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# The gravitational field of spacetime of constant curvature[1]

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**Abstract.** The purpose of the presented paper is to study a gravitational field on a pseudo-Riemannian manifold $M$ of constant curvature with a metric tensor $g$. The type of gravitational field is determined by the characteristic of $\lambda$-tensor.

**Keywords.** Gravitational field, pseudo-Riemannian manifold, constant curvature.

**MS classification.** 58A05, 58D17, 58D30.

Let $M$ and $M'$ be pseudo-Riemannian manifolds with metrics $g$ and $g'$, respectively. Then a differentiable map $f : M \to M'$ is a local isometry if each of the tangent maps $f_* : T_x M \to T_{f(x)} M'$ is a linear isometry (isomorphism of vector spaces which preserves inner products), where $x \in M$.

**Theorem 1.** *Let $\mathbb{R}^n_s$, $0 \leq s \leq n$, denote the vector space of real $n$-tuples $x = (x^1, \dots, x^n)$ with the bilinear form*

$$\mathbf{b}^n_s(x, y) = -\sum_{i=1}^{s} x^i y^i + \sum_{j=s+1}^{n} x^j y^j .$$

*Then*

(i) $\mathbf{b}^n_s$ *is a non-degenerate symmetric bilinear form.*
(ii) *If $s \neq t$ then $\mathbb{R}^n_s$ is not linear isometric to $\mathbb{R}^n_t$.*
(iii) *If $V$ is an $n$-dimensional real vector space with a non-degenerate symmetric bilinear form, then there is a linear isometry of $V$ onto some $\mathbb{R}^n_s$.*

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

**Theorem 2.** *Let $K$ be a non-zero real number. Define $r > 0$ by $K = er^{-2}$, $e = \pm 1$. Let $\Sigma$ be the quadratic*

$$\Sigma = \left\{ x \in \mathbb{R}^n_s : \mathbf{b}^n_s(x, x) = er^2 \right\}, \quad n \geq 3.$$

*If $x \in \Sigma$ then $T_x\Sigma$ is a non-degenerate subspace of $T_x\mathbb{R}^n_s$, so $\mathbb{R}^n_s$ induces a pseudo-Riemannian metric on $\Sigma$.*

*With this metric, $\Sigma$ is a complete pseudo-Riemannian manifold of constant curvature $K$ and signature $(s,\ n - s - 1)$ if $e = 1$ or $(s - 1,\ n - s)$ if $e = -1$.*

Given integers $0 \leq s \leq n$ with $n \geq 2$ and a number $r > 0$, we define

$$\mathbb{S}^n_s = \left\{ x \in \mathbb{R}^{n+1}_s : \mathbf{b}^{n+1}_s(x, x) = r^2 \right\},$$
$$\mathbb{H}^n_s = \left\{ x \in \mathbb{R}^{n+1}_{s+1} : \mathbf{b}^{n+1}_{s+1}(x, x) = -r^2 \right\}.$$

Thus Theorem 2 says that $\mathbb{S}^n_s$ and $\mathbb{H}^n_s$ are complete pseudo-Riemannian manifold of signature $(s, n - s)$ and of constant curvatures $r^{-2}$ and $-r^{-2}$, respectively. They are called the pseudo-Riemannian spheres and hyperbolic spaces, respectively.

**Theorem 3** (Riemann). *Let $M$ be a pseudo-Riemannian manifold of dimension $n \geq 2$ and let $K$ be a real number.*

*Then the following conditions are equivalent.*

(i) *$M$ is of constant curvature $K$.*

(ii) *If $x \in M$, then there are local coordinates $x^i$ on a neighbourhood of $x$ in which the metric is given by*

$$ds^2 = \frac{e_1\, dx^1 \otimes dx^1 + \cdots + e_n\, dx^n \otimes dx^n}{\left\{ 1 + \frac{K}{4} \sum_i e_i (x^i)^2 \right\}^2}, \quad e_i = \pm 1.$$

(iii) *If $x \in M$, then $x$ has a neighbourhood which is isometric to an open set on some $\mathbb{S}^n_s$ if $K > 0$, $\mathbb{R}^n_s$ if $K = 0$, $\mathbb{H}^n_s$ if $K < 0$.*

The proofs of the above theorems one can find in ([3]).

Now, let $M$ be a four-dimensional real differentiable manifold equipped with a metric tensor of signature $(1, 3)$ and $X = (X_1, \ldots, X_4)$ be a moving frame on an open $U \subset M$. Since dual coframe $\{dx^a\}$ is defined by $dx^a(X_b) = \delta^a_b$, then traditionally a pseudo-Riemannian metric on $M$ is written

$$ds^2 = g_{ab}(x)\, dx^a \otimes dx^b. \tag{1}$$

From Theorem 3(ii) in our case is

$$ds^2 = \frac{-dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3}{\left\{ 1 + \frac{K}{4} e_i (x^i)^2 \right\}^2} \tag{2}$$

and differentiable functions $g_{ab}(x)$ on $U$ have the form

$$g_{ab}(x) = \begin{pmatrix} -D^2(x) & & & 0 \\ & D^2(x) & & \\ & & D^2(x) & \\ 0 & & & D^2(x) \end{pmatrix}, \tag{3}$$

where $D(x) = 1/(1 + (K/4)e_i(x^i)^2)$, $x = (x^0, x^1, x^2, x^3) \in U$, $e_0 = -1$, $e_i = +1$ for $i = 1, 2, 3$.

Now, let us recall the well-known formulas

$$\Gamma^c_{ad} = \frac{1}{2} g^{cb}\left(\frac{\partial g_{ab}}{\partial x^d} + \frac{\partial g_{bd}}{\partial x^a} - \frac{\partial g_{da}}{\partial x^b}\right), \quad \Gamma^c_{ad} = \Gamma^c_{da} \tag{4}$$

and

$$\begin{aligned} R^c_{abd} &= \frac{\partial}{\partial x^d}\Gamma^c_{ab} - \frac{\partial}{\partial x^b}\Gamma^c_{ad} + \Gamma^l_{ab}\Gamma^c_{ld} - \Gamma^l_{ad}\Gamma^c_{lb}, \\ R_{acbd} &= g_{cl} R^l_{abd}, \\ R_{acbd} &= -R_{acdb} = -R_{cabd} = R_{bdac}. \end{aligned} \tag{5}$$

From (4) the Christoffel symbols for the metric tensor (3) are

$$\begin{array}{lll}
\Gamma^0_{00} = \frac{1}{2} Kx^0 D(x), & \Gamma^0_{10} = -\frac{1}{2} Kx^1 D(x), & \Gamma^0_{20} = -\frac{1}{2} Kx^2 D(x), \\
\Gamma^1_{00} = -\frac{1}{2} Kx^1 D(x), & \Gamma^1_{10} = \frac{1}{2} Kx^0 D(x), & \Gamma^1_{20} = 0, \\
\Gamma^2_{00} = -\frac{1}{2} Kx^2 D(x), & \Gamma^2_{10} = 0, & \Gamma^2_{20} = \frac{1}{2} Kx^0 D(x), \\
\Gamma^3_{00} = -\frac{1}{2} Kx^3 D(x), & \Gamma^3_{10} = 0, & \Gamma^3_{20} = 0, \\
\Gamma^0_{30} = -\frac{1}{2} Kx^3 D(x), & \Gamma^0_{11} = \frac{1}{2} Kx^0 D(x), & \Gamma^0_{21} = 0, \\
\Gamma^1_{30} = 0, & \Gamma^1_{11} = -\frac{1}{2} Kx^1 D(x), & \Gamma^1_{21} = -\frac{1}{2} Kx^2 D(x), \\
\Gamma^2_{30} = 0, & \Gamma^2_{11} = \frac{1}{2} Kx^2 D(x), & \Gamma^2_{21} = -\frac{1}{2} Kx^1 D(x), \\
\Gamma^3_{30} = \frac{1}{2} Kx^0 D(x), & \Gamma^3_{11} = \frac{1}{2} Kx^3 D(x), & \Gamma^3_{21} = 0,
\end{array}$$

$$\begin{array}{ll}
\Gamma^0_{31} = 0, & \Gamma^0_{22} = \frac{1}{2} Kx^0 D(x), \\
\Gamma^1_{31} = -\frac{1}{2} Kx^3 D(x), & \Gamma^1_{22} = \frac{1}{2} Kx^1 D(x), \\
\Gamma^2_{31} = 0, & \Gamma^2_{22} = -\frac{1}{2} Kx^2 D(x), \\
\Gamma^3_{31} = -\frac{1}{2} Kx^1 D(x), & \Gamma^3_{22} = \frac{1}{2} Kx^3 D(x), \\
\Gamma^0_{32} = 0, & \Gamma^0_{33} = \frac{1}{2} Kx^0 D(x), \\
\Gamma^1_{32} = 0, & \Gamma^1_{33} = \frac{1}{2} Kx^1 D(x), \\
\Gamma^2_{32} = -\frac{1}{2} Kx^3 D(x), & \Gamma^2_{33} = \frac{1}{2} Kx^2 D(x), \\
\Gamma^3_{32} = -\frac{1}{2} Kx^2 D(x), & \Gamma^3_{33} = -\frac{1}{2} Kx^3 D(x).
\end{array}$$

From Definition (5) the components, different from zero of the Riemann tensor

$R^c_{abd}$ are

$$R^1_{010}(x) = R^2_{020}(x) = R^3_{030}(x) = K D(x) - \tfrac{1}{4} K^2 D^2(x)\, e_i (x^i)^2,$$

$$\begin{aligned} R^0_{101}(x) &= R^2_{121}(x) = R^3_{131}(x) = R^0_{202}(x) = R^1_{212}(x) = R^3_{232}(x) \\ &= R^0_{303}(x) = R^1_{313}(x) = R^2_{323}(x) \\ &= -K D(x) + \tfrac{1}{4} K^2 D^2(x)\, e_i (x^i)^2. \end{aligned}$$

Since $e_i (x^i)^2 = (4/K)(1/D(x) - 1)$, thus

$$\begin{aligned} K D(x) - \frac{1}{4} K^2 D^2(x)\, e_i (x^i)^2 &= K D(x) - \frac{1}{4} K^2 D^2(x) \cdot \frac{4}{K} \left( \frac{1}{D(x)} - 1 \right) \\ &= K D(x) - K D(x) + K D^2(x) = K D^2(x). \end{aligned}$$

Hence and from (5) we have components, different from zero, of the Riemann tensor $R_{acbd}$

$$\begin{aligned} R_{0110}(x) &= R_{0220}(x) = R_{0330}(x) = R_{1212}(x) = R_{3131}(x) = R_{2323}(x) \\ &= K D^4(x). \end{aligned}$$

Let us consider spacetime $(M, g)$ as a four dimensional real manifold with a metric tensor of signature (1,3). Thus at each point $x \in M$ there exists a basic of the space $T^*_x M$ in which the metric tensor $g$ has a diagonal form

$$\eta_{ab} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}.$$

It is convenient to introduce a 6-dimensional formalism in the pseudo-Euclidean space $\mathbb{R}^6$. The rule for changing to the 6-dimensional formalism is the following (see [2])

| | | | | | | |
|---|---|---|---|---|---|---|
| $ab$ : | 23 | 31 | 12 | 10 | 20 | 30 |
| $A$ : | 1 | 2 | 3 | 4 | 5 | 6. |

Let us notice that the tensor

$$\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc} = \eta_{abcd} \to \eta_{AB} = \begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ 0 & & & & & -1 \end{pmatrix}$$

has the signature (3,3).

Now, we introduce the metric tensor defined by

$$g_{ac}g_{bd} - g_{ad}g_{bc} = g_{abcd} \to g_{AB},$$

where $g_{ab}$ are components of a metric tensor at an arbitrary point of the manifold $M$ and collective indices are the skew-symmetric pairs $ab \to A$, $cd \to B$. The tensor $g_{AB}$ $(A, B = 1, \ldots, 6)$ is symmetric and non-singular.

Thus tensors $g_{AB}$ and $R_{AB}$ are

$$g_{AB}(x) = \begin{pmatrix} D^4(x) & & & & & 0 \\ & D^4(x) & & & & \\ & & D^4(x) & & & \\ & & & -D^4(x) & & \\ & & & & -D^4(x) & \\ 0 & & & & & -D^4(x) \end{pmatrix},$$

$$R_{AB}(x) = \begin{pmatrix} KD^4(x) & & & & & 0 \\ & KD^4(x) & & & & \\ & & KD^4(x) & & & \\ & & & -KD^4(x) & & \\ & & & & -KD^4(x) & \\ 0 & & & & & -KD^4(x) \end{pmatrix}.$$

Therefore, it only remains to find canonical forms of the λ-tensor $R_{AB} - \lambda g_{AB}$. The canonical forms of symmetric λ-tensors have been given in paper ([1]).

Since in our case the determinant λ-matrix $R_{AB} - \lambda g_{AB}$ is equal to zero if and only if $\lambda = K$, we have

$$g_{A'B'} = \begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ 0 & & & & & -1 \end{pmatrix},$$

$$R_{A'B'} = \begin{pmatrix} K & & & & & 0 \\ & K & & & & \\ & & K & & & \\ & & & -K & & \\ & & & & -K & \\ 0 & & & & & -K \end{pmatrix}.$$

It is the type $G_7[(111111)]$ in the Segre symbols (see [1]).

Let us notice that this consideration has shown that the Riemann tensor

$$R_{abcd}(x) = K\big(g_{ac}(x)g_{bd}(x) - g_{ad}(x)g_{bc}(x)\big)$$

where $g_{ab}$ is the metric tensor of the pseudo-Riemannian manifold $M$ of constant curvature $K$.

## References

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