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# $\mathcal{E}^1(M)$-Dirac structures and Jacobi structures

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**Abstract.** Using $\mathcal{E}^1(M)$-Dirac structures, a notion introduced by A. Wade, we obtain conditions under which a submanifold of a Jacobi manifold has an induced Jacobi structure, generalizing the result obtained by Courant for Dirac structures and submanifolds of a Poisson manifold.

**Keywords.** Dirac structures, Poisson manifolds, Jacobi manifolds

**MS classification.** 17B63, 17B66, 53C12, 53D10, 53D17.

## 1. Introduction

The notion of a Poisson manifold was introduced by Lichnerowicz in [7], see also [13]. A Poisson structure on a manifold $M$ is a Lie bracket on the space of $C^\infty$ real valued functions on $M$ such that it is a derivation on each argument with respect to the usual product of functions. One of the main motivations for the introduction of this notion comes from Classical Mechanics. In fact, it appears in a natural way in the study of some mechanical systems, particularly systems with constraints or in the reduction of systems with symmetry groups. Given a Poisson manifold $M$, A. Weinstein indicated two different cases in which there exists on a submanifold $Q$ of $M$ a Poisson structure naturally induced by the one of $M$ (see [13]).

In [1], T. Courant introduced the so-called Dirac structures, which encompass both Poisson and presymplectic structures on a manifold. A Dirac structure on a manifold $M$ is a vector sub-bundle $\tilde{L}$ of $TM \oplus T^*M$ that is maximally isotropic under the natural symmetric pairing on $TM \oplus T^*M$ and such that the space of sections of $\tilde{L}$, $\Gamma(\tilde{L})$, is closed under the Courant bracket $[\cdot,\cdot]\tilde{}$ on $\Gamma(TM \oplus T^*M)$, see Examples 2.2. Moreover, using Dirac structures, he stated sufficient conditions for a submanifold $Q$ of $M$ to inherit a Poisson structure, generalizing some results in [13].

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On the other hand, a Jacobi structure on a manifold $M$ is a local Lie algebra structure, in the sense of Kirillov [5], on the space $C^\infty(M,\mathbb{R})$, see [2, 3, 8]. Apart from Poisson manifolds, interesting examples of Jacobi manifolds are contact and locally conformal symplectic manifolds. Moreover, if $M$ is endowed with a Jacobi structure, then the bundle $T^*M \times \mathbb{R} \to M$ of 1-jets is a Lie algebroid, which generalizes the cotangent Lie algebroid associated with a Poisson manifold. Dazord et al. study (in [2]) distinguished submanifolds of a Jacobi manifold $M$, see also [9].

In [12], A. Wade introduced a proper definition of a Dirac structure on the vector bundle $\mathcal{E}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ (a $\mathcal{E}^1(M)$-Dirac structure in our terminology) as a vector sub-bundle $L$ of $\mathcal{E}^1(M)$ which is maximally isotropic under the natural symmetric pairing of $\mathcal{E}^1(M)$ and such that the space $\Gamma(L)$ is closed under a suitable bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(\mathcal{E}^1(M))$. Apart from $\mathcal{E}^1(M)$-Dirac structures associated with Dirac structures on $M$, other interesting examples can be obtained from Jacobi structures on $M$, from a 1-form on $M$ (a precontact structure in our terminology) or from a locally conformal presymplectic structure, that is, a pair $(\Omega,\omega)$, where $\Omega$ is a 2-form on $M$, $\omega$ is a closed 1-form and $d\Omega = \omega \wedge \Omega$, see [12].

The aim of this paper is, using $\mathcal{E}^1(M)$-Dirac structures, to give sufficient conditions for a submanifold $Q$ of a Jacobi manifold $M$ to inherit a Jacobi structure. The paper is organized as follows. In Section 2, we recall the definition of a $\mathcal{E}^1(M)$-Dirac structure and we also present some examples that were obtained in [12]. In Section 3, we show sufficient conditions for a submanifold $Q$ of a manifold $M$ to inherit a $\mathcal{E}^1(M)$-Dirac structure. On the other hand, we prove, in Section 4, when a $\mathcal{E}^1(M)$-Dirac structure is induced by a Jacobi structure on $M$. Using the results obtained in Sections 3 and 4, we give a process to claim that a Jacobi structure on a manifold $M$ is able to pass to a submanifold $Q$ of $M$. As an application, we directly deduce a result previously proved in [1] about Dirac and Poisson structures on a manifold $M$ and some results about distinguished submanifolds of a Jacobi manifold (see [2, 10]).

## 2. $\mathcal{E}^1(M)$-Dirac structures

### 2.1. Definitions and examples

If $M$ is a differentiable manifold, we will denote by $\mathcal{E}^1(M)$ the vector bundle $(TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}) \to M$. Note that the space of global sections $\Gamma(\mathcal{E}^1(M))$ of $\mathcal{E}^1(M)$ can be identified with the direct sum

$$\big(\mathfrak{X}(M) \times C^\infty(M,\mathbb{R})\big) \oplus \big(\Omega^1(M) \times C^\infty(M,\mathbb{R})\big).$$

Along this section, we will recall the definition of a $\mathcal{E}^1(M)$-Dirac structure, which was introduced by A. Wade in [12]. We will also give several examples related to this notion.

The natural symmetric and skew-symmetric pairings $\langle\cdot,\cdot\rangle_+$ and $\langle\cdot,\cdot\rangle_-$ on $V \oplus V^*$, $V$ being a real vector space of finite dimension, can be extended, in a natural way, to the Whitney sum $A \oplus A^*$, where $A \to M$ is a real vector bundle over a manifold $M$. We also denote by $\langle\cdot,\cdot\rangle_+$ and $\langle\cdot,\cdot\rangle_-$ the resultant pairings on $\Gamma(A\oplus A^*)\cong\Gamma(A)\oplus\Gamma(A^*)$. In the particular case when $A=TM\times\mathbb{R}$, the explicit expressions of $\langle\cdot,\cdot\rangle_+$ and $\langle\cdot,\cdot\rangle_-$ on $\Gamma(\mathcal{E}^1(M))$ are

$$\begin{aligned}&\big\langle(X_1,f_1)+(\alpha_1,g_1),(X_2,f_2)+(\alpha_2,g_2)\big\rangle_+=\tfrac{1}{2}\,(i_{X_2}\alpha_1+f_2g_1+i_{X_1}\alpha_2+f_1g_2),\\ &\big\langle(X_1,f_1)+(\alpha_1,g_1),(X_2,f_2)+(\alpha_2,g_2)\big\rangle_-=\tfrac{1}{2}\,(i_{X_2}\alpha_1+f_2g_1-i_{X_1}\alpha_2-f_1g_2),\end{aligned}\tag{1}$$

for $(X_i,f_i)+(\alpha_i,g_i)\in\Gamma(\mathcal{E}^1(M))$, $i\in\{1,2\}$.

On the other hand, in [12] A. Wade introduced a suitable $\mathbb{R}$-bilinear bracket $[\![\cdot,\cdot]\!]:\Gamma(\mathcal{E}^1(M))\times\Gamma(\mathcal{E}^1(M))\to\Gamma(\mathcal{E}^1(M))$ on the space $\Gamma(\mathcal{E}^1(M))$ given by

$$\begin{aligned}[\![(X_1,f_1)+(\alpha_1,g_1),(X_2,f_2)+(\alpha_2,g_2)]\!]&=\big([X_1,X_2],X_1(f_2)-X_2(f_1)\big)\\ &+\big(\mathcal{L}_{X_1}\alpha_2-\mathcal{L}_{X_2}\alpha_1+\tfrac{1}{2}\,d(i_{X_2}\alpha_1-i_{X_1}\alpha_2)+f_1\alpha_2-f_2\alpha_1\\ &\quad+\tfrac{1}{2}\,(g_2\,df_1-g_1\,df_2-f_1\,dg_2+f_2\,dg_1),\\ &\quad X_1(g_2)-X_2(g_1)+\tfrac{1}{2}\,(i_{X_2}\alpha_1-i_{X_1}\alpha_2-f_2g_1+f_1g_2)\big),\end{aligned}$$

for $(X_i,f_i)+(\alpha_i,g_i)\in\Gamma(\mathcal{E}^1(M))$, $i\in\{1,2\}$, where $[\cdot,\cdot]$ is the usual Lie bracket of vector fields and $\mathcal{L}$ is the Lie derivative operator on $M$. This bracket is skew-symmetric but it is not, in general, a Lie bracket, since the Jacobi identity does not hold see [12].

Using the symmetric pairing and the bracket $[\![\cdot,\cdot]\!]$ we introduce the notion of a $\mathcal{E}^1(M)$-Dirac structure as follows.

**Definition 2.1** ([12]). A $\mathcal{E}^1(M)$-*Dirac structure* on $M$ is a sub-bundle $L$ of $\mathcal{E}^1(M)$ which is maximally isotropic under the symmetric pairing $\langle\cdot,\cdot\rangle_+$ and such that $\Gamma(L)$ is closed under $[\![\cdot,\cdot]\!]$.

We note that the restriction of $[\![\cdot,\cdot]\!]$ to $\Gamma(L)$ satisfies the Jacobi identity, see [12]. Some examples of $\mathcal{E}^1(M)$-Dirac structures are the following.

**Examples 2.2.** a) *Dirac structures* – Let $\tilde{L}$ be a vector sub-bundle of $TM\oplus T^*M$ and consider the vector sub-bundle $L$ of $\mathcal{E}^1(M)$ whose sections are

$$\Gamma(L)=\big\{(X,0)+(\alpha,f)\big|X+\alpha\in\Gamma(\tilde{L}),\ f\in C^\infty(M,\mathbb{R})\big\}.\tag{2}$$

Then, *$\tilde{L}$ is a Dirac structure on $M$* in the sense of Courant [1] if and only if $L$ is a $\mathcal{E}^1(M)$-Dirac structure (see [12]). We recall that a vector sub-bundle $\tilde{L}$ of $TM\oplus T^*M$ is a Dirac structure on $M$ if $\tilde{L}$ is maximally isotropic under the natural symmetric pairing $\langle\cdot,\cdot\rangle_+$ on $TM\oplus T^*M$ and, in addition, the space of sections of $L$, $\Gamma(L)$, is closed under the *Courant bracket* $[\cdot,\cdot]\tilde{}$ which is defined by

$$[X_1+\alpha_1,X_2+\alpha_2]\tilde{}=[X_1,X_2]+\big(\mathcal{L}_{X_1}\alpha_2-\mathcal{L}_{X_2}\alpha_1+\tfrac{1}{2}\,d(i_{X_2}\alpha_1-i_{X_1}\alpha_2)\big),$$

for $X_1+\alpha_1,X_2+\alpha_2\in\mathfrak{X}(M)\oplus\Omega^1(M)\cong\Gamma(TM\oplus T^*M)$.

b) *Jacobi structures* – A *Jacobi structure* on a manifold $M$ is a pair $(\Lambda,E)$, where $\Lambda$ is a 2-vector and $E$ is a vector field on $M$, such that $[\Lambda,\Lambda]=2E\wedge\Lambda$ and

$[E, \Lambda] = 0$, $[\cdot, \cdot]$ being the Schouten–Nijenhuis bracket. If the vector field $E$ identically vanishes then $(M, \Lambda)$ is a *Poisson manifold*. Jacobi and Poisson manifolds were introduced by Lichnerowicz [7, 8], see also [2, 3, 5, 6, 11, 13].

Now, given a 2-vector $\Lambda$ and a vector field $E$ on a manifold $M$, we can consider the vector sub-bundle $L_{(\Lambda,E)}$ of $\mathcal{E}^1(M)$ whose sections are

$$\Gamma(L_{(\Lambda,E)}) = \Big\{(\#_\Lambda(\alpha) + fE, -i_E\alpha) + (\alpha, f) \Big| \\ (\alpha, f) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})\Big\}, \tag{3}$$

where $\#_\Lambda : \Omega^1(M) \to \mathfrak{X}(M)$ is the homomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $\beta(\#_\Lambda(\alpha)) = \Lambda(\alpha, \beta)$, for $\alpha, \beta \in \Omega^1(M)$. Note that the vector bundles $L_{(\Lambda,E)}$ and $T^*M \times \mathbb{R}$ are isomorphic. Moreover, we have that $L_{(\Lambda,E)}$ is a $\mathcal{E}^1(M)$-Dirac structure if and only if $(\Lambda, E)$ is a Jacobi structure (see [12]).

### 2.2. $\mathcal{E}^1(M)$-Dirac structures and submanifolds of the base space

Let $L$ be a vector sub-bundle of $\mathcal{E}^1(M)$ which is maximally isotropic under the symmetric pairing $\langle \cdot, \cdot \rangle_+$ and $Q$ be a submanifold of $M$. If $x$ is a point of $Q$, we may define the vector space $(L_Q)_x$ by

$$(L_Q)_x = \frac{L_x \cap ((T_xQ \times \mathbb{R}) \oplus (T_x^*M \times \mathbb{R}))}{L_x \cap \big(\{0\} \oplus ((T_xQ)^\circ \times \{0\})\big)}, \tag{4}$$

where $(T_xQ)^\circ$ is the annihilator of $T_xQ$, that is, $(T_xQ)^\circ = \{\alpha \in T_x^*M \mid \alpha|_{T_xQ} = 0\}$. We have that the linear map $(L_Q)_x \to (T_xQ \times \mathbb{R}) \oplus (T_x^*Q \times \mathbb{R})$ given by $[(u, \lambda) + (\alpha, \mu)] \mapsto (u, \lambda) + (\alpha|_{T_xS}, \mu)$ is a monomorphism and thus $(L_Q)_x$ can be identified with a subspace of $(T_xQ \times \mathbb{R}) \oplus (T_x^*Q \times \mathbb{R})$. Moreover, using the results of [1, Section 1.4], we deduce that $(L_Q)_x$ is a maximally isotropic subspace of $(T_xQ \times \mathbb{R}) \oplus (T_x^*Q \times \mathbb{R})$ under the symmetric pairing $\langle \cdot, \cdot \rangle_+$. In particular, this implies that $\dim(L_Q)_x = \dim Q + 1$, for all $x \in Q$. In addition, we have the following proposition.

**Proposition 2.3** ([4]). *Let $L$ be a $\mathcal{E}^1(M)$-Dirac structure and $Q$ be a submanifold of $M$. If the dimension of*

$$L_x \cap \big((T_xQ \times \mathbb{R}) \oplus (T_x^*M \times \mathbb{R})\big)$$

*keeps constant for all $x \in Q$ (or, equivalently, the dimension of $L_x \cap (\{0\} \oplus ((T_xQ)^\circ \times \{0\}))$ keeps constant for all $x \in Q$) then*

$$L_Q = \bigcup_{x \in Q} (L_Q)_x$$

*is a vector sub-bundle of $\mathcal{E}^1(Q)$ and, furthermore, $L_Q$ is a $\mathcal{E}^1(Q)$-Dirac structure.*

## 3. $\mathcal{E}^1(M)$-Dirac structures and Jacobi structures

In this Section we will give a sufficient condition for a $\mathcal{E}^1(M)$-Dirac structure to be associated with a Jacobi structure $(\Lambda, E)$ on $M$.

Let $L$ be a vector sub-bundle of $\mathcal{E}^1(M)$ which is isotropic under the canonical symmetric pairing of $\mathcal{E}^1(M)$. We consider the bundle map $\bar{\rho}_L^* : L \to T^*M \times \mathbb{R}$ given by

(5) $$\bar{\rho}_L^*(e_x) = (\alpha, \mu),$$

for $e_x = (u, \lambda) + (\alpha, \mu) \in L_x$ and $x \in M$. Then, we may define the 2-vector $P_L(x)$ on the vector space $\bar{\rho}_L^*(L_x)$ by

(6) $$P_L(x)\big(\bar{\rho}_L^*((e_1)_x), \bar{\rho}_L^*((e_2)_x)\big) = -\langle (e_1)_x, (e_2)_x \rangle_-,$$

for $(e_1)_x, (e_2)_x \in L_x$. Since $L$ is a isotropic vector sub-bundle of $\mathcal{E}^1(M)$ under the symmetric pairing $\langle \cdot, \cdot \rangle_+$, we deduce that the 2-vector $P_L(x)$ is well defined. Note that if $e_1, e_2 \in \Gamma(L)$ then one may consider the function $P_L(\bar{\rho}_L^*(e_1), \bar{\rho}_L^*(e_2)) \in C^\infty(M, \mathbb{R})$ given by

$$P_L(\bar{\rho}_L^*(e_1), \bar{\rho}_L^*(e_2))(x) = P_L(x)\big(\bar{\rho}_L^*((e_1)_x), \bar{\rho}_L^*((e_2)_x)\big),$$

for all $x \in M$. In fact, if $e_i = (X_i, f_i) + (\alpha_i, g_i)$, with $i \in \{1, 2\}$, we have that

(7) $$P_L\big((\alpha_1, g_1), (\alpha_2, g_2)\big) = i_{X_1}\alpha_2 + f_1 g_2.$$

**Theorem 3.1.** *Let $L$ be a $\mathcal{E}^1(M)$-Dirac structure such that*

(8) $$\bar{\rho}_L^*(L_x) = T_x^*M \times \mathbb{R}, \qquad \textit{for all } x \in M.$$

*Then, there exists a Jacobi structure $(\Lambda, E)$ on $M$ such that $L$ is associated with $(\Lambda, E)$, that is, $L = L_{(\Lambda,E)}$.*

**Proof.** Using (8) and the fact that $\dim L_x = \dim T_x^*M \times \mathbb{R}$ for all $x \in M$, we obtain that the map

$$\bar{\rho}_L^*(L_x)|_{L_x} : L_x \longrightarrow T_x^*M \times \mathbb{R}$$

is a linear isomorphism, for all $x \in M$, see (5). Consequently, $P_L$ defines a section of the vector bundle $\bigwedge^2(TM \times \mathbb{R}) \to M$, that is, a pair $(\Lambda, E) \in \mathcal{V}^2(M) \times \mathfrak{X}(M)$. The relation between $P_L$ and $(\Lambda, E)$ is given by

(9) $$P_L(x)\big((\alpha_1,\mu_1), (\alpha_2,\mu_2)\big) = \Lambda(x)(\alpha_1,\alpha_2) + \mu_1\alpha_2\big(E(x)\big) - \mu_2\alpha_1\big(E(x)\big),$$

for all $x \in M$ and $(\alpha_1, \mu_1), (\alpha_2, \mu_2) \in T_x^*M \times \mathbb{R}$.

Now, suppose that $(u, \lambda) + (\alpha, \mu) \in L_x$, with $x \in M$. From (6), (8) and (9), it follows that

(10) $$\beta(\#_\Lambda(x)(\alpha) + \mu E(x)) - \nu\alpha(E(x)) = \beta(u) + \lambda\nu,$$

for all $(\beta, \nu) \in T_x^*M \times \mathbb{R}$, that is, $u = \#_\Lambda(x)(\alpha) + \mu\, E(x)$ and $\lambda = -\alpha(E(x))$.

Using (10) and that $\dim L_x = \dim M + 1$, we deduce that

$$L_x = \left\{ \left(\#_\Lambda(x)(\alpha) + \mu E(x), -\alpha(E(x))\right) + (\alpha, \mu) \middle| (\alpha, \mu) \in T_x^*M \times \mathbb{R} \right\}.$$

Thus,

$$\Gamma(L) = \left\{ (\#_\Lambda(\alpha) + fE, -i_E\alpha) + (\alpha, f) \middle| (\alpha, f) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \right\}. \tag{11}$$

Finally, using (11) and the fact that $L$ is a $\mathcal{E}^1(M)$-Dirac structure, we conclude that (see Examples 2.2) $(\Lambda, E)$ is a Jacobi structure and $L = L_{(\Lambda,E)}$. □

**Remark 3.2.** Let $\tilde{L} \subseteq TM \oplus T^*M$ be a Dirac structure such that $\rho^*_{\tilde{L}}(\tilde{L}_x) = T_x^*M$ for all $x \in M$, where $\rho^*_{\tilde{L}} : \tilde{L} \to T^*M$ is given by

$$\rho^*_{\tilde{L}}(e_x) = \alpha,$$

for $e_x = u + \alpha \in \tilde{L}_x$ and $x \in M$. Then, using (2), we have that the corresponding $\mathcal{E}^1(M)$-Dirac structure satisfies the hypotheses of Theorem 3.1. Furthermore, from (2) and (3), we deduce that the vector field $E$ of the resulting Jacobi structure $(\Lambda, E)$ is zero. Therefore, we conclude that there exists a Poisson structure $\Lambda$ on $M$ such that $\tilde{L}$ is associated to $\Lambda$, that is,

$$\Gamma(\tilde{L}) = \left\{ (\#_\Lambda(\alpha), \alpha) \middle| \alpha \in \Omega^1(M) \right\}.$$

## 4. On submanifolds of a Jacobi manifold and $\mathcal{E}^1(M)$-Dirac structures

If $(\Lambda, E)$ is a Jacobi structure on a manifold $M$ and $Q$ is a submanifold of $M$, we will show sufficient conditions to be imposed in order to obtain a Jacobi structure on $Q$.

Let $L_{(\Lambda,E)}$ be the $\mathcal{E}^1(M)$-Dirac structure associated with the Jacobi structure $(\Lambda, E)$. For all $x \in Q$ define the vector space $((L_{(\Lambda,E)})_Q)_x$ as in (4), that is,

$$((L_{(\Lambda,E)})_Q)_x = \frac{(L_{(\Lambda,E)})_x \cap ((T_xQ \times \mathbb{R}) \oplus (T_x^*M \times \mathbb{R}))}{(L_{(\Lambda,E)})_x \cap \left(\{0\} \oplus ((T_xQ)^\circ \times \{0\})\right)}.$$

From Proposition 2.3, we deduce that if the dimension of

$$(L_{(\Lambda,E)})_x \cap \left(\{0\} \oplus ((T_xQ)^\circ \times \{0\})\right)$$

keeps constant for all $x \in Q$, then $(L_{(\Lambda,E)})_Q$ is a $\mathcal{E}^1(Q)$-Dirac structure. Moreover, we have that

$$(L_{(\Lambda,E)})_x \cap \left(\{0\} \oplus ((T_xQ)^\circ \times \{0\})\right) = (T_xQ)^\circ \cap \ker \#_\Lambda(x) \cap \langle E(x) \rangle^\circ \tag{12}$$

for all $x \in Q$.

Since $(L_{(\Lambda,E)})_Q$ is maximally isotropic under the canonical symmetric pairing of $\mathcal{E}^1(M)$, we have that $\bar{\rho}^*_{(L_{(\Lambda,E)})_Q}((L_{(\Lambda,E)})_Q) = ((L_{(\Lambda,E)})_Q \cap (TQ \times \mathbb{R}))^\circ$, see [1], p. 635. Therefore, to prove that $\bar{\rho}^*_{(L_{(\Lambda,E)})_Q}((L_{(\Lambda,E)})_Q) = T^*Q \times \mathbb{R}$ is equivalent to show that $(L_{(\Lambda,E)})_Q \cap (TQ \times \mathbb{R}) = \{(0,0)\}$. Furthermore, for any $L$ which is maximally isotropic under the canonical symmetric pairing of $\mathcal{E}^1(M)$ and any submanifold $S$ of $M$, we have that (see [1, p. 641]),

$$(L_S)_x \cap (T_xS \times \mathbb{R}) = \frac{L_x \cap \big((T_xS \times \mathbb{R}) \oplus ((T_xS)^\circ \times \{0\})\big)}{L_x \cap \big(\{0\} \oplus ((T_xS)^\circ \times \{0\})\big)}$$

for any $x \in S$. Therefore, $(L_{(\Lambda,E)})_Q \cap (TQ \times \mathbb{R}) = \{(0,0)\}$ if and only if the following conditions hold:

$$TQ \cap \#_\Lambda(TQ^\circ) = \{0\}, \qquad E|_Q \text{ is tangent to } Q. \tag{13}$$

Note that if $E|_Q$ is tangent to $Q$, then $(T_xQ)^\circ \subseteq \langle E(x)\rangle^\circ$, for all $x \in Q$.

Summing up, we conclude that

**Proposition 4.1.** *Let $(M, \Lambda, E)$ be a Jacobi structure and $Q$ be a submanifold of $M$ such that:*

(i) *$TQ^\circ \cap \ker \#_\Lambda$ is a a vector bundle over $Q$,*
(ii) *$TQ \cap \#_\Lambda(TQ^\circ) = \{0\}$,*
(iii) *$E|_Q$ is tangent to $Q$.*

*Then, $L_Q$ is a $\mathcal{E}^1(Q)$-Dirac structure which induces a Jacobi structure on $Q$.*

Using (2), (3), (4), Remark 3.2 and Proposition 4.1, we deduce a result which was first stated in [1].

**Corollary 4.2.** *Let $(M, \Lambda)$ be a Poisson manifold and $Q$ be a submanifold of $M$ such that:*

(i) *$TQ^\circ \cap \ker \#_\Lambda$ is a vector bundle over $Q$,*
(ii) *$TQ \cap \#_\Lambda(TQ^\circ) = \{0\}$.*

*Then, $L_Q$ is a $\mathcal{E}^1(Q)$-Dirac structure coming from a Dirac structure in the sense of Courant which induces a Poisson structure on $Q$.*

Next, let us see three examples where Proposition 4.1 and Corollary 4.2 holds.

**Examples 4.3.** a) Let $(M, \Lambda, E)$ be a Jacobi structure and $Q$ be a submanifold of $M$ such that

$$\#_\Lambda(x)(T_x^*M) + \big\langle E(x)\big\rangle \subseteq T_xQ, \qquad \text{for all } x \in Q. \tag{14}$$

This condition is equivalent to the two following ones:

$$\#_\Lambda(x)(T_x^*M) \subseteq T_xQ, \qquad E(x) \in T_xQ,$$

for all $x \in Q$. From the first condition, we deduce that

$$TQ^\circ \cap \ker \#_\Lambda = TQ^\circ. \tag{15}$$

Moreover, using (15), we obtain that

$$TQ \cap \#_\Lambda(TQ^\circ) = TQ \cap \#_\Lambda(\ker \#_\Lambda \cap TQ^\circ) = \{0\}.$$

Therefore, the hypotheses of Proposition 4.1 hold.

This example was first treated in [2], see also [10]. More precisely, in [2], it is proved that given a Jacobi manifold $(M, \Lambda, E)$ and a submanifold $Q$ of $M$, there exists a Jacobi structure on $Q$ such that the canonical inclusion $i_Q : Q \to M$ is a Jacobi morphism if and only if (14) holds. Moreover, in that case the Jacobi structure on $Q$ is unique. On the other hand, if $(M, \Lambda)$ is a Poisson manifold and $Q$ is a submanifold of $M$ such that $\#_\Lambda(x)(T_x^*M) \subseteq T_xQ$, for all $x \in Q$, then $Q$ is called a Poisson submanifold of the first kind (see [9, 13]).

b) Let $(M, \Lambda, E)$ be a Jacobi manifold and $Q$ be a submanifold of $M$ such that

$$T_xQ + \#_\Lambda(x)\big((T_xQ)^\circ\big) = T_xM, \quad E(x) \in T_xQ, \quad \text{for all } x \in Q. \tag{16}$$

From [2, Lemma 3.8], we have that the first condition in (16) implies the two following ones

$$TQ^\circ \cap \ker \#_\Lambda = \{0\}, \quad TQ \cap \#_\Lambda(TQ^\circ) = \{0\}.$$

Thus, Proposition 4.1 holds in this case (for a different point of view, see [2, 10]).

In the particular case that $E$ identically vanishes, that is, $(M, \Lambda)$ is a Poisson manifold, a submanifold $Q$ of $M$ satisfying first condition of (16) is called a Poisson submanifold of the second kind (see [9, 13]).

c) Let $(M, \Lambda)$ be a Poisson manifold and $Q$ be a submanifold which is transverse to any leaf, that is,

$$TQ \oplus \#_\Lambda(T_Q^*M) = T_QM.$$

Since $TQ + \#_\Lambda(T_Q^*M) = T_QM$, we obtain that $TQ^\circ \cap ker\, \#_\Lambda = \{0\}$. On the other hand, $TQ \cap \#_\Lambda(T_Q^*M) = \{0\}$ implies that $TQ \cap \#_\Lambda(TQ^\circ) = \{0\}$. Therefore, conditions of Corollary 4.2 hold and there exists a Poisson structure on $Q$. For another treatment of this example, see [1, 13].

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