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Towards a solution of the inverse problem of the calculus of variations for scalar ordinary differential equations[1]

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Abstract. We state a combinatorial conjecture, which, if true, yields a simple solution to the multiplier version of the inverse problem of the calculus of variations. With the help of a computer, we verified our conjecture for $3 \le n \le 100$, and thus found necessary and sufficient conditions for a $2n$th-order scalar ordinary differential equation $\partial^{2n}u/\partial x^{2n} = f(x, u, \partial u/\partial x, \dots, \partial^{2n-1}u/\partial x^{2n-1})$, $3 \le n \le 100$, to admit a variational multiplier.

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0. Introduction

We consider the following problem: Is there a non-degenerate Lagrangian L of order n such that the Euler–Lagrange equations $E(L)$ on the equation manifold of a $2n$th-order scalar ordinary differential equation

$$(1) \qquad \frac{\partial^{2n}u}{\partial x^{2n}} = f\left(x, u, \frac{\partial u}{\partial x}, \dots, \frac{\partial^{2n-1}u}{\partial x^{2n-1}}\right)$$

vanish?

This problem is referred to as *the multiplier version of the inverse problem of the calculus of variations.* Equations satisfying the above condition are called *multiplier variational.* Darboux ([3]) observed that every second-order scalar equation is multiplier variational. The inverse problem for a fourth-order equation was solved

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by Fels, see [4]. The inverse problem for 6 th and 8 th-order equations was solved in ([5]). In this paper we study the inverse problem for higher-order equations. We state a combinatorial conjecture and show that if the conjecture is true it implies a simple solution to the multiplier version of the inverse problem. Using MAPLE, we verified our conjecture for $3 \le n \le 100$, and hence found necessary and sufficient conditions for a $2n$th-order scalar ordinary differential equation

$$\frac{\partial^{2n}u}{\partial x^{2n}} = f\left(x, u, \frac{\partial u}{\partial x}, \dots, \frac{\partial^{2n-1}u}{\partial x^{2n-1}}\right), \qquad 3 \le n \le 100,$$

to admit a variational multiplier. We approach the problem using the methods of the variational bicomplex ([1]). Finally note that our work is based on the general results obtained in ([2]) and ([5]).

1. Formulation of the problem and preliminaries

For a given ordinary differential equation of order $2n$

(2) $$u_{2n} = f(x, u, u_1, u_2, \dots, u_{2n-1}),$$

where $u_1 = \partial u/\partial x, u_2 = \partial^2 u/\partial x^2$, etc., and f is a smooth function, we want to determine whether there is a *non-degenerate* nth-*order Lagrangian* $L = L(x, u, u_1, \dots, u_n)$,

(3) $$\frac{\partial^2 L}{\partial u_n^2} \neq 0 \quad \text{such that} \quad E(L) = 0,$$

where E is the *Euler–Lagrange operator*

(4) $$E = \sum_{i=0}^{n} (-D_x)^i \frac{\partial}{\partial u_i}$$

and where

$$D_x = \frac{\partial}{\partial x} + \sum_{i=0}^{2n-2} u_{i+1} \frac{\partial}{\partial u_i} + f \frac{\partial}{\partial u_{2n-1}},$$

is the *total derivative operator* restricted to the equation manifold. In the formulas above $u_0 = u$. We will use this notation whenever convenient. We also assume that all geometric objects (i.e., functions, forms, vector fields, etc.) in this article are smooth and all our considerations are local. If there is an nth-order Lagrangian satisfying conditions (3), we say that (2) is *(multiplier) variational* or that (2) *admits a variational multiplier.* The cases $n = 1$ and $n = 2$ of the inverse problem require special treatment and were solved in ([3, 4]) as noted in the introduction. Therefore we will assume that $n \ge 3$. For this case we obtained some general results ([5]), which will be recalled at the end of this section.

Equation (2) may be written as the Pfaffian system

$$\theta^0 = du - u_1\,dx = 0,$$
$$\theta^i = du_i - u_{i+1}\,dx = 0, \qquad \text{for} \quad i = 1, \dots, 2n-2,$$
$$\theta^{2n-1} = du_{2n-1} - f\,dx = 0,$$

on the space of variables $(x, u, u_1, \dots, u_{2n-1})$. The forms $\theta^0, \theta^1, \dots, \theta^{2n-1}$ are referred to as *contact forms*. The differential ideal generated by these forms is called *contact ideal.* Let Ω^p denotes the ring of smooth p-forms over the space of variables $(x, u, u_1, \dots, u_{2n-1})$, and let

$$\Omega^* = \Omega^0 \cup \Omega^1 \cup \cdots \cup \Omega^{2n+1}.$$

The *order* of the function $g \in \Omega^0$ is the largest integer $k \geq 0$ such that $\partial g/\partial u_k \neq 0$. We define the *contact order* of θ^i to be i, for $i = 0, \dots, 2n-1$, and the *contact order* of dx to be 0. The non-zero forms

(5) $$g\,\theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_p} \quad \text{and} \quad g\,dx \wedge \theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_{p-1}},$$

where $g \in \Omega^0$ have *contact order* $\sum_{k=1}^{p} i_k$ and $\sum_{k=1}^{p-1} i_k$, respectively. Let $\omega \in \Omega^p$, $p \geq 1$, be a sum of terms of type (5) that are linearly independent over the ring of functions Ω^0. The *contact order* of ω is defined to be the maximum contact order of its terms. Recall that a *classical contact transformation* is a transformation that preserves the contact ideal and the contact order of a form (via pull-back).

We now briefly recall few basic definitions and facts from the theory of the *variational bicomplex* (see [1,2]), that will be needed later in the text. For $\omega \in \Omega^*$, let $D_x\omega$ denotes the Lie derivative of ω with respect to the total vector field D_x. The differential d splits into two parts d_H and d_V called the *horizontal differential* and *vertical differential,* respectively. We define

(6) $$d_H\omega = dx \wedge D_x\omega \quad \text{and} \quad d_V\omega = d\omega - d_H\omega.$$

Both d_H and d_V are linear anti-derivations and ${d_H}^2 = 0$, $d_H d_V + d_V d_H = 0$, ${d_V}^2 = 0$. It is easy to see that $d\omega = 0$ if and only if $d_H\omega = 0$ and $d_V\omega = 0$. For a smooth function $g \in \Omega^0$ we have

$$d_H g = D_x g\,dx \quad \text{and} \quad d_V g = \sum_{i=0}^{2n-1} \frac{\partial g}{\partial u_i}\theta^i.$$

The d_H structure equations for the coframe $(dx, \theta^0, \dots, \theta^{2n-1})$, are given by

$$d_H(dx) = 0, \qquad d_H\theta^i = dx \wedge \theta^{i+1},$$

for $i = 0, \dots, 2n-2$, and

$$d_H\theta^{2n-1} = dx \wedge d_V f.$$

The d_V structure equations of this coframe all vanish. Note that the order of a function $g \in \Omega^0$ is equal to the contact order of $d_V g$.

We now define the two-form Π, introduced in [5]. Π is essentially a candidate for the differential of the Euler–Poincaré form (see Theorem 1 (ii) below). Let

$$A = \frac{1}{n}\frac{\partial f}{\partial u_{2n-1}}, \tag{7}$$

and define the functions $A_{i,j}$, $i = 0, \ldots, n-1$, and $j = i+1, \ldots, 2n-i-1$, recursively by

$$A_{i,2n-i-1} = (-1)^{n-i-1} \tag{8a}$$

for $i = 0, \ldots, n-1$,

$$A_{0,j} = -D_x(A_{0,j+1}) + AA_{0,j+1} + (-1)^n \frac{\partial f}{\partial u_{j+1}} \tag{8b}$$

for $j = 2n-2, \ldots, 1$,

$$A_{i,j} = -D_x(A_{i,j+1}) + AA_{i,j+1} - A_{i-1,j+1}, \tag{8c}$$

for $i = 1, \ldots, n-2$, and $j = 2n-i-2, \ldots, i+1$. We set

$$\Pi = \sum_{i=0}^{n-1}\sum_{j=i+1}^{2n-i-1} A_{i,j}\theta^i \wedge \theta^j.$$

Taking the horizontal differential of Π we arrive at

$$\begin{aligned} d_H\Pi &= dx \wedge \Big(A\Pi - \sum_{i=1}^{n-1} I_i\theta^{i-1}\wedge\theta^i\Big) \\ D_x\Pi &= A\Pi - \sum_{i=1}^{n-1} I_i\theta^{i-1}\wedge\theta^i, \end{aligned} \tag{9}$$

where

$$I_1 = -D_x(A_{0,1}) + AA_{0,1} + (-1)^n\frac{\partial f}{\partial u_1}, \tag{10a}$$

$$I_{i+1} = -D_x(A_{i,i+1}) + AA_{i,i+1} - A_{i-1,i+1}, \qquad i = 1, \ldots, n-2, \tag{10b}$$

The basic general results of ([5]) can be summarized as follows.

Theorem 1. *For $n \geq 3$, the following statements are equivalent:*
(i) *Equation* (2) *is multiplier variational.*
(ii) *There is a function $a \in \Omega^0$, $a \neq 0$, such that $d(a\Pi) = 0$.*
(iii) *$d\Pi = \lambda \wedge \Pi$, for some one-form $\lambda \in \Omega^1$.*

Moreover, provided the above conditions are satisfied, the function a in (ii) *is unique up to a multiplication by a constant, λ in* (iii) *is uniquely determined, and*

$$d(\ln a) = -\lambda. \tag{11}$$

2. The necessary conditions

In this section we assume that Equation (2) is multiplier variational. From Theorem 1 and (9) we derive our *first necessary condition*

$$I_1 = I_2 = \cdots = I_{n-1} = 0.$$

By [2, Theorem 1], it follows that

$$p(t) = f(x, u, u_1, \ldots, u_n, tu_{n+1}, t^2 u_{n+2} \cdots, t^{n-1} u_{2n-1})$$

is a polynomial in t of degree $\leq n$. This is our *second necessary condition.* In particular we have

$$\text{(12)} \qquad \frac{\partial f}{\partial u_{j+1}} \quad \text{is of order} \leq \begin{cases} 3n-1-j & \text{for } n < j \leq 2n-2, \\ 2n-1 & \text{for } j \leq n. \end{cases}$$

Using the induction argument we deduce from equations (12) and (8a–8c) that

$$\text{(13)} \qquad \text{the order of } A_{i,j} \text{ is} \leq \begin{cases} 3n-1-(i+j) & \text{for } n < i+j \leq 2n-2, \\ 2n-1 & \text{for } i+j \leq n. \end{cases}$$

for $0 \leq i \leq n-1$ and $i < j \leq 2n-i-1$. Taking the vertical differential

$$d_V \Pi = \sum_{i=0}^{n-1} \sum_{j=i+1}^{2n-i-1} d_V A_{i,j} \wedge \theta^i \wedge \theta^j,$$

we conclude that $d_V \Pi$ *is of contact order* $\leq 3n-1$. This simple result will later play a key role in our arguments. Notice that to prove this it is not necessary for the Equation (2) to be variational. We only need the fact that $p(t)$ is a polynomial of degree $\leq n$.

By Theorem 1 (ii), there is a function $a \in \Omega^0$, $a \neq 0$, such that $d(a\Pi) = 0$. Thus, $d_V(a\Pi) = 0$, and so

$$d_V \Pi = -d_V(\ln a) \wedge \Pi.$$

The left-hand side of this equation is of contact order $\leq 3n-1$. By Lemma 3 below the one-form $d_V(\ln a)$ is of contact order $\leq n$ and therefore a is of order $\leq n$. Since $d_H(a\Pi) = 0$, then $D_x \Pi = -D_x(\ln a) \wedge \Pi$, and so by (9)

$$D_x(\ln a) = -A.$$

Because a is of order n

$$D_x^\infty(\ln a) = D_x(\ln a) = -A,$$

where D_x^∞ denotes the *total derivative operator* (not restricted to the equation manifold)

$$D_x^\infty = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u_{i+1} \frac{\partial}{\partial u_i}.$$

By a straightforward computation using

$$\left[\frac{\partial}{\partial u}, D_x^\infty\right] = 0$$

and

$$\left[\frac{\partial}{\partial u_i}, D_x^\infty\right] = \frac{\partial}{\partial u_{i-1}}, \qquad \text{for} \quad i = 1, 2, \ldots$$

where $[\,\cdot\,,\cdot\,]$ denotes the Lie bracket, it follows

$$E^\infty(A) = -E^\infty\big(D_x^\infty(\ln a)\big) = 0.$$

Here

$$E^\infty = \sum_{i=0}^{\infty}(-D_x^\infty)^i\left(\frac{\partial}{\partial u_i}\right) \tag{14}$$

denotes the *Euler–Lagrange operator*.

This is our *third necessary condition.* Summarizing we have proved

Theorem 2. *Let $n \geq 3$ and let Equation* (2) *be variational. Then the following three conditions are satisfied:*

(i) *The function $p(t) = f(x, u, u_1, \ldots, u_n, tu_{n+1}, t^2u_{n+2}\cdots, t^{n-1}u_{2n-1})$ is a polynomial in t of degree $\leq n$.*

(ii) $I_1 = \cdots = I_{n-1} = 0$.

(iii) $E^\infty(A) = 0$.

Lemma 3. *Let $n \geq 3$. If $\omega \in \Omega^1$ and $\omega \wedge \Pi$ is a form of contact order $\leq$ $2n + k - 1$, $k \geq 0$, then ω is a one-form of contact order $\leq k$.*

Proof. For convenience denote $\xi_i = \theta^i$ and

$$\xi_{2n-i-1} = \sum_{j=i+1}^{2n-i-1} A_{i,j}\theta^j \quad \text{for} \quad i = 0, 1, \ldots, n-1.$$

Then ξ_i has contact order i for $i = 1, \ldots, 2n-1$, and

$$\Pi = \sum_{i=0}^{n-1} \xi_i \wedge \xi_{2n-i-1}.$$

For $\omega \in \Omega^1$ we write

$$\omega = b\,dx + \sum_{j=0}^{2n-1} a_j\xi_j,$$

for some functions $b, a_i \in \Omega^0$. We have

$$\omega \wedge \Pi = \sum_{i=0}^{n-1} b\,dx \wedge \xi_i \wedge \xi_{2n-i-1} + \sum_{i=0}^{n-1}\sum_{j=0}^{2n-1} a_j\xi_j \wedge \xi_i \wedge \xi_{2n-i-1}.$$

Let $\omega \wedge \Pi$ be of contact order $\leq 2n+k-1$, then

$$\sum_{i=0}^{n-1} \sum_{j=k+1}^{2n-1} a_j \xi_j \wedge \xi_i \wedge \xi_{2n-i-1} = 0.$$

Fix $j_0 > k$ and choose i_0, $0 \leq i_0 \leq n-1$, such that $\xi_{j_0} \wedge \xi_{i_0} \wedge \xi_{2n-i_0-1} \neq 0$. Then

$$\sum_{i=0}^{n-1} \sum_{j=k+1}^{2n-1} a_j \xi_j \wedge \xi_i \wedge \xi_{2n-i-1} \wedge \bigwedge_l \xi_l = \pm a_{j_0} \xi_0 \wedge \cdots \wedge \xi_{2n-1} = 0,$$

where l is taken over all l, $l = 0, \ldots, 2n-1$, $l \neq j_0$, $l \neq i_0$, and $l \neq 2n-i_0-1$. Hence $a_j = 0$ for all $j > k$ and so ω is of contact order $\leq k$. $\square$

3. Necessary = sufficient?

In this section assume the conditions (i), (ii) and (iii) of Theorem 2 are satisfied. In many cases we will show that these conditions are sufficient for the equation (2) to be multiplier variational.

By (iii) $E^\infty(A) = 0$. By the local exactness of the free variational bicomplex, [1, Proposition 4.3], there (locally) exists a function a such that $D_x^\infty(\ln a) = -A$. From (7) and Theorem 2 (i) we deduce that A is of order $\leq n+1$ and so a is of order $\leq n$. Hence

$$D_x(\ln a) = D_x^\infty(\ln a) = -A.$$

From the last equation, Theorem 2 (ii), and (9) follows

$$d_H(a\Pi) = 0. \tag{15}$$

Taking the vertical differential of $a\Pi$ we obtain

$$d_V(a\Pi) = d_V a \wedge \Pi + a\, d_V \Pi.$$

By the results of Section 2, $d_V \Pi$ is of contact order $\leq 3n-1$ and so $d_V(a\Pi)$ is of contact order $\leq 3n-1$. We will show that $d_V(a\Pi) = 0$, provided the following conjecture is satisfied.

Let Ω_t^3, $3 \leq t \leq 6n-6$, be a module over the ring Ω^0 with a basis $\mathcal{B}_t^3 = \{\theta^i \wedge \theta^j \wedge \theta^k;\ 0 \leq i < j < k \leq 2n-1, i+j+k = t\}$. Let $\omega \in \Omega_t^3$,

$$\omega = \sum_{\substack{0 \leq i < j < k \leq 2n-1 \\ i+j+k=t}} g_{i,j,k}\, \theta^i \wedge \theta^j \wedge \theta^k$$

for some functions $g_{i,j,k} \in \Omega^0$. Consider the Lie derivative

$$D_x\omega = \sum_{\substack{0 \leq i < j < k \leq 2n-1 \\ i+j+k=t}} g_{i,j,k}(\theta^{i+1} \wedge \theta^j \wedge \theta^k + \theta^i \wedge \theta^{j+1} \wedge \theta^k + \theta^i \wedge \theta^j \wedge \theta^{k+1})$$
$$+ \{\text{form of contact order} \leq t\},$$

where $\theta^{2n} = 0$. Thus $D_x : \Omega^3_t \to \Omega^3$ defines the linear map $D^{(t)}_x : \Omega^3_t \to \Omega^3_{t+1}$, given by

$$D^{(t)}_x(\theta^i \wedge \theta^j \wedge \theta^k) = \theta^{i+1} \wedge \theta^j \wedge \theta^k + \theta^i \wedge \theta^{j+1} \wedge \theta^k + \theta^i \wedge \theta^j \wedge \theta^{k+1} \tag{16}$$

for $0 \leq i < j < k \leq 2n-1,\ i + j + k = t,$ and here again $\theta^{2n} = 0$.

Conjecture 4. *Let $n \geq 3$. The linear map*

$$D^{(t)}_x : \Omega^3_t \to \Omega^3_{t+1}$$

for $t = 3, \dots, 3n-1$, is an injection.

Using a computer software we verified our conjecture for $n = 3, \dots, 100$.

The proof of the conjecture for $t = 3, \dots, 2n-1$, is simple. Consider the ordering $<$ of the set $B^3 = \{\theta^i \wedge \theta^j \wedge \theta^k;\ 0 \leq i < j < k \leq 2n-1\}$ given by $\theta^{i_1} \wedge \theta^{j_1} \wedge \theta^{k_1} < \theta^{i_2} \wedge \theta^{j_2} \wedge \theta^{k_2}$ if and only if

$$\begin{aligned} &i_1 < i_2 \quad \text{or} \\ &i_1 = i_2, \quad \text{and} \quad j_1 < j_2 \quad \text{or} \\ &i_1 = i_2, \quad j_1 = j_2, \quad \text{and} \quad k_1 < k_2. \end{aligned}$$

This ordering induces an ordering of the basis $\mathcal{B}^3_t$ of Ω^3_t. Assume that the kernel of $D^{(t)}_x$ contains a non-zero element ω,

$$\omega = \sum_{\substack{0 \leq i < j < k \leq 2n-1 \\ i+j+k=t}} a_{i,j,k}\, \theta^i \wedge \theta^j \wedge \theta^k.$$

Let $\theta^{i_0} \wedge \theta^{j_0} \wedge \theta^{k_0}$ be the least element of $\mathcal{B}^3_t$, such that $a_{i_0,j_0,k_0} \neq 0$. Assuming $k_0 < 2n-1$, we obtain

$$\begin{aligned} D^{(t)}_x \omega = {} & a_{i_0,j_0,k_0} \theta^{i_0} \wedge \theta^{j_0} \wedge \theta^{k_0+1} \\ & + \{\text{linear combination of elements of } \mathcal{B}^3_{t+1} \text{ over } \Omega^0 \text{ greater then} \\ & \quad \theta^{i_0} \wedge \theta^{j_0} \wedge \theta^{k_0+1}\}, \end{aligned}$$

and so $a_{i_0,j_0,k_0} = 0$, which is a contradiction. Thus, $k_0 = 2n-1$ and so

$$t = i_0 + j_0 + k_0 > k_0 = 2n-1.$$

We conclude that the conjecture holds for $t = 3, \dots, 2n-1$. □

Now we are ready to conclude the proof of the main result. Let t be the contact order of $d_V(a\Pi)$. Recall that in the beginning of this section we have proved $t \leq 3n-1$.

We write

$$d_V(a\Pi) = \sum_{\substack{0 \leq i < j < k \leq 2n-1 \\ i+j+k \leq t}} a_{i,j,k}\, \theta^i \wedge \theta^j \wedge \theta^k,$$

for some functions $a_{i,j,k} \in \Omega^0$. Since d_H and d_V anticommute, by (15) we have

$$d_H d_V(a\Pi) = -d_V d_H(a\Pi) = 0.$$

On the other hand

$$d_H d_V(a\Pi) = dx \wedge \sum_{\substack{0\le i<j<k\le 2n-1\\ i+j+k=t}} a_{i,j,k}(\theta^{i+1}\wedge\theta^j\wedge\theta^k+\theta^i\wedge\theta^{j+1}\wedge\theta^k+\theta^i\wedge\theta^j\wedge\theta^{k+1})$$

$$+\{\text{ form of contact order} \le t\}$$

$$= dx\wedge D_x^{(t)}\Big(\sum_{\substack{0\le i<j<k\le 2n-1\\ i+j+k=t}} a_{i,j,k}\,\theta^i\wedge\theta^j\wedge\theta^k\Big)$$

$$+\{\text{ form of contact order} \le t\}.$$

In the above formulas, again, $\theta^{2n}=0$. Comparing the last two equations we have

$$D_x^{(t)}\Big(\sum_{\substack{0\le i<j<k\le 2n-1\\ i+j+k=t}} a_{i,j,k}\,\theta^i\wedge\theta^j\wedge\theta^k\Big) = 0.$$

Assume now that Conjecture 4 holds. It follows that $a_{i,j,k}=0$ for all $0\le i<j<k\le 2n-1,\ i+j+k=t$. We conclude that $d_V(a\Pi)$ is of contact order $t-1$. By induction argument on the contact order of $d_V(a\Pi)$ we deduce that

$$d_V(a\Pi) = 0.$$

Using (15) we obtain

$$d(a\Pi) = d_H(a\Pi) + d_V(a\Pi) = 0,$$

and so, by Theorem 1 (ii), is (2) variational. Summarizing we have proved

Theorem 5. *Assume that Conjecture* 4 *is satisfied for some* $n\ge 3$. *Then the equation*

$$u_{2n} = f(x,u,u_1,\dots,u_{2n-1})$$

is variational if and only if the following conditions are satisfied:

(i) *The function* $p(t) = f(x,u,u_1,\dots,u_n,tu_{n+1},t^2u_{n+2}\cdots,t^{n-1}u_{2n-1})$ *is a polynomial in* t *of degree* $\le n$.

(ii) $I_1=\cdots=I_{n-1}=0$.

(iii) $E^\infty(A)=0$.

We know that conditions (i), (ii), (iii), are sufficient for the Equation (2) to be (multiplier) variational for $n=3,\dots,100$. We conjecture that these conditions are sufficient for any $n\ge 3$.

Finally note that in ([5]) we conjectured that the two conditions (i) and (ii) alone are sufficient for equation (2) to be variational and we proved this conjecture for 6 th and 8 th-order equations. We still support this conjecture.

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