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A new geometric proposal for the Hamiltonian description of classical field theories[1]

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Abstract. We consider the geometric formulation of the Hamiltonian formalism for field theory in terms of *Hamiltonian connections* and *multisymplectic forms*. In this framework the covariant Hamilton equations for mechanics and field theory are defined in terms of multisymplectic $(n+2)$-forms, where n is the dimension of the basis manifold, together with connections on the configuration bundle. We provide a new geometric Hamiltonian description of field theory, based on the introduction of a suitable *composite fibered bundle* which plays the role of an *extended configuration bundle*. Instead of fibrations over an n-dimensional base manifold $\boldsymbol{X}$, we consider *fibrations over a line bundle* Θ *fibered over* $\boldsymbol{X}$. The concepts of *extended Legendre bundle, Hamiltonian connection, Hamiltonian form* and *covariant Hamilton equations* are introduced and put in relation with the corresponding standard concepts in the polymomentum approach to field theory.

Keywords. Fiber bundles, jets, connections, Hamilton equations.

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1. Introduction

A geometric formulation of the Hamiltonian formalism for field theory in terms of *Hamiltonian connections* and *multisymplectic forms* was developed in ([19, 20, 21]). We recall that, in this framework, the covariant Hamilton equations for Mechanics and field theory are defined in terms of multisymplectic $(n+2)$-forms, where n is the dimension of the basis manifold, together with connections on the configuration bundle.

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We provide here a new geometric Hamiltonian description of field theory, based on the introduction of a suitable *composite fibered bundle* which plays the role of an *extended configuration bundle*. One of the main features of this approach is that one can describe the polymomenta and other objects appearing in the *polymomentum* formulation of field theory (see, e.g., [2, 7, 9, 10, 13, 15, 16] and references therein) in terms of differential forms with values in the vertical tangent bundle of an appropriate line bundle Θ. The introduction of the line bundle Θ can be here understood as a suitable way of describing the *gauge* character appearing in the Hamiltonian formalism for field theory (see [11] for a nice introduction to this topic). Instead of bundles over an n-dimensional base manifold $\boldsymbol{X}$, we consider *fibrations over a line bundle* Θ *fibered over* $\boldsymbol{X}$; the concepts of *event bundle*, *configuration bundle* and *Legendre bundle* are then introduced following the analogous setting introduced (in [19, 20, 21]) for Mechanics and for the polymomentum approach to field theory. Moreover, *Hamiltonian connections, Hamiltonian forms* and *covariant Hamilton equations* can be suitably described in this framework. This new approach takes into account the existence of more than one independent variable in field theory, but enables us to keep as far as possible most of the nice features of time-dependent Hamiltonian Mechanics.

Already in the seventies, Kijowski stressed the prominent role of the symplectic structures in field theories, [10, 11, 12, 13]. Our approach can provide a suitable geometric interpretation of the canonical theory of gravity and gravitational energy, as presented in ([8]), where the local line bundle coordinate τ plays the role of a *parameter* and enables one to consider the gravitational energy as a '*gravitational charge*'.

In Section 2 we state the general framework of composite fiber bundles, their jet prolongations and composite connections. Section 3 contains the main result of this note, i.e., Theorem 3.11, which relates the *abstract Hamiltonian dynamics* introduced here with the standard Hamilton–De Donder equations (see [15, 16] for a detailed review on the topic and recent developments). Furthermore, since it stresses the underlying algebraic structure of field theory, this 'extended' approach turns out to be very promising, e.g., *via* the application of some results concerned with a new K-theory for vector bundles carrying this special kind of multisymplectic structure (see [24]). However, this topic will be developed elsewhere.

2. Jets of fibered manifolds and connections

The general framework is a fibered bundle $\pi : \boldsymbol{Y} \to \boldsymbol{X}$, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n + m$ and, for $r \geq 0$, its jet manifold $J_r\boldsymbol{Y}$. We recall the natural fiber bundles $\pi^r_s : J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$, $r \geq s$, $\pi^r : J_r\boldsymbol{Y} \to \boldsymbol{X}$, and, among these, the *affine* fiber bundles π^r_{r-1}. We denote by $V\boldsymbol{Y}$ the vector subbundle of the tangent bundle $T\boldsymbol{Y}$ formed by vectors on $\boldsymbol{Y}$ which are vertical with respect to the fibering π (see, e.g., [22]).

Greek indices, $\lambda, \mu, \ldots$, run from 1 to n and they label base coordinates, while Latin indices, $i, j, \ldots$, run from 1 to m and label fiber coordinates, unless otherwise specified. We denote multi-indices of dimension n by boldface Greek letters such as $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, with $0 \le \alpha_\mu$, $\mu = 1, \dots, n$; by an abuse of notation, we denote with λ the multi-index such that $\alpha_\mu = 0$ if $\mu \neq \lambda$, $\alpha_\mu = 1$ if $\mu = \lambda$. We also set $|\boldsymbol{\alpha}| := \alpha_1 + \dots + \alpha_n$. The charts induced on $J_r\boldsymbol{Y}$ are denoted by $(x^\lambda, y^i_{\boldsymbol{\alpha}})$, with $0 \le |\boldsymbol{\alpha}| \le r$; in particular, we set $y^i_{\boldsymbol{0}} \equiv y^i$. The local bases of vector fields and 1-forms on $J_r\boldsymbol{Y}$ induced by the coordinates above are denoted by $(\partial_\lambda, \partial^{\boldsymbol{\alpha}}_i)$ and $(d^\lambda, d^i_{\boldsymbol{\alpha}})$, respectively.

The *contact maps* on jet spaces, [17], induce the natural complementary fibered morphisms over the affine fiber bundle $J_r\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y}$

$$\text{Д}_r \colon J_r\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X} \to TJ_{r-1}\boldsymbol{Y}, \tag{1}$$
$$\vartheta_r : J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} TJ_{r-1}\boldsymbol{Y} \to VJ_{r-1}\boldsymbol{Y},$$

$r \ge 1$, with coordinate expressions, for $0 \le |\boldsymbol{\alpha}| \le r-1$, given by

$$\text{Д}_r = d^\lambda \otimes \text{Д}_\lambda = d^\lambda \otimes (\partial_\lambda + y^j_{\boldsymbol{\alpha}+\lambda}\partial^{\boldsymbol{\alpha}}_j), \tag{2}$$
$$\vartheta_r = \vartheta^j_{\boldsymbol{\alpha}} \otimes \partial^{\boldsymbol{\alpha}}_j = (d^j_{\boldsymbol{\alpha}} - y^j_{\boldsymbol{\alpha}+\lambda} d^\lambda) \otimes \partial^{\boldsymbol{\alpha}}_j,$$

and the natural fibered splitting, [17, 19, 22],

$$J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y} = J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} (T^*\boldsymbol{X} \oplus V^*J_{r-1}\boldsymbol{Y}). \tag{3}$$

Let us consider the following dual exact sequences of vector bundles over $\boldsymbol{Y}$:

$$0 \to V\boldsymbol{Y} \hookrightarrow T\boldsymbol{Y} \to \boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X} \to 0, \tag{4}$$
$$0 \to \boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T^*\boldsymbol{X} \hookrightarrow T^*\boldsymbol{Y} \to V^*\boldsymbol{Y} \to 0\,.$$

Definition 2.1. A *connection* on the fiber bundle $\boldsymbol{Y} \to \boldsymbol{X}$ is defined by the dual linear bundle morphisms over $\boldsymbol{Y}$

$$\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X} \to T\boldsymbol{Y}, \qquad V^*\boldsymbol{Y} \to T^*\boldsymbol{Y} \tag{5}$$

which split the exact sequences (4).

Remark 2.2. Let Γ^i_λ be the local components of the connection Γ. The above linear morphisms over $\boldsymbol{Y}$ yield uniquely a horizontal tangent-valued 1-form on $\boldsymbol{Y}$, which we denote by $\Gamma = d^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i)$ and which projects over the soldering form on $\boldsymbol{X}$. Dually, a connection Γ on $\boldsymbol{Y}$ can be also represented by the vertical-valued 1-form $\Gamma = (d^i - \Gamma^i_\lambda d^\lambda) \otimes \partial_i$, see [19]. Taking this into account, the canonical splitting (3) provides the horizontal splitting

$$J_1\boldsymbol{Y} \underset{\boldsymbol{Y}}{\times} T\boldsymbol{Y} \simeq J_1\boldsymbol{Y} \underset{\boldsymbol{Y}}{\times} (V\boldsymbol{Y} \underset{\boldsymbol{Y}}{\oplus} H\boldsymbol{Y}).$$

Proposition 2.3 ([5, 22]). *There is a one-to-one correspondence between the connections Γ on a fiber bundle $\boldsymbol{Y} \to \boldsymbol{X}$ and the global sections $\Gamma : \boldsymbol{Y} \to J_1\boldsymbol{Y}$ of the affine jet bundle $J_1\boldsymbol{Y} \to \boldsymbol{Y}$.*

In the following a relevant role is played by the composition of fiber bundles

$$Y \to \Theta \to X, \tag{6}$$

where $\pi_{YX} : Y \to X$, $\pi_{Y\Theta} : Y \to \Theta$ and $\pi_{\Theta X} : \Theta \to X$ are fiber bundles. The above composition was introduced under the name of *composite fiber bundle*, in [5, 18, 20], and shown to be useful for physical applications, e.g., for the description of mechanical systems with time-dependent parameters. We recall some structural properties of composite fiber bundles, [19].

Proposition 2.4. *Given a composite fiber bundle* (6)*, let h be a global section of the fiber bundle $\pi_{\Theta X}$. Then the restriction $Y_h := h^*Y$ of the fiber bundle $\pi_{Y\Theta}$ to $h(X) \subset \Theta$ is a subbundle $i_h : Y_h \hookrightarrow Y$ of the fiber bundle $Y \to X$.*

Proposition 2.5. *Given a section h of the fiber bundle $\pi_{\Theta X}$ and a section s_Θ of the fiber bundle $\pi_{Y\Theta}$ their composition $s = h \circ s_\Theta$ is a section of the composite bundle $Y \to X$. Conversely, every section s of the fiber bundle $Y \to X$ is the composition of the section $h = \pi_{Y\Theta} \circ s$ of $\pi_{\Theta X}$ and some section s_Θ of $\pi_{Y\Theta}$ over the closed submanifold $h(X) \subset \Theta$.*

2.1. Connections on composite bundles

We shall be concerned here with the description of connections on composite fiber bundles. We will follow the notation and main results stated in ([19]); see also ([1]).

We shall denote by $J_1\Theta$, $J_1^\Theta Y$ and $J_1 Y$, the jet manifolds of the fiber bundles $\Theta \to X$, $Y \to \Theta$ and $Y \to X$ respectively.

Let γ be a connection on the composite bundle π_{YX} projectable over a connection Γ on $\pi_{\Theta X}$, i.e., such $J_1\pi_{Y\Theta} \circ \gamma = \Gamma \circ \pi_{Y\Theta}$. Let $\mathfrak{H}_\Theta$ be a connection on the fiber bundle $\pi_{Y\Theta}$. Given a connection Γ on $\pi_{\Theta X}$, there exists a canonical morphism over Y, [19, 22],

$$\rho : J_1\Theta \underset{X}{\times} J_1^\Theta Y \to J_1 Y,$$

which sends $(\Gamma, \mathfrak{H}_\Theta)$, into the *composite connection* $\gamma := \mathfrak{H}_\Theta \circ \Gamma$ on π_{YX}, projectable over Γ.

Remark 2.6. Let h be a section of $\pi_{\Theta X}$. Every connection $\mathfrak{H}_\Theta$ induces the pullback connection $\mathfrak{H}_h$ on the subbundle $Y_h \to X$. We recall that the composite connection $\gamma = \mathfrak{H}_\Theta \circ \Gamma$ is reducible to $\mathfrak{H}_h$ if and only if h is an integral section of Γ.

We have the following exact sequences of *vector bundles over a composite bundle Y*:

$$\begin{aligned} &0 \to V_\Theta Y \hookrightarrow VY \to Y \underset{\Theta}{\times} V\Theta \to 0, \\ &0 \to Y \underset{\Theta}{\times} V^*\Theta \hookrightarrow V^*Y \to V^*_\Theta Y \to 0, \end{aligned} \tag{7}$$

where $V_\Theta \boldsymbol{Y}$ and $V^*_\Theta \boldsymbol{Y}$ are the vertical tangent and cotangent bundles to the bundle $\pi_{\boldsymbol{Y}\Theta}$.

Remark 2.7. Every connection $\mathfrak{H}_\Theta$ on $\pi_{\boldsymbol{Y}\Theta}$ provides the dual splittings

$$VY = V_\Theta \boldsymbol{Y} \underset{\boldsymbol{Y}}{\oplus} \mathfrak{H}_\Theta(\boldsymbol{Y} \underset{\Theta}{\times} V\Theta), \qquad V^*Y = \boldsymbol{Y} \underset{\Theta}{\times} V^*\Theta \underset{\boldsymbol{Y}}{\oplus} \mathfrak{H}_\Theta(V^*_\Theta \boldsymbol{Y}), \tag{8}$$

of the above exact sequences.

By means of these splittings we can construct the *vertical covariant differential* on the composite bundle $\pi_{\boldsymbol{YX}}$, i.e., the first order differential operator

$$\Delta_{\mathfrak{H}} : J_1\boldsymbol{Y} \to T^*\boldsymbol{X} \underset{\boldsymbol{Y}}{\oplus} V^*_\Theta \boldsymbol{Y}\,. \tag{9}$$

The restriction of $\Delta_{\mathfrak{H}}$, induced by a section h of $\pi_{\Theta\boldsymbol{X}}$, coincides with the covariant differential on $\boldsymbol{Y}_h$ relative to the pull-back connection $\mathfrak{H}_h$, [19].

3. Hamiltonian formalism for field theory

We recall now that the covariant Hamiltonian field theory can be conveniently formulated in terms of Hamiltonian connections and Hamiltonian forms, [5, 20]. Here we shall construct a Hamiltonian formalism for field theory as a theory on the composite bundle $\boldsymbol{Y} \to \Theta \to \boldsymbol{X}$, with $\pi_{\Theta\boldsymbol{X}} : \Theta \to \boldsymbol{X}$ a *line bundle* having local fibered coordinates (x^λ, τ).

Let us now consider the *extended Legendre bundle*

$$\Pi_\Theta := V^*\boldsymbol{Y} \wedge \Big(\overset{n-1}{\bigwedge} T^*\Theta\Big) \to \boldsymbol{X}.$$

There exists the canonical isomorphism

$$\Pi_\Theta \simeq \overset{n}{\bigwedge} T^*\Theta \underset{\boldsymbol{Y}}{\otimes} V^*\boldsymbol{Y} \underset{\boldsymbol{Y}}{\otimes} T\Theta\,. \tag{10}$$

Definition 3.1. We call the fiber bundle $\pi_{\boldsymbol{Y}\Theta} : \boldsymbol{Y} \to \Theta$ the *abstract event space* of the field theory. The *configuration space* of the field theory is then the first order jet manifold $J^\Theta_1 \boldsymbol{Y}$.

The *abstract Legendre bundle* of the field theory is the fiber bundle $\Pi_\Theta \to \Theta$.

Let now $\mathfrak{H}_\Theta$ be a connection on $\pi_{\boldsymbol{Y}\Theta}$ and Γ_Θ be a connection on $\pi_{\Theta\boldsymbol{X}}$. We have the following non-canonical isomorphism

$$\Pi_\Theta \simeq_{(\mathfrak{H}_\Theta, \Gamma_\Theta)} \overset{n}{\bigwedge} T^*\Theta \underset{\boldsymbol{Y}}{\otimes} \big[(\boldsymbol{Y} \underset{\Theta}{\oplus} V^*\Theta) \underset{\boldsymbol{Y}}{\oplus} \mathfrak{H}_\Theta(V^*_\Theta \boldsymbol{Y})\big] \underset{\boldsymbol{Y}}{\otimes} (V\Theta \underset{\Theta}{\oplus} H\Theta). \tag{11}$$

In this perspective, we consider the canonical bundle monomorphism over $\boldsymbol{Y}$ providing the tangent-valued Liouville form on Π_Θ, i.e.,

$$\vartheta_{\boldsymbol{Y}} : \Pi_\Theta \hookrightarrow \overset{n+1}{\bigwedge} T^*\boldsymbol{Y} \underset{\boldsymbol{Y}}{\otimes} (V\Theta \underset{\Theta}{\oplus} H\Theta), \tag{12}$$

the coordinate expression of which is

$$\vartheta_{\boldsymbol{Y}} = p^{\lambda}_{i} d^{i} \wedge \omega \otimes \partial_{\lambda} \otimes \partial_{\tau} \simeq p^{\lambda}_{i} \vartheta^{i} \wedge \omega_{\lambda} \otimes \partial_{\tau} , \tag{13}$$

where ϑ^i are generators of vertical 1-forms (i.e., contact forms) on $\boldsymbol{Y}$ and '$\simeq$' is the isomorphism defined by (11).

The polysymplectic form $\Omega_{\boldsymbol{Y}}$ on Π_{Θ} is then intrinsically defined by

$$\Omega_{\boldsymbol{Y}} \rfloor \psi = d(\vartheta_{\boldsymbol{Y}} \rfloor \psi) ,$$

where ψ is an arbitrary 1-form on Θ; its coordinate expression is given by

$$\Omega_{\boldsymbol{Y}} = dp^{\lambda}_{i} \wedge d^{i} \wedge \omega \otimes \partial_{\lambda} \otimes \partial_{\tau} \simeq dp^{\lambda}_{i} \wedge \vartheta^{i} \wedge \omega_{\lambda} \otimes \partial_{\tau} \, . \tag{14}$$

Remark 3.2. The polysymplectic form (14) is related to the kind of *'special' multisymplectic structures* on vector bundles studied from a topological point of view in ([24]).

Let $J_1\Pi_{\Theta}$ be the first order jet manifold of the extended Legendre bundle $\Pi_{\Theta} \to \boldsymbol{X}$. By Proposition 2.3 a connection γ on the extended Legendre bundle is in one-to-one correspondence with global sections of the affine bundle $J_1\Pi_{\Theta} \to \Pi_{\Theta}$.

Definition 3.3. A connection γ on the extended Legendre bundle Π_{Θ} is said to be a *Hamiltonian connection* iff the exterior form $\gamma \rfloor \Omega_{\boldsymbol{Y}}$ is closed.

As a straightforward application of the relative Poincaré lemma we have then, [19], the following.

Proposition 3.4. *Let γ be a Hamiltonian connection on Π_{Θ} and $\boldsymbol{U}$ be an open subset of Π_{Θ}. Locally, we have*

$$\gamma \rfloor \Omega_{\boldsymbol{Y}} = dp^{\lambda}_{i} \wedge \vartheta^{i} \wedge \omega_{\lambda} \otimes \partial_{\tau} - d\mathcal{H} \wedge \omega := dH, \tag{15}$$

where $\mathcal{H} : \boldsymbol{U} \subset \Pi_{\Theta} \to V\Theta$.

Definition 3.5. The local mapping $\mathcal{H} : \boldsymbol{U} \subset \Pi_{\Theta} \to V\Theta$ is called a *Hamiltonian*. The form H on the extended Legendre bundle Π_{Θ} is called a *Hamiltonian form*.

Every Hamiltonian form H admits a Hamiltonian connection γ_H such that

$$\gamma_H \rfloor \Omega_{\boldsymbol{Y}} = dH \, . \tag{16}$$

Let now set $\bar{p}^{\lambda}_{i} := p^{\lambda}_{i}\partial_{\tau}$. Then the Hamiltonian form H is the Poincaré–Cartan form of the *Lagrangian* $L_H = (\bar{p}^{\lambda}_{i} y^{i}_{\lambda} - \mathcal{H})\omega$ on $J_1\Pi_{\Theta}$, with values in $V\Theta$.

Definition 3.6. The *Hamilton operator* for H is defined as the Euler–Lagrange operator associated with L_H, namely:

$$\mathcal{E}_H : J_1\Pi_{\Theta} \to T^*\Pi_{\Theta} \wedge \bigwedge^{n} T^*\boldsymbol{X} \, . \tag{17}$$

The kernel of the Hamilton operator (17), i.e., the Euler–Lagrange equations for

L_H, is an affine closed embedded subbundle of $J_1\Pi_\Theta \to \Pi_\Theta$, locally given by the equations

$$y^i_\lambda = \partial^\lambda_i \mathcal{H}, \qquad \bar{p}^\lambda_{\lambda\, i} = -\partial_i \mathcal{H}\,. \tag{18}$$

Definition 3.7. The kernel of Hamilton operator defines the *covariant Hamilton equations* (18) on the extended Legendre bundle $\Pi_\Theta \to \boldsymbol{X}$.

Remark 3.8. Notice that a global section of

$$\ker \mathcal{E}_H \to \Pi_\Theta$$

is a Hamiltonian connection γ_H satisfying relation (16).

In the sequel we state the main result of this note, which points out the relation with the standard polysymplectic approach (for a review of the topic see, e.g., [2, 7, 9, 10, 13, 15, 16] and references quoted therein). The basic idea is that the present geometric formulation can be interpreted as a suitable generalization to field theory of the so-called *homogeneous formalism* for mechanics.

Let $\Delta_{\tilde{\mathfrak{H}}}$ be the vertical covariant differential, see (9), relative to the connection $\tilde{\mathfrak{H}}_\Theta$ on the abstract Legendre bundle $\Pi_\Theta \to \Theta$.

Definition 3.9. We define the *abstract covariant Hamilton equations* to be the kernel of the first order differential operator $\Delta_{\tilde{\mathfrak{H}}}$.

Lemma 3.10. *Let γ_H be a Hamiltonian connection on $\Pi_\Theta \to \boldsymbol{X}$. Let $\tilde{\mathfrak{H}}_\Theta$ and Γ be connections on $\Pi_\Theta \to \boldsymbol{Y}$ and $\Theta \to \boldsymbol{X}$, respectively. Let σ and h be sections of the bundles $\pi_{\boldsymbol{Y}\Theta}$ and $\pi_{\Theta\boldsymbol{X}}$, respectively.*

Then the standard Hamiltonian connection on $\Pi_\Theta \to \boldsymbol{X}$ turns out to be the pull-back connection $\tilde{\mathfrak{H}}_\phi$ induced on the subbundle $\Pi_{\Theta\phi} \hookrightarrow \Pi_\Theta \to \boldsymbol{X}$ by the section $\phi = h \circ \sigma$ of $\boldsymbol{Y} \to \boldsymbol{X}$.

Proof. The abstract Legendre bundle is in fact a composite bundle $\Pi_\Theta \to \boldsymbol{Y} \to \Theta$, so that it is possible to apply the results concerning connections on composite bundles recalled in Subsection 2.1, for any connection $\tilde{\mathfrak{H}}_\Theta$.

Our claim then follows for any section ϕ of the composite bundle $\boldsymbol{Y} \to \Theta \to \boldsymbol{X}$ of the type $\phi = h \circ \sigma$, since the extended Legendre bundle $\Pi_\Theta \to \boldsymbol{X}$ can be also seen as the composite bundle $\Pi_\Theta \to \boldsymbol{Y} \to \boldsymbol{X}$. □

We can then state our main result as follows.

Theorem 3.11. *Let $\Delta_{\tilde{\mathfrak{H}},\phi}$ be the covariant differential on the subbundle*

$$\Pi_{\Theta\phi} \hookrightarrow \Pi_\Theta \to \boldsymbol{X}$$

relative to the pull-back connection $\tilde{\mathfrak{H}}_\phi$.

The kernel of $\Delta_{\tilde{\mathfrak{H}},\phi}$ coincides with the Hamilton–De Donder equations of the standard polysymplectic approach to field theories.

Proof. It is a straightforward consequence of Lemma 3.10 together with Definitions 3.7 and 3.9. □

Remark 3.12. Our approach provides a suitable geometric interpretation of the canonical theory of gravity and gravitational energy, as presented in ([8]), where τ plays the role of a *parameter* and enables one to consider the gravitational energy as a '*gravitational charge*'. This topic is currently under investigation and it will be developed in a separate forthcoming paper.

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