of a compact manifold could be locally contractible in the C^∞-topology, since all derivatives of diffeomorphisms are bounded. However, by restricting only on displacement distance and the first-order derivative, we will give a criterion that a diffeomorphism could be isotopic to the identity.

Remark 2. The local contractibility of the group of diffeomorphisms of hyperbolic spaces has been studied, in quit different ways, by Earle and Elles (see [2]) for 2-dimensional case, and by Hatcher for 3-dimensional case, cf. [3].

2. Preliminaries

Let M be a complete, connected smooth Riemannian manifold of dimension n (≥ 2). We denote by (v, w) the inner product of vectors v and w. The readers refer to the book ([1]) for Riemannian geometric materials. Let f be a diffeomorphism of M that satisfies the condition (1) in the introduction. Then we have a vector field W on M defined by

$$W(x) = \exp_x^{-1}\big(f(x)\big)$$

for $x \in M$. It is well-defined since $\exp_x : \{v \in T_xM : |v| < i(x)\} \to M$ is injective. Let $x \in M$ be a point with $f(x) \neq x$, and $c : [0,1] \to M$ a unique geodesic connecting $x = c(0)$ to $f(x) = c(1)$, parametrized proportionally to arc-length. For a unit tangent vector v at x, let $\alpha : (-\epsilon, \epsilon) \to M$ be a smooth curve with $\alpha(0) = x$ and $\dot\alpha(0) = v$. We consider a variation of c, $F : [0,1] \times (-\epsilon, \epsilon) \times T_xM \to M$ defined by $F(t, s; v) = \exp_{\alpha(s)} tW(\alpha(s))$, where $W(\alpha(s)) = \exp^{-1}_{\alpha(s)}(f(\alpha(s)))$. We observe, by the definition, that the variation vector field

$$Y_v(t) = \left.\frac{\partial F(t, s; v)}{\partial s}\right|_{s=0} = (dF_t)_x(v)$$

is the Jacobi field along c. Also

$$\left.\frac{\partial F(0, s; v)}{\partial s}\right|_{s=0} = v$$

and

$$\left.\frac{\partial F(1, s : v)}{\partial s}\right|_{s=0} = (df)_x(v).$$

Since $[\partial/\partial t, \partial/\partial s] = 0$, it follows that $\nabla_{\dot c} Y_v(0) = \nabla_v W,$ where ∇ denotes the covariant derivative.

The above observation yields the following local result:

Proposition 1. *Let M be a complete connected Riemannian manifold with constant curvature -1, and f a diffeomorphism of M that satisfies the condition* (1). *Set $W(x) = \exp_x^{-1} f(x)$. Suppose that* $\max|\nabla_v W| \leqq 1$ *holds for any $x \in M$ and any unit tangent vector v at x. Define a map $F_t : M \to M$* by $F_t(x) = \exp_x tW(x)$. *Then F_t is a local diffeomorphism for each t, where F_0 = identity and $F_1 = f$.*

Proof. Suppose that $W(x) \neq 0$. For a vector v at x, we denote by $v^\top$ (resp. $v^\perp$) the component of v tangent, normal respectively, to $W(x)$. Since M has constant negative curvature -1, the Jacobi field $Y_v(t)$ can be written as

$$Y_v(t) = P(t)\cosh t + Q(t)\sinh t + (a+bt)\dot{c}(t)/h^{-1},$$

where $h = |W(x)|$, $a = (v, \dot{c}(0)/h^{-1})$, $b = (\nabla_v W, \dot{c}(0)/h^{-1})$, and P, Q are parallel vector fields along c with $P(0) = v^\perp$, $Q(0) = (\nabla_v W)^\perp$. By the above expression of $Y_v(t)$ it follows that, if either a or b is zero, then $Y_v(t)$ never vanishes since M has no conjugate points. Similarly, if either $|P|$ or $|Q|$ is zero, then $Y_v(t)$ never vanishes.

If $|P(0)| = |v^\perp| \neq 0$ and $|P(0)| \geqq |Q(0)|$ holds, we have

$$\big|P(t)\cosh t + Q(t)\sinh t\big| \geqq |P(0)| \cdot \sinh t \cdot \left|\coth t - \frac{\big|Q(0)\big|}{\big|P(0)\big|}\right| > 0 \tag{2}$$

for t, $0 \leqq t \leqq 1$, since $\coth t > 1$. While, if $|v^\top| \neq 0$ and $|a| \geqq |b|$ holds, then

$$|a+bt| \geqq \left|1 - t\frac{|b|}{|a|}\right| |a| > 0 \tag{3}$$

for $0 \leqq t < 1$. Moreover, we have the following lemma.

Lemma 1. *If* $\max|\nabla_v W| \leqq 1$ *holds for any x in M and any unit tangent vector v at x, one cannot have two inequalities*

$$|(\nabla_v W)^\perp| > |v^\perp|, \qquad |(\nabla_v W)^\top| > |v^\top|$$

simultaneously.

Assuming that the lemma is true, we see that the Jacobi field $Y_v(t)$ never vanishes for t, $0 \leqq t \leqq 1$ in the case $ab|P||Q| \neq 0$ also. Thus, due to the inverse function theorem, the statement of the Proposition holds. □

Proof of the Lemma 1. Suppose that both inequalities hold simultaneously. Then

$$\begin{aligned} |\nabla_v W| &= |(\nabla_v W)^\perp|^2 + |(\nabla_v W)^\top|^2 \\ &> |v^\perp|^2 + |v^\top|^2 \\ &= 1, \end{aligned} \tag{4}$$

a contradiction. □

3. Proof of the theorem

Due to the proposition the map $F_t : M \to M$ defined by $F_t(x) = \exp_x tW(x)$ is a local diffeomorphism and F_t defines a homotopy between the identity and f.

If, further, F_t were surjective, then it turns out to be a covering map. And, since f has mapping of degree one, so is F_t. Thus F_t is a diffeomorphism for each t.

Suppose that F_t is not surjective. Then there is a point y in M such that $F_t(M) \subseteq M - \{y\}$. It is known that $H_n(M - \{y\} : \mathbb{Z}) = 0$, while $H_n(M : \mathbb{Z}) = \mathbb{Z}$. So, the homomorphism $\iota_* \cdot (F_t)_* : H_n(M : \mathbb{Z}) \to H_n(M - \{y\} : \mathbb{Z}) \to H_n(M : \mathbb{Z})$ is a zero-map, where $\iota : M - \{y\} \to M$ is the inclusion map. On the other hand, since F_t is homotopic to the identity, $(F_t)_* : H_n(M : \mathbb{Z}) \to H_n(M : \mathbb{Z})$ is the identity. Thus we have a contradiction. □

References

[1] J. Cheeger and D.G. Ebin, *Comparison Theorems in Riemannian Geometry* (American Elsevier, North Holand, 1975).

[2] C.J. Earle and J. Eells Jr., A fiber bundle description of Teichmuller theory, *J. Diff. Geom.* 3 (1969) 19–43.

[3] W.P. Thurston, The Topology and Geometry of 3-manifolds (Princeton University, Lecture Note, 1978.

Midori Goto
Faculty of Information Engineering
Fukuoka Institute of Technology
Fukuoka 811-0295
Japan
E-mail: m-gotou@fit.ac.jp