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Ricci tensor of pseudo-cylindric metrics[1]

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Abstract. Let $\mathcal{P}$ the class of compact Riemannian manifolds with parallel Ricci tensor and $\mathcal{H}$ the class of such manifolds with harmonic curvature. The inclusion $\mathcal{P} \subset \mathcal{H}$ holds. Let us consider the Riemannian cylindric product $(S^1 \times S^{n-1}, dt^2 + d\xi^2)$, where S^1 is the circle of length T and $(S^{n-1}, d\xi^2)$ is the standard sphere. This metric is known to have a parallel Ricci curvature tensor.

We also consider the manifold $S^n - \Lambda_k$, where Λ_k is a finite point-set of the standard sphere $(S^n, d\xi^2)$.

We show that unless the trivial ones, all the pseudo-cylindric metrics in the conformal class $[dt^2 + d\xi^2]$ as well as the Yamabe metric of $(S^n - \Lambda_k, d\xi^2)$ belong $\mathcal{H}$ but not belong $\mathcal{P}$.

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1. Introduction

It is well known that the class $\mathcal{P}$ of compact Riemannian manifolds with parallel Ricci tensor is included in the class $\mathcal{C}$ of manifolds with constant scalar curvature. Let $\mathcal{H}$ the class of such manifolds with harmonic curvature. The following inclusion holds

$$\mathcal{P} \subset \mathcal{H}.$$

Moreover, this inclusion is strict according to examples of Riemannian metrics given by A. Derdzinski ([1, 5]) and A. Gray ([6]). In other words, there exist metrics with harmonic curvature and Ricci tensor non-parallel. But examples of such metrics are very few.

Let (M, g) be a Riemannian manifold of dimension $n \geq 3$. M is said to have harmonic curvature if the divergence of its curvature tensor $\mathcal{R}$ vanishes (in local

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coordinates: $\nabla^i \mathcal{R}_{ijkl} = 0$). That means the Ricci tensor r is a Codazzi tensor ($\nabla_k r_{hj} - \nabla_j r_{hk} = 0$). In other words, in the compact case of the manifold the Riemannian connection is a Yang–Mills potential in the tangent bundle.

Answering to the question on the parallelism of the Ricci tensor of the Riemannian metrics, A. Derdzinski gave examples of compact manifolds with harmonic curvature but non-parallel Ricci tensor: $\delta\mathcal{R} = 0$ and $\nabla r \neq 0$. Moreover, he obtains some classification results, [5].

The corresponding manifolds are bundles with fibers N over the circle S^1, parametrized by arc length t and of length $T = \int_{S^1} dt$, equipped with the warped metrics $dt^2 + h^{4/n}(t) g_0$ on the product $S^1 \times N$. Here, (N, g_0) is an Einstein manifold of dimension $n \geq 3$, and the function $h(t)$ on the prime factor is a periodic solution of the ODE, established by Derdzinski

$$(1) \qquad h'' - \frac{nR}{4(n-1)} h^{1-4/n} = -\frac{n}{4} C h$$

for some constant $C > 0$. This function must be non-constant, otherwise the corresponding metric has a parallel Ricci tensor.

We know that the number of such metric naturally must depend on the geometry of S^1 and N. In particular, we proved there is a positive bound T_0 depending on the S^1 length such that when

$$T < T_0,$$

the above equation may have only constant solutions. The corresponding Ricci tensor cannot be non-parallel. We are interested in the following problem:

When can g have a harmonic curvature and non-parallel Ricci tensor?

This concerns specially the conformally flat manifolds. Let the Riemannian cylindric product $(S^1 \times S^{n-1}, dt^2 + d\xi^2)$, where S^1 is the circle of length T and $(S^{n-1}, d\xi^2)$ is the standard sphere. A such metric has a parallel Ricci tensor. Moreover, we know the number of Yamabe metrics is finite in the conformal class of the cylindric metric $[dt^2 + d\xi^2]$, see [2, 3, 7]. So, we call *pseudo-cylindric metric* any non-trivial Yamabe metric g_c in $[dt^2 + d\xi^2]$.

Consider also the manifolds $S^n - \Lambda_k$, where Λ_k is a finite point-set of the standard sphere $(S^n, d\xi^2)$.

We show that except the trivial ones, all the pseudo-cylindric metrics in the conformal class $[dt^2 + d\xi^2]$ as well as the complete Yamabe metrics of $(S^n - \Lambda_k, d\xi^2)$ belong $\mathcal{H}$ but not belong $\mathcal{P}$.

2. Harmonic Riemannian curvature and Ricci tensor non-parallel

Let a compact Riemannian manifold (M, g), $\mathcal{R}$ the curvature associated to the metric g, $r = \mathrm{Ric}(g)$ its Ricci tensor, and D the Riemannian connection. This curvature is harmonic if its formal divergence vanishes $\delta\mathcal{R} = 0$. In local coordinates we get $D_k \mathcal{R}_{ij} = D_i \mathcal{R}_{kj}$. In particular, any Riemannian metric with parallel Ricci tensor $Dr = 0$ (i.e., $D_i \mathcal{R}_{kj} = 0$) has an harmonic curvature. Actually, in this case the Levi-Civita connection D is a Yang–Mills potential on the tangent bundle of M. In this way, the connection D is a critical point of the Yang–Mills functional

$$\mathcal{YM}(\nabla) = \frac{1}{2}\int_M \|R^\nabla\|\, dv,$$

where R^∇ is the curvature associated to the connection ∇.

The Riemannian curvature must be harmonic for all Einstein manifolds and for all conformally flat manifolds with constant scalar curvature; this can be deduced from the orthogonal decomposition of the curvature tensor. Moreover, the condition of the curvature harmonicity is, in a way, a generalization of the Einstein condition of the metric. Indeed, every Einstein metric has a parallel Ricci tensor $Dr = 0$. However, in the 3-dimensional case only, we get an identity between harmonic curvature metrics and conformally flat metrics with constant scalar curvature.

One wondered about the existence of compact metric with harmonic curvature and non-parallel Ricci curvature, as it was well known in the non-compact case.

In answer, Derdzinski ([5]) gave examples of such metrics with $Dr \neq 0$. The corresponding manifolds are bundles with fibers (N, g_0), a $(n-1)$-dimensional Einstein manifold with positive (constant) scalar curvature, over the circle S^1 (parametrized by arc length t). The allowed metric is the warped product $dt^2 + h^{4/n}(t)g_0$ on $S^1 \times N$. Here $h(t)$ is a function on the prime factor periodic solution of the ODE:

$$h'' - \frac{nR}{4(n-1)}h^{1-\frac{4}{n}} = -\frac{n}{4}Ch \tag{2}$$

for some constant $C > 0$.

This function must be non-constant, otherwise the corresponding metric has a parallel Ricci tensor. Notice that the manifolds $S^1 \times N$ are not conformally flat, unless when (N, g_0) has constant sectional curvature.

3. Curvature property of pseudo-cylindric metrics

Let $(S^n, d\xi^2)$ be the standard sphere with radius 1. Consider metrics of the conformal class $g \in [d\xi^2]$ on the domain $S^n - \Lambda_k$, where Λ_k is a finite point-set of S^n, $\Lambda_k = \{p_1, p_2, \dots, p_k\}$.

For $k = 2$, there is a conformal diffeomorphism between $S^n - \{p_1, p_2\}$ and $(S^1 \times S^{n-1}, dt^2 + d\xi^2)$, where S^1 is the circle of length T. The non-trivial Yamabe metrics on $(S^1 \times S^{n-1}, dt^2 + d\xi^2)$, are called pseudo-cylindric metrics. There are metrics of the form $g = u^{4/(n-1)}(dt^2 + d\xi^2)$ where the C^∞-function u is a non-constant positive solution of the Yamabe equation.

Furthermore, [5] established a classification of the compact n-dimensional Riemannian manifolds (M_n, g), $n \geq 3$, with harmonic curvature. If the Ricci tensor $\mathrm{Ric}(g)$ is not parallel and has less than three distinct eigenvalues at each point, then (M, g) is isometrically covered by a manifold $(S^1(T) \times N, dt^2 + h^{4/n}(t)g_0)$, where the non-constant positive periodic solutions h verify (1).

The following result gives other metrics with the same curvature property.

Theorem 1. *Consider the product manifold* $(S^1(T) \times S^{n-1}, g)$, $g = dt^2 + d\xi^2$ *where* (S^1, dt^2) *is the circle of length* T *and* $(S^{n-1}, d\xi^2)$ *is the standard sphere with radius* 1. *Under the condition*

$$T > T_1 = \frac{2\pi}{\sqrt{n-2}},$$

on the circle length, the Riemannian curvatures of the associated pseudo-cylindric metrics $g_c = u_c^{4/(n-2)} g$ *are harmonic and their Ricci tensors are non-parallel.*

Moreover, any pseudo-cylindric metric may be identified to a Derdzinski metric up to a conformal transformation.

Proof. The condition $T > T_1 = 2\pi/\sqrt{n-2}$, ensures the existence of a non-trivial Yamabe metric, see [3]. Since the product $(S^1(T) \times S^{n-1}, g)$ is conformally flat, then g_c has an harmonic curvature.

Consider the local coordinates $(x^0, x^1, \ldots, x^{n-1})$, where $x^0 = t$ is the coordinate corresponding to the circle factor S^1. Any metric tensor in the conformal class $[g]$ can be written $\bar{g} = u^{4/(n-2)} g$, where the C^∞-function u is defined on the circle.

We then obtain

$$\bar{g}_{00} = u^{4/(n-2)}, \quad \bar{g}_{0i} = 0, \quad \bar{g}_{ij} = u^{4/(n-2)} (g_0)_{ij},$$

where $i, j = 1, 2, \ldots, n-1$.

We get the Christoffel symbols $\bar{\Gamma}^i_{jk}$ associated to the metric $\bar{g}$.

$$\bar{\Gamma}^0_{jk} = -\frac{2}{n-2}\frac{u'}{u}\bar{g}_{jk}, \quad \bar{\Gamma}^i_{j0} = \frac{2}{n-2}\frac{u'}{u}\bar{\delta}^i_j,$$

$$\bar{\Gamma}^0_{00} = \frac{2}{n-2}\frac{u'}{u}, \quad \bar{\Gamma}^i_{00} = 0.$$

The Ricci tensor components are

$$\bar{\mathcal{R}}_{ij} = \mathcal{R}_{ij} - \frac{2\nabla_{ij}u}{u} + \frac{2n}{n-2}\frac{\nabla_i u \nabla_j u}{u^2} - \frac{2}{n-2}\frac{|\nabla u|^2 + u\Delta u}{u^2} g_{ij}.$$

We then have

$$\bar{\mathcal{R}}_{0i} = 0, \qquad \bar{\mathcal{R}}_{00} = 2\frac{n-1}{n-2}\left[\frac{u''}{u} + \frac{u'^2}{u^2}\right].$$

The associated scalar curvatures are $R(g) = (n-1)(n-2)$ and $\bar{R}(u^{4/(n-2)} g) = n(n-1)$. D denotes the Riemannian connection associated to the metric g. Since u should be a non-constant periodic function, so we have

$$D_0 \bar{\mathcal{R}}_{00} = \frac{d\bar{\mathcal{R}}_{00}}{dt} \neq 0.$$

The Ricci curvature of the pseudo-cylindric metrics is not parallel, except for the cylindric one.

For the second part of Theorem 1. One just remark that in the locally conformally flat case of this product only, the factor N may be identified with the standard sphere S^{n-1}. Hence, these metrics must be conformally flat and will be conformal to the cylindric metric, as S.T. Yau ([8]) has shown:

Lemma 1. *Any warped metric $dt^2 + f^2(t)g_0$ on the product Riemannian manifold $(S^1(T) \times S^{n-1}, dt^2 + g_0)$ must be conformally flat and conformal to a Riemannian metric product $d\theta^2 + d\xi^2$ where θ is a S^1-parametrization with length $\int_{S^1}(dt/f(t))$.*

Thus, any Derdzinski metric may be identified with a pseudo-cylindric metric up to a conformal diffeomorphism, say F. Then the metrics are relied

$$dt^2 + f^2(t)\, d\xi^2 = F^*[(u_c^j)^{4/(n-2)}(dt^2 + d\xi^2)],$$

where u_c^j are the Yamabe solutions belong to the (same) conformal class. Indeed, a warped metric can be written $dt^2 + f^2(t)\, d\xi^2 = f^2(t)[(dt/f)^2 + d\xi^2]$. After a change of variables and by using the conformal flatness, we get

$$dt^2 + f^2(t)g_0 = \phi^2(\theta)[d\theta^2 + d\xi^2]$$

which is conformally flat.

Remark 1. Parallelism property of the Ricci tensor have particularly an interest for the conformally flatness case. Indeed, consider a Riemannian manifold (M, g) with a parallel Ricci tensor. This implies in particular, that its Weyl tensor is harmonic: $\delta W = 0$ (in local coordinates $D_k W^h_{ijk} = 0$). For a metric $\tilde{g}$ in the conformal class and with harmonic Weyl tensor $[g] : \tilde{g} = e^{2\rho} g$, we get the following equality

$$\tilde{\delta}\widetilde{W} = \delta W - \tfrac{1}{2}(n-3)W(\nabla\rho, \cdot, \cdot, \cdot).$$

Since, the corresponding Ricci tensor is parallel, we then obtain in local coordinates

$$(n-3)W^l_{ijk} D_l \rho = 0.$$

Thus, necessarily we have $\rho = 0$ when (M, g) is not conformally flat.

4. Ricci tensor of the asymptotic pseudo-cylindric metrics

We are interested in curvature properties of the manifolds $M = S^n - \Lambda_k$ where $\Lambda_k = \{p_1, p_2, \ldots, p_k\}$ and $k > 2$. The allowed metrics are complete and have a (positive) constant scalar curvature. Which property must have their Ricci tensor?

Consider positive solutions u, on the cylinder of the singular Yamabe equation

$$4\frac{n-1}{n-2}\Delta_{g_0} u + R(g_0)u - R(g)u^{(n+2)/(n-2)} = 0,$$

$g = u^{4/(n-2)} g_0$ is complete on $S^n - \Lambda_k$ and $R(g) = $ const bigger than 0. By using a reflection argument, it is known ([7]) that any solution $u(x)$ of the above equation, with a singularity at p_l, is asymptotic to one of the pseudo-cylindric functions near the point p_l. In fact, any such solution $u(x)$ corresponds to a solution $u(t, \xi)$ on a domain of $\mathbb{R} \times S^{n-1}$.

Moreover, the corresponding pseudo-cylindric metric is unique.

Let $\bar{g} = [u(t, \xi)]^{4/(n-2)} g_0$ be a Yamabe metric conformal to the standard metric and ${u_c}^{j,l}(t)$ be the pseudo-cylindric solution at a point $p_l \in \Lambda_k$. The index j is corresponding to one periodic solution with period T. More precisely, we have

Lemma 2. *As $t \to \infty$ we get the following estimate of the Yamabe singular solution on $S^n - \Lambda_k$ near the point p_l*

$$u(t, \xi) = {u_c}^{j,l}(t) + {u_c}^{j,l}(t) \mathcal{O}(e^{-\beta t}),$$

where ${u_c}^{j,l}(t)$ is the corresponding pseudo-cylindric solution and β is a positive constant.

In other words, this solution $u(t, \xi)$, asymptotically closed to a pseudo-cylindric function $u_c^j(t)$ around a singular point p_l, can be written as $t \to \infty$

$$u(t, \xi) = u_c^j(t) + e^{-\alpha t} v(t, \xi). \tag{3}$$

Here α is a positive constant depending only on the cylindric bound and $v(t, \xi)$ is a bounded function.

Thus, we may estimate the Yamabe metrics $\bar{g} = [u(t, \xi)]^{4/(n-2)} g_0$ on the manifold $S^n - \Lambda_k$, where Λ_k is a finite point-set of the standard sphere (S^n, g_0). The corresponding metrics on $S^n - \Lambda_k$ are complete with constant scalar curvature and are asymptotic at each singular point to a pseudo-cylindric metric.

More precisely, a singular Yamabe metric $\bar{g}$ on $S^n - \Lambda_k$ is asymptotic to a unique pseudo-cylindric metric g around a singular point p_l. We get the following result

Theorem 2. *Let $\bar{g}$ be a complete metric of positive constant scalar curvature conformal to the standard metric g_0 on the manifold $S^n - \Lambda_k$, where $\Lambda_k = \{p_1, p_2, \ldots, p_k\}$, $k > 2$. Then $\bar{g}$ has a harmonic curvature and non-parallel Ricci tensor, except for the standard one.*

Proof. Notice that, this manifold $(S^n - \Lambda_k, \bar{g})$ is locally conformally flat. The condition of constant scalar curvature implies that its Riemannian curvature is harmonic. We get the converse for the dimension $n = 3$.

We will use the asymptotic property given by Lemma 2. More exactly, we have

$$u(t, \xi) = {u_c}^j(t) + e^{-\alpha t} v(t, \xi).$$

We proceed as in the proof of Theorem 1. Consider a local chart (t, ξ) of a point in $S^n - \Lambda_k$. Here is $t = x^0$ and $\xi = (x^1, x^2, ..., x^{n-1})$.

In this local coordinate system, we obtain the components of the tensor metric

$$\bar{g}_{00} = u^{4/(n-2)}, \qquad \bar{g}_{0i} = 0, \qquad \bar{g}_{ij} = u^{4/(n-2)} (g_0)_{ij}.$$

Then we get the first component of the Ricci tensor

$$\bar{\mathcal{R}}_{00} = 2\frac{n-1}{n-2}\left[\frac{1}{u}\frac{\partial^2 u}{\partial t^2} + \frac{1}{u^2}\left(\frac{\partial u}{\partial t}\right)^2\right].$$

D denotes the Riemannian connection associated to the metric $\bar{g}$. By Lemma 2, we may deduce that

$$\frac{\partial u}{\partial x^i} = e^{-\alpha t}\frac{\partial v}{\partial x^i} \quad \text{and} \quad \nabla_{i0}u = -\alpha e^{-\alpha t}\frac{\partial v}{\partial x^i} + e^{-\alpha t}\frac{\partial^2 v}{\partial x^i \partial t}.$$

Recall that $v = v(t, \xi)$ is a bounded function, and α is a positive constant. Thus, we can write $\bar{\mathcal{R}}_{0i} = e^{-\alpha t} f(t, x_1, x_2, \ldots, x_{n-1})$ where f is bounded. Consequently, we may calculate the components of the covariant derivative

$$D_0\bar{\mathcal{R}}_{00} = 2\frac{n-1}{n-2}\frac{u_c'''}{u_c} + 2\frac{n+2}{n-2}\frac{u_c' u_c''}{u_c^2} - 4\frac{{u'}_c^3}{u_c^3} - e^{-\alpha t}g(t,x_1,x_2,\ldots,x_{n-1})$$

where g is a bounded function. Therefore, since t is large, $D_0\bar{\mathcal{R}}_{00} \neq 0$. Hence, the Ricci tensor of the singular Yamabe metrics are not parallel.

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