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# Euler's superequations[1]

Valentin Gabriel Cristea

**Abstract.** Let $L$ define a regular problem in the calculus of variations on a supermanifold. A necessary condition for a piecewise superdifferentiable supercurve $C$ in the sense of Rogers be a week local minimum for $L$ is that $C$ be superdifferentiable and $\widetilde{C}$ be an integral supercurve of $X$, where $X$ is defined by $X \lrcorner\ d\sigma = 0$, $\omega = dL$, $\langle X, dt\rangle = 1$, the superform $\sigma = \mathcal{L}^*\omega$ is defined on $T(M) \times B_L$ and $\mathcal{L}$ is an immersion of $T(M) \times B_L$ into $T^*(M) \times B_L$, (a Legendre supertransformation).

**Keywords.** Supervector space, supermanifold, superbundle, superhamiltonian, Legendre supertransformation.

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Let $V$ be a supervector space (as in [3]), $V^*$ be the dual supervector space (as in [4]), $M$ be a supermanifold in the sense of Rogers (see [6]) and $T(M)$ be the tangent superspace or superbundle ([4]) over $M$.

Let us consider only algebras over the real numbers. For each positive integer $L$, $B_L$ (see [6]) will denote the Grassmann algebra over the reals with generators $1^L, \beta_1^L, \dots, \beta_L^L$ and relations

$$1^L \cdot \beta_i^L = \beta_i^L \cdot 1^L = \beta_i^L, \qquad i = 1, \dots, L,$$

$$\beta_i^L \cdot \beta_j^L = -\beta_j^L \cdot \beta_i^L, \qquad i, j = 1, \dots, L.$$

$B_L$ is a graded algebra ([7]) and can be written as a direct sum

$$B_L = (B_L)_0 \oplus (B_L)_1,$$

where $(B_L)_0$ and $(B_L)_1$ are the even and odd part of $(B_L)$ respectively (see [6]). We consider the $(m, n)$-dimensional supereuclidean space as $B_L^{m,n} = (B_L)_0^m \oplus (B_L)_1^n$, see [6], with $L > n$. Let $M_L$ denote (due to Kostant [5]) the set of finite sequences of positive integers $\mu = (\mu_1, \dots, \mu_k)$ with $1 \le \mu_1 < \dots < \mu_k \le L$.

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

$M_L$ includes the sequence with no elements, denoted $\phi$. As it follows in [5] for each $\mu$ in $M_L$,

$$\beta_\mu^{(L)} = \beta_{\mu_1}^{(L)} \cdots \beta_{\mu_k}^{(L)}, \qquad k = 1, \ldots, L$$

and $\beta_\phi^{(L)} = 1^{(L)}$. A typical element $b$ of $B_L$ may be expressed as

$$b = \sum_{\mu \in M_L} b^\mu \cdot \beta_\mu^{(L)},$$

where the coefficients $b^\mu$ are real numbers. With the norm on $B_L$ defined by

$$\|b\| = \sum_{\mu \in M_L} |b^\mu|,$$

$B_L$ is a Banach algebra, [6].

We consider the body map (in DeWitt's terminology, see [4]) $\varepsilon_L : B_L \to \mathbb{R}$ given by $\varepsilon_L(b) = b^\phi$.

For the first time, I have introduced, see [3], $B_L^{m+n} = B_L^{m,n} \oplus B_L^{n,m}$ with $n = 2r$, the scalar product

$$\langle v, w\rangle = x^1 \cdot y^1 + \cdots + x^m \cdot y^m + \theta^1 \cdot \theta'^{r+1} + \cdots + \theta^r \cdot \theta'^n - \theta^{r+1} \cdot \theta'^1 - \cdots - \theta^n \cdot \theta'^r$$

for every $v = (x^1, \ldots, x^m, \theta^1, \ldots, \theta^n)$, $w = (y^1, \ldots, y^m, \theta'^1, \ldots, \theta'^n) \in B_L^{m+n}$.

**Definition 1** ([6]). A function $f : B_L^{m,n} \to B_L$ is called a superdifferentiable function if there exist $f_\mu \in C^\infty(\mathbb{R}^m, \mathbb{R})$ such that:

$$f(x, \theta) = \sum_{\mu \in M_n} f_\mu(x)\theta^\mu,$$

where $M_n = \{(\mu_1, \ldots, \mu_n) \mid 1 \leq \mu_1 < \cdots < \mu_n \leq n\}$, see [5].

Let $M$ be a Hausdorff topological space, [6].

(a) An $(m, n)$-chart on $M$ over $B_L$ is a pair $(U, \psi)$ with $U$ an open set of $M$ and $\psi$ a homeomorphism of $U$ onto an open subset of ${B_L}^{m,n}$.

(b) An $(m, n)$-superdifferentiable structure on $M$ over $B_L$ is a collection of $(m, n)$-charts

$$\left\{(U_\alpha, \psi_\alpha)\middle|\alpha \in \Lambda\right\}$$

on $M$ such that $(i)$ $M = \cup_{\alpha\in\Lambda} U_\alpha$ and $(ii)$ for each pair $\alpha, \beta \in \Lambda$ the mapping $\psi_\beta \circ \psi_\alpha^{-1}$ is a superdifferentiable function of $\psi_\alpha(U_\alpha \cap U_\beta)$ onto $\psi_\beta(U_\alpha \cap U_\beta)$, $(iii)$ the collection $\{(U_\alpha, \psi_\alpha) \mid \alpha \in \Lambda\}$ is a maximal collection of open charts for which $(i)$ and $(ii)$ hold.

**Definition 2** ([6]). An $(m, n)$-dimensional superdifferentiable supermanifold over $B_L$ is a Hausdorff topological space $M$ with an $(m, n)$-superdifferentiable structure over $B_L$.

**Definition 3.** The function $C : [a, b] \to M$ is called a superdifferentiable supercurve, see [3], if the functions $x^i \circ C$, for every $1 \le i \le m$ and $\theta^\alpha \circ C$ for every $1 \le \theta \le n$ are superdifferentiable, see [6], the functions $\varepsilon_L \circ x^i \circ C$ for every $1 \le i \le m$ and $\varepsilon_L \circ \theta^\alpha \circ C$ for every $1 \le \alpha \le n$ are differentiable in $\mathbb{R}$ and $(x^i, \theta^\alpha)$ are the coordinates of a point $p \in M$.

**Definition 4.** Let $L$ be a superdifferentiable function on $T(M) \times B_L$ and we make distinction between this superdifferentiable function $L$ and the positive integer $L$. Then $L$ defines a superdifferentiable map $\mathcal{L} : T(M) \times B_L \to T^*(M) \times B_L$ called the Legendre supertransformation, given in local coordinates by

$$\begin{aligned} &x^i \circ \mathcal{L} = x^i \quad \forall 1 \le i \le m, && \theta^\alpha \circ \mathcal{L} = \theta^\alpha \quad \forall 1 \le \alpha \le n, \\ &y^i \circ \mathcal{L} = \frac{\partial L}{\partial x^i} \quad \forall 1 \le i \le m, && \delta^\alpha \circ \mathcal{L} = \frac{\partial L}{\partial \theta^\alpha} \quad \forall 1 \le \alpha \le n, \\ &t \circ \mathcal{L} = t. \end{aligned}$$

**Definition 5.** If the Legendre supertransformation is an immersion, cf. [4], of $T(M) \times B_L$ into $T^*(M) \times B_L$ , then the function $L$ will be called a regular super-Lagrangian.

**Definition 6.** If the Legendre supertransformation is an immersion, the map $\mathcal{L}^{-1}$ comes locally in a similar way from a function $H$ on $T^*(M) \times B_L$ called the super-Hamiltonian:

$$H(y, \delta) = \left\langle \mathcal{L}^{-1}(y, \delta), (y, \delta) \right\rangle - L \circ \mathcal{L}^{-1}(y, \delta).$$

The function $E = H \circ \mathcal{L}$ is globally well defined on $T(M) \times B_L$.

**Theorem 7.** *If $M$ is a Riemann supermanifold,* [4]*, and*

$$L(v, t) = \tfrac{1}{2} \cdot \langle v, v \rangle$$

*then $L$ is a regular super-Lagrangian and $\mathcal{L}$ coincides on each tangent superspace,* [4]*, with the map of $T_q(M) \to T_q^*(M)$ given by the scalar product, see* [3]*. Furthermore, in this case,*

$$E = H \circ \mathcal{L} = L.$$

**Definition 8.** Let $C : [a, b] \to M$ be a superdifferentiable supercurve on $M$. Then $C$ determines a supercurve, $\widetilde{C}$ on $T(M) \times B_L$ defined by $\widetilde{C}(t) = (C'(t), t)$ for each $t \in [a, b]$. Therefore, we can consider the integral

$$I(C) = \int_a^b L(\widetilde{C}(t))\, dt.$$

Let $C_j$ and $C_j^1$ be the restrictions of $C$, $C^1$ respectively, to the interval $[s_j, s_{j+1}]$, where $a = s_0 < \cdots < s_r = b$, $W \subset M$ $C_j$ and $C_j^1$ be superdifferentiable supercurves of $(s_j + \varepsilon, s_{j+1} - \varepsilon)$ into $W$.

**Definition 9.** A supercurve $C$ is called weak local minimum if there are $W$ and $\varepsilon > 0$ such that $\varepsilon_L(I(C)) \leq \varepsilon_L(I(C^1))$ for all piecewise superdifferentiable supercurves satisfying

$$C^1(a) = C(a) \quad \text{and} \quad C^1(b) = C(b).$$

**Proposition 10.** *Let $C$ be a weak local minimum of $L$. Then at every point $t$ where $C$ is differentiable the tangent supervector $Y_q = C'(t)$ satisfies*

$$(1) \qquad Y_q \lrcorner\, d\omega_q = 0$$

*for $\theta^\alpha(t) = t \cdot (\bar{\delta}^\alpha(t) + \bar{\delta}^{\alpha+r}(t))$ and $\theta^{\alpha+r}(t) = t \cdot (\bar{\delta}^{\alpha+r}(t) - \bar{\delta}^\alpha(t))$ for every $\alpha \in \{1, \dots, r\}$ and $(y, \bar{\delta})$ are coordinates on $({B_L}^{m+n})^*$ where*

$$e_i \lrcorner\, e^{j_1} \wedge \cdots \wedge e^{j_r} = \begin{cases} 0 & \text{if } i \neq j_k \text{ for any } k \\ (-1)^{(i)+k-1} \cdot e^{j_1} \wedge \cdots \wedge e^{j_{k-1}} \wedge e^{j_{k+1}} \wedge \cdots \wedge e^{j_r} & \text{if } i = j_k \end{cases}$$

*and $(i)$ is 0 if $e_i \in {B_L}^{m,n}$ and is 1 if $e_i \in {B_L}^{n,m}$ and where $(e_i)_{i=1,\dots,m+n}$ is a basis of ${B_L}^{m+n}$ and $(e^j)_{j=1,\dots,m+n}$ is a basis of $(B_L^{m+n})^*$.*

**Proof.** One can use Lemma 2.2 from Sternberg, [8], and setting

$$H(y, \delta) = \sum_{i=1}^{m} \dot{x}^i \cdot y^i + \sum_{\alpha=1}^{r} (\dot{\theta}^\alpha \cdot \delta^{\alpha+r} - \dot{\theta}^{\alpha+r} \cdot \delta^\alpha)$$
$$- L(\dot{x}^1, \dots, \dot{x}^m, \dot{\theta}^1, \dots, \dot{\theta}^n)$$

the superform $d\omega$ has the expression

$$d\omega = \sum_{i=1}^{m} dy^i \wedge dx^i + \sum_{\alpha=1}^{n} d\delta^\alpha \wedge d\theta^\alpha - dH \wedge dt$$

where $\omega = dL$. From the relation (2.12) from Sternberg [8]we get

$$(2) \qquad \dot{\theta}^\alpha(t) \cdot \left(\bar{\delta}^\alpha(t) + \bar{\delta}^{\alpha+r}(t)\right) + \dot{\theta}^{\alpha+r}(t) \cdot \left(\bar{\delta}^{\alpha+r}(t) - \bar{\delta}^\alpha(t)\right)$$

for all $\alpha \in \{1, \dots, r\}$ and $(y, \bar{\delta})$ are coordinates on $({B_L}^{m+n})^*$. From the relation (2) we obtain $\theta^\alpha(t) = t \cdot (\bar{\delta}^\alpha(t) + \bar{\delta}^{\alpha+r}(t))$ and $\theta^{\alpha+r}(t) = t \cdot (\bar{\delta}^{\alpha+r}(t) - \bar{\delta}^\alpha(t))$ for all $\alpha \in \{1, \dots, r\}$. Thus we proved

$$Y_q \lrcorner\, d\omega_q = 0$$

for $\theta^\alpha(t) = t \cdot (\bar{\delta}^\alpha(t) + \bar{\delta}^{\alpha+r}(t))$ and $\theta^{\alpha+r}(t) = t \cdot (\bar{\delta}^{\alpha+r}(t) - \bar{\delta}^\alpha(t))$ for all $\alpha \in \{1, \dots, r\}$ and $(y, \bar{\delta})$ are coordinates on $({B_L}^{m+n})^*$. □

**Lemma 11.** *If $C$ is a local weak minimum then $C$ is superdifferentiable.*

**Proof.** One can use [8, Lemma 2.3] for each curve $C_0(t), C_1(t), \dots, C_{12\cdots n}$ from a piecewise superdifferentiable supercurve

$$C(t) = C_0(t) + C_1(t)\theta^1 + \cdots + C_{12\cdots n}(t) \cdot \theta^1 \cdots \theta^n. \quad \square$$

**Definition 12.** Given $(m, n)$-dimensional supermanifold $M$, a supervector field $X$ on $T(M) \times B_L$ is called Euler supervector field on $T(M) \times B_L$ if it is defined by:

$$X \lrcorner\, d\sigma = 0, \qquad \langle X, dt \rangle = 1. \tag{3}$$

**Definition 13.** Equations (3) are called *Euler's superequations*. Equations that are written locally as (1) are called *Euler's superequations in Hamiltonian form.*

**Definition 14.** A piecewise superdifferentiable supercurve $C$ on $M$ such that $\widetilde{C}$ is an integral supercurve of $X$ is called an *extremal*. In case of kinetic energy of a Riemann supermanifold the extremals are called *geodesics*.

In case of kinetic energy of a Riemann supermanifold for the extremals, we can get the equations of supergeodesics given by DeWitt, see [4]. We only study the case of a-supercurve with

$$C(t) = C_1(t)\theta^1 + \cdots + C_{123}(t) \cdot \theta^1 \cdot \theta^2 \cdot \theta^3 + \cdots$$

If $L$ is the energy of a Riemann supermetric [4], [1], $L$ is given by

$$L = \tfrac{1}{2} \sum_{i,j=1}^{m} g_{ij} \cdot \dot{x}^i \cdot \dot{x}^j + \tfrac{1}{2} \sum_{\alpha,\beta=1}^{n} \bar{g}_{\alpha\beta} \cdot \dot{\theta}^\alpha \cdot \dot{\theta}^\beta$$

so that $\partial L / \partial \dot{\theta}^\alpha = \sum_{\beta=1}^{n} \bar{g}_{\alpha\beta} \cdot \dot{\theta}^\beta$. Then from the equations

$$\frac{d\theta^\alpha}{dt} = \dot{\theta}^\alpha \quad \text{and} \quad \frac{d \dfrac{\partial L}{\partial \dot{\theta}}^\alpha}{dt} = \frac{\partial L}{\partial \theta^\alpha}, \tag{4}$$

we get

$$\frac{d \sum_{\beta=1}^{n} \bar{g}_{\alpha\beta} \cdot \dot{\theta}^\beta}{dt} = \frac{1}{2} \sum_{\mu,\eta=1}^{n} \frac{\partial \bar{g}_{\mu\eta}}{\partial \theta^\alpha} \cdot \dot{\theta}^\mu \cdot \dot{\theta}^\eta$$

or

$$\sum_{\beta=1}^{n} \bar{g}_{\alpha\beta} \frac{d^2\theta^\beta}{dt^2} + \sum_{\beta,\mu=1}^{n} \frac{\partial \bar{g}_{\alpha\beta}}{\partial \theta^\mu} \frac{d\theta^\mu}{dt} \frac{d\theta^\beta}{dt} = \frac{1}{2} \sum_{\mu,\eta=1}^{n} \frac{\partial \bar{g}_{\mu\eta}}{\partial \theta^\alpha} \frac{d\theta^\mu}{dt} \frac{d\theta^\eta}{dt}.$$

But

$$\sum_{\mu,\eta=1}^{n} \frac{\partial \bar{g}_{\alpha\mu}}{\partial \theta^\eta} \cdot \dot{\theta}^\mu \cdot \dot{\theta}^\eta = - \sum_{\mu,\eta=1}^{n} \frac{\partial \bar{g}_{\alpha\eta}}{\partial \theta^\mu} \cdot \dot{\theta}^\mu \cdot \dot{\theta}^\eta$$

so we can write

$$\sum_{\beta=1}^{n} \bar{g}_{\alpha\beta} \frac{d^2\theta^\beta}{dt^2} = \frac{1}{2} \sum_{\mu,\eta=1}^{n} \left( \frac{\partial \bar{g}_{\mu\eta}}{\partial \theta^\alpha} - \frac{\partial \bar{g}_{\alpha\mu}}{\partial \theta^\eta} + \frac{\partial \bar{g}_{\alpha\eta}}{\partial \theta^\mu} \right) \frac{d\theta^\mu}{dt} \frac{d\theta^\eta}{dt}.$$

Let $(\bar{g}^{\alpha\beta})$, see [4] or [1], be the inverse matrix to $\bar{g}_{\alpha\beta}$ because $\bar{g}_{\alpha\beta}$ is non-singular and set

$$\Gamma^{\beta}_{\mu\eta} = \frac{1}{2}\sum_{\alpha=1}^{n} \bar{g}^{\alpha\beta} \cdot \left(\frac{\partial \bar{g}_{\mu\eta}}{\partial\theta^{\alpha}} - \frac{\partial \bar{g}_{\alpha\mu}}{\partial\theta^{\eta}} + \frac{\partial \bar{g}_{\alpha\eta}}{\partial\theta^{\mu}}\right).$$

Then the equations (4) for supergeodesics take the form

$$\frac{d^2\theta^{\beta}}{dt^2} = \sum_{\mu,\eta=1}^{n} \Gamma^{\beta}_{\mu\eta}\frac{d\theta^{\mu}}{dt}\frac{d\theta^{\eta}}{dt}.$$

**Theorem 15.** *Let $L$ define a regular problem in the calculus of variations on supermanifolds. A necessary condition that a piecewise superdifferentiable supercurve $C$ in the sense of Rogers be a week local minimum for $L$ is that $C$ be superdifferentiable and $\widetilde{C}$ be an integral supercurve of $X$ where $X$ is defined by*

$$X \lrcorner\, d\sigma = 0, \quad \omega = dL, \quad \langle X, dt\rangle = 1,$$

*the superform $\sigma = \mathcal{L}^*\omega$ is well defined on $T(M) \times B_L$ and $\mathcal{L}$ is an immersion of $T(M) \times B_L$ into $T^*(M) \times B_L$.*

**Proof.** Equation (1) is satisfied by the supervector $X = \bar{C}' = \mathcal{L}_*\widetilde{C}'$. Thus we have $\widetilde{C}' \lrcorner\, d\sigma = 0$ and $\langle\widetilde{C}', dt\rangle = 1$. If $C$ is a local minimum then $C$ is superdifferentiable and $\widetilde{C}$ is an integral supercurve of $X$. Suppose

$$X = \sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{n} X^{\alpha}\frac{\partial}{\partial\theta^{\alpha}} + \sum_{i=1}^{m} \dot{X}^i\frac{\partial}{\partial\dot{x}^i} + \sum_{\alpha=1}^{n}\dot{X}^{\alpha}\frac{\partial}{\partial\dot{\theta}^{\alpha}} + \frac{\partial}{\partial t}$$

in terms of local coordinates. Using Proposition 10 and Lemma 11, one can prove this theorem. □

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Valentin Gabriel Cristea
University “Valachia”
Targoviste
Romania
E-mail: valentin_cristea@yahoo.com