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The natural operators lifting 1-forms to the r-jet prolongation of the cotangent bundle[1]

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Abstract. First, we classify all natural operators $T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(J^rT^*)$ transforming vector fields to functions on the r-jet prolongation J^rT^* of the cotangent bundle. Next, we classify natural operators $T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ lifting 1-forms from n-manifolds to J^rT^*. As an application we prove that for $r \geq 1$ there is no canonical symplectic structure on J^rT^*. We also solve similar problems with $J^r(\bigwedge^p T^*)$, $J^r(\bigodot^p T^*)$ and $J^r(\bigotimes^p T^*)$ playing the role of J^rT^*.

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0. Introduction

In [1], the authors studied the naturality problem how a 1-form ω on an n-manifold M can induce a 1-form $B(\omega)$ on the cotangent bundle T^*M. This problem is reflected in natural operators $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*T^*$ in the sense of Kolář, Michor and Slovák ([3]). The classification result of ([1]) says that every natural operator $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*T^*$ is of the form $B(\omega) = a\omega^V + b\lambda$, for some $a, b \in \mathbb{R}$, where ω^V is the vertical lifting of ω to T^*M and λ is the canonical Liouville 1-form on T^*M.

In this paper we study similar problem with T^* replaced by its r-jet prolongation J^rT^*. First, we classify all natural operators $T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(J^rT^*)$ transforming vector fields to functions on J^rT^*. Next, we prove that every natural operator $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ transforming a 1-form ω from an n-manifold M into a 1-form $B(\omega)$ on J^rT^*M is of the form $B(\omega) = a\omega^V + b(\pi^r_0)^*\lambda$ for some uniquely determined real numbers a and b, where ω^V is the vertical lifting of ω to J^rT^*M

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

and $\pi_0^r : J^rT^*M \to T^*M$ is the target projection. Here $(\pi_0^r)^*\lambda$ denotes the pull--back of the Liouville 1-form λ on T^*M with respect to π_0^r. As an application we deduce that for $r \geq 1$ there is no canonical symplectic structure on J^rT^*. For $r = 0$ we have $J^0T^*M = T^*M$ and we reobtain the above cited result from ([1]). At the end of this note we solve similar problems with $J^r(\bigwedge^p T^*)$, $J^r(\bigodot^p T^*)$ and $J^r(\bigotimes^p T^*)$ playing the role of J^rT^*.

Natural operators lifting functions, vector fields and 1-form to some natural bundles were used practically in all papers in which the problem of prolongations of geometric structures was studied, e.g., in [10]. That is why such natural operators are classified, see [1], [2]–[9], etc.

From now on the usual coordinates on $\mathbb{R}^n$ are denoted by $x^1, \dots, x^n$.

All manifolds are assumed to be finite dimensional and smooth, i.e., of class $\mathcal{C}^\infty$. Maps between manifolds are assumed to be smooth.

1. Natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(J^rT^*)$

The r-jet prolongation of the cotangent bundle T^*M is the vector bundle (over M) of all r-jets of 1-forms on M: $J^rT^*M = \{j^r_x\eta \mid \eta \in \Omega^1(M),\ x \in M\}$. Every embedding $\varphi : M \to N$ induces a vector bundle map $J^rT^*\varphi : J^rT^*M \to J^rT^*N$ defined by

$$J^rT^*\varphi(j^r_x\eta) = j^r_{\varphi(x)}(\varphi_*\eta),$$

where $j^r_x\eta \in (J^rT^*M)_x$ and $x \in M$. The correspondence $J^rT^* : \mathcal{M}f_n \to \mathcal{VB}$ is a vector bundle functor from the category $\mathcal{M}f_n$ of n-manifolds and their embeddings into the category $\mathcal{VB}$ of vector bundles and their vector bundle maps.

We have the following natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(J^rT^*)$.

Example 1. Let $j = 0, \dots, r$. Let X be a vector field on an n-manifold M. We define $A^{\langle j\rangle}(X) : J^rT^*M \to \mathbb{R}$, $A^{\langle j\rangle}(X)(j^r_x\omega) = X^{\langle j\rangle}(\langle\omega, X\rangle)_x$, $j^r_x\omega \in J^rT^*M$, $\omega \in \Omega^1(M)$, $x \in M$, $X^{\langle j\rangle} = X \circ \cdots \circ X$ (j-times). The correspondence $A^{\langle j\rangle} : T_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(J^rT^*)$ is a natural operator.

Proposition 1. *Every natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(J^rT^*)$ is of the form*

$$A = H(A^{\langle 0\rangle}, \dots, A^{\langle r\rangle})$$

for some uniquely determined smooth map $H \in \mathcal{C}^\infty(\mathbb{R}^{r+1})$.

Proof. For A as above we define $H : \mathbb{R}^{r+1} \to \mathbb{R}$,

$$H(t_0, \dots, t_r) = A\left(\frac{\partial}{\partial x^1}\right)_{j^r_0(\sum_{j=0}^r t_j(x^1)^j\,dx^1)}.$$

We prove that $A = H(A^{\langle 0\rangle}, \dots, A^{\langle r\rangle})$. Since any non-vanishing vector field X is locally $\partial/\partial x^1$ in some local coordinates on M, it is sufficient to show that $A(\partial/\partial x^1)_\sigma = H(A^{\langle 0\rangle}(\partial/\partial x^1)_\sigma, \dots, A^{\langle r\rangle}(\partial/\partial x^1)_\sigma)$ for any $\sigma \in (J^rT^*\mathbb{R}^n)_0$. Using the invariance of A and $A^{\langle j\rangle}$ with respect to $(x^1, (1/t)x^2, \dots, (1/t)x^n) : \mathbb{R}^n \to \mathbb{R}^n$ for $t \neq 0$ and next putting $t \to 0$, we can assume that $\sigma = j^r_0(\sum_{j=0}^r t_j (x^1)^j dx^1)$. Now it remains to observe that $A^{\langle j\rangle}(\partial/\partial x^1)_\sigma = t_j$ for $j = 0, \dots, r$.

The uniqueness of H is clear as $(A^{\langle j\rangle}(\partial/\partial x^1))^r_{j=0}$ is a surjection onto $\mathbb{R}^{r+1}$. □

We have the following immediate corollary.

Corollary 1. *Every natural function on $J^r T^*$ is constant.*

We see that $A^{\langle 0\rangle}(X)$ depends linearly on X, i.e., $A^{\langle 0\rangle}$ is a linear operator. We have the next corollary.

Corollary 2. *Every natural linear operator $A : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(J^r T^*)$ is of the form*

$$A = aA^{\langle 0\rangle}$$

for some uniquely determined real number a.

Proof. The corollary is a consequence of Proposition 1 and the homogeneous function theorem, [3]. □

2. A decomposition proposition

Example 2. If $\omega : TM \to \mathbb{R}$ is a 1-form on an n-manifold M, we have its vertical lifting $B^V(\omega) = \omega \circ T\pi : T(J^r T^* M) \to \mathbb{R}$ to $J^r T^* M$, where $\pi : J^r T^* M \to M$ is the projection. The correspondence

$$B^V : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^r T^*)$$

is a natural operator.

Proposition 2 (Decomposition Proposition). *Consider a natural operator $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^r T^*)$. Then there exists the uniquely determined real number a such that*

$$B = aB^V + \theta$$

for some canonical 1-form θ on $J^r T^*$.

Lemma 1. *We have $B(\omega)|V(J^r T^*\mathbb{R}^n) = 0$ for $\omega \in \Omega^1(\mathbb{R}^n)$, where $V(J^r T^*\mathbb{R}^n)$ is the vertical subbundle in $T(J^r T^*\mathbb{R}^n)$.*

Proof. We use the invariance of B with respect to the homotheties $(1/t)\,\mathrm{id}_{\mathbb{R}^n}$ for $t \neq 0$ and next we put $t \to 0$. □

Proof of Proposition 2. Because of Lemma 1 B is uniquely determined by the values $\langle B(\omega)_\sigma, J^r T^*(\partial/\partial x^1)_\sigma\rangle$ for any $\omega = \sum \omega_i\, dx^i \in \Omega^1(\mathbb{R}^n)$ and any $\sigma \in (J^r T^*\mathbb{R}^n)_0$, where $J^r T^*(\partial/\partial x^1)$ is the complete lifting of $\partial/\partial x^1$ to $J^r T^*\mathbb{R}^n$.

Using the invariance of B with respect to the homotheties $(1/t)\,\mathrm{id}_{\mathbb{R}^n}$ for $t \neq 0$ we get the homogenity condition

$$t\left\langle B(\omega)_\sigma,\, J^rT^*\left(\frac{\partial}{\partial x^1}\right)_\sigma\right\rangle = \left\langle B((t\,\mathrm{id}_{\mathbb{R}^n})^*\omega)_{J^rT^*((1/t)\,\mathrm{id}_{\mathbb{R}^n})(\sigma)},\right.$$
$$\left. J^rT^*\left(\frac{\partial}{\partial x^1}\right)_{J^rT^*((1/t)\,\mathrm{id}_{\mathbb{R}^n})(\sigma)}\right\rangle.$$

Then by the non-linear Petree theorem, [3], and homogeneous function theorem we deduce that $\langle B(\omega)_\sigma, J^rT^*(\partial/\partial x^1)_\sigma\rangle$ depends linearly on $\omega(0)$ and σ.

Then using the invariance of B with respect to $(x^1, (1/t)x^2, \dots, (1/t)x^n) : \mathbb{R}^n \to \mathbb{R}^n$ we deduce that $\langle B(\omega)_\sigma, J^rT^*(\partial/\partial x^1)_\sigma\rangle$ depends linearly on $\omega(0)$ and σ.

These facts complete the proof of Proposition 2. □

3. On canonical 1-forms on J^rT^*

Example 3. Let $\pi_0^r : J^rT^*M \to T^*M$ be the target projection. Let λ be the canonical Liouville 1-form on T^*M. By the pull-back we have the 1-form $\lambda^r = (\pi_0^r)^*\lambda$ on J^rT^*M.

Proposition 3. *Every canonical* 1*-form on* J^rT^* *is a constant multiple of* λ^r.

Proof. Every canonical 1-form θ on J^rT^* induces a linear natural operator $A^{(\theta)} : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(J^rT^*)$ such that $A^{(\theta)}(X)_\sigma = \langle\theta_\sigma, J^rT^*(X)_\sigma\rangle$, $\sigma \in J^rT^*M$, $X \in \mathcal{X}(M)$. The correspondence $\theta \to A^{(\theta)}$ is a linear injection. The injectivity is a consequence of Lemma 1. Then Corollary 2 ends the proof. □

4. The main theorem

The main result of this note is the following classification theorem.

Theorem 1. *Every natural operator* $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ *is of the form*

$$B = aB^V + b\lambda^r$$

for some uniquely determined real numbers a and b.

Proof. It is an immediate consequence of Propositions 2 and 3. □

We have the following immediate corollary.

Corollary 3. *Every linear natural operator* $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ *is a constant multiple of the vertical lifting.*

5. The natural operators $T|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ of closed type

A natural operator $C : T|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ transforming a vector field X on an n-manifold M into a 1-form $C(X)$ on J^rT^*M is of closed type if $C(X)$ is closed for every vector field X on an n-manifold M.

If $A : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(J^rT^*)$ is a natural operator then $dA : T|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$, $(dA)(X) := d(A(X))$, $X \in \mathcal{X}(M)$, is a natural operator of closed type. We prove that every natural operator $T|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ of closed type is of this form.

Proposition 4. *Every natural operator $C : T|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ of closed type is of the form*

$$C = dA$$

for some natural operator $A : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(J^rT^*)$.

Proof. We have the natural operator $\Theta^*C : T|_{\mathcal{M}f_n} \rightsquigarrow T^*$, where Θ is the zero section of J^rT^*. By the well-known result it must be 0.

Let $X \in \mathcal{X}(\mathbb{R}^n)$. Since $C(X)$ is closed, by Poincaré lemma, $C(X) = d(A(X))$ for some $A(X) : J^rT^*\mathbb{R}^n \to \mathbb{R}$. By $\Theta^*C = 0$ we can assume that $A(X) \circ \Theta = 0$. Then

$$A(\varphi_*X) = (J^rT^*\varphi)_*A(X)$$

for any embedding $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, i.e., A corresponds to a natural operator $A : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(J^rT^*)$.

Then $C = dA$. □

Corollary 4. *Every linear natural operator $C : T|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ of closed type is a constant multiple of $dA^{\langle 0\rangle}$.*

Proof. It is a consequence of Corollary 2 and Proposition 4. □

Corollary 5. *There are no canonical closed 1-forms on J^rT^*.*

Proof. It is a consequence of Propositions 1 and 4. It is also a consequence of Proposition 3 because λ^r is not closed. □

6. Canonical closed 2-forms on J^rT^* and non-existence of a canonical symplectic structure on J^rT^*

Proposition 5. *Every canonical closed 2-form Ω on J^rT^* is a constant multiple of $d\lambda^r$.*

Proof. Using Ω we define a linear natural operator $C : T|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ of closed type such that $C(X) = \Omega(J^rT^*(X))$, $X \in \mathcal{X}(M)$. By Corollary 4 this operator is proportional to $dA^{\langle 0\rangle}$.

On the other hand, similarly as in Lemma 1, $\Omega|(V(J^rT^*M)\times_{J^rT^*M}V(J^rT^*M)) = 0$, i.e., Ω is uniquely determined by C.
These facts complete the proof. □

We have the following interesting corollary of Proposition 5.

Corollary 6. *There are no canonical symplectic structures on J^rT^* for $r \geq 1$.*

Remark 1. In ([8]) we proved Corollary 6 in another way. (We proved that there were no canonical volume forms on J^rT^* for $r \geq 1$.)

7. A generalization

Using similar methods as in Sections 1–6 one can deduce the following results. We leave the details to the reader.

7.1. The case of $\bigwedge^p T^*$

The r-jet prolongation of the skew p-power $\bigwedge^p T^*M$ of the cotangent bundle is the vector bundle $J^r(\bigwedge^p T^*M) = \{j^r_x\eta \mid \eta \in \Omega^p(M),\ x \in M\}$ over M of all r-jets of p-forms on M. Every embedding $\varphi : M \to N$ induces a vector bundle map $J^r(\bigwedge^p T^*)\varphi : J^r(\bigwedge^p T^*M) \to J^r(\bigwedge^p T^*N)$, $J^r(\bigwedge^p T^*)\varphi(j^r_x\eta) = j^r_{\varphi(x)}(\varphi_*\eta)$, $j^r_x\eta \in (J^r(\bigwedge^p T^*M))_x$, $x \in M$. The correspondence $J^r(\bigwedge^p T^*) : \mathcal{M}f_n \to \mathcal{VB}$ is a vector bundle functor.

Proposition 6. *For $p \geq 2$ every natural operator*

$$A : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}\Big(J^r\Big(\bigwedge^p T^*\Big)\Big)$$

lifting a vector field on an n-manifold M into a function $A(X) : J^r(\bigwedge^p T^*M) \to \mathbb{R}$ is of the form $A = c$ for $c \in \mathbb{R}$.

Proposition 7 (Decomposition Proposition). *Let $p \geq 2$. Consider a natural operator $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^r(\bigwedge^p T^*))$ lifting a 1-form ω on an n-manifold M into a 1-form $B(\omega)$ on $J^r(\bigwedge^p T^*M)$. Then there exists the uniquely determined real number a such that $B = aB^V + \theta$ for some canonical 1-form θ on $J^r(\bigwedge^p T^*)$.*

Proposition 8. *For $p \geq 2$ every canonical 1-form on $J^r(\bigwedge^p T^*)$ is zero.*

Theorem 2. *For $p \geq 2$ every natural operator $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^r(\bigwedge^p T^*))$ is a constant multiple of the vertical lifting.*

Proposition 9. *For $p \geq 2$ every natural operator*

$$C : T|_{\mathcal{M}f_n} \rightsquigarrow T^*\Big(J^r\Big(\bigwedge^p T^*\Big)\Big)$$

of closed type is zero.

Proposition 10. *For $p \geq 2$ every canonical closed 2-form on $J^r(\bigwedge^p T^*)$ is zero. In particular there is no canonical symplectic structure on $J^r(\bigwedge^p T^*)$.*

7.2. The case of $\bigodot^p T^*$

The r-jet prolongation of $\bigodot^p T^*M$ is the vector bundle

$$J^r\Big(\bigodot^p T^*M\Big) = \{j^r_x\sigma | \sigma \in \mathrm{Sym}\,\mathcal{T}^{(0,p)}(M),\ x \in M\}$$

over M of all r-jets of symmetric tensor fields σ of type $(0, p)$ on M. Every embedding $\varphi : M \to N$ induces a vector bundle map $J^r(\bigodot^p T^*)\varphi : J^r(\bigodot^p T^*M) \to J^r(\bigodot^p T^*N)$, $J^r(\bigodot^p T^*)\varphi(j^r_x\sigma) = j^r_{\varphi(x)}(\varphi_*\sigma)$, $j^r_x\sigma \in (J^r(\bigodot^p T^*M))_x$, $x \in M$. The correspondence $J^r(\bigodot^p T^*) : \mathcal{M}f_n \to \mathcal{VB}$ is a vector bundle functor.

Example 4. Let $j = 0, \dots, r$. Let X be a vector field on an n-manifold M. We define $A^{\langle j\rangle}(X) : J^r(\bigodot^p T^*M) \to \mathbb{R}$, $A^{\langle j\rangle}(X)(j^r_x\sigma) = X^{\langle j\rangle}(\langle\sigma, \bigodot^p X\rangle)_x$, $j^r_x\sigma \in J^r(\bigodot^p T^*M)$, $\sigma \in \mathrm{Sym}\,\mathcal{T}^{(0,p)}(X)$, $x \in M$, $X^{\langle j\rangle} = X \circ \dots \circ X$ (j-times). The correspondence $A^{\langle j\rangle} : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(J^r(\bigodot^p T^*))$ is a natural operator.

Proposition 11. *For $p \geq 2$ every natural operator*

$$A : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}\Big(J^r\Big(\bigodot^p T^*\Big)\Big)$$

is of the form

$$A = H(A^{\langle 0\rangle}, \dots, A^{\langle r\rangle})$$

for some uniquely determined smooth map $H \in \mathcal{C}^\infty(\mathbb{R}^{r+1})$.

Proposition 12 (Decomposition Proposition). *Let $p \geq 2$. Consider a natural operator $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^r(\bigodot^p T^*))$ lifting a 1-form ω on an n-manifold M into a 1-form $B(\omega)$ on $J^r(\bigodot^p T^*M)$. Then there exists the uniquely determined real number a such that $B = aB^V + \theta$ for some canonical 1-form θ on $J^r(\bigodot^p T^*)$.*

Proposition 13. *For $p \geq 2$ every canonical 1-form on $J^r(\bigodot^p T^*)$ is zero.*

Theorem 3. *For $p \geq 2$ every natural operator $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^r(\bigodot^p T^*))$ is a constant multiple of the vertical lifting.*

Proposition 14. *For $p \geq 2$ every natural operator*

$$C : T|_{\mathcal{M}f_n} \rightsquigarrow T^*\Big(J^r\Big(\bigodot^p T^*\Big)\Big)$$

of closed type is zero.

Proposition 15. *For $p \geq 2$ every canonical closed 2-form on $J^r(\bigodot^p T^*)$ is zero. In particular there is no canonical symplectic structure on $J^r(\bigodot^p T^*)$.*

7.3. The case of $\bigotimes^p T^*$

The r-jet prolongation of $\bigotimes^p T^*M$ is the vector bundle

$$J^r\Big(\bigotimes^p T^*M\Big) = \{j^r_x\tau \,|\, \tau \in \mathcal{T}^{(0,p)}(M),\ x \in M\}$$

over M of all r-jets of tensor fields τ of type $(0, p)$ on M. Every embedding $\varphi : M \to N$ induces a vector bundle map $J^r(\bigotimes^p T^*)\varphi : J^r(\bigotimes^p T^*M) \to J^r(\bigotimes^p T^*N)$, $J^r(\bigotimes^p T^*)\varphi(j^r_x\tau) = j^r_{\varphi(x)}(\varphi_*\tau)$, $j^r_x\tau \in (J^r(\bigotimes^p T^*M))_x$, $x \in M$. The correspondence

$$J^r\Big(\bigotimes^p T^*\Big) : \mathcal{M}f_n \to \mathcal{VB}$$

is a vector bundle functor.

Example 5. Let $j = 0, \dots, r$. Let X be a vector field on an n-manifold M. We define $A^{\langle j\rangle}(X) : J^r(\bigotimes^p T^*M) \to \mathbb{R}$, $A^{\langle j\rangle}(X)(j^r_x\tau) = X^{\langle j\rangle}(\langle\tau, \bigotimes^p X\rangle)_x$, $j^r_x\tau \in J^r(\bigotimes^p T^*M)$, $\tau \in \mathcal{T}^{(0,p)}(X)$, $x \in M$, $X^{\langle j\rangle} = X \circ \cdots \circ X$ (j-times). The correspondence $A^{\langle j\rangle} : T_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(J^r(\bigotimes^p T^*))$ is a natural operator.

Proposition 16. *For $p \geq 2$ every natural operator*

$$A : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}\Big(J^r\Big(\bigotimes^p T^*\Big)\Big)$$

is of the form

$$A = H(A^{\langle 0\rangle}, \dots, A^{\langle r\rangle})$$

for some uniquely determined smooth map $H \in \mathcal{C}^\infty(\mathbb{R}^{r+1})$.

Proposition 17 (Decomposition Proposition). *Let $p \geq 2$. Consider a natural operator $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^r(\bigotimes^p T^*))$ lifting a 1-form ω on an n-manifold M into a 1-form $B(\omega)$ on $J^r(\bigotimes^p T^*M)$. Then there exists the uniquely determined real number a such that $B = aB^V + \theta$ for some canonical 1-form θ on $J^r(\bigotimes^p T^*)$.*

Proposition 18. *For $p \geq 2$ every canonical 1-form on $J^r(\bigotimes^p T^*)$ is zero.*

Theorem 4. *For $p \geq 2$ every natural operator $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(J^r(\bigotimes^p T^*))$ is a constant multiple of the vertical lifting.*

Proposition 19. *For $p \geq 2$ every natural operator*

$$C : T|_{\mathcal{M}f_n} \rightsquigarrow T^*\Big(J^r\Big(\bigotimes^p T^*\Big)\Big)$$

of closed type is zero.

Proposition 20. *For $p \geq 2$ every canonical closed 2-form on $J^r(\bigotimes^p T^*)$ is zero. In particular there is no canonical symplectic structure on $J^r(\bigotimes^p T^*)$.*

7.4. The case of $\bigwedge^p(J^rT^*)$

For every n-manifold M we have $\bigwedge^p(J^rT^*M)$, the skew-symmetric tensor p-power of J^rT^*M. Every embedding $\varphi : M \to N$ induces

$$\bigwedge^p(J^rT^*\varphi) : \bigwedge^p(J^rT^*M) \to \bigwedge^p(J^rT^*N).$$

The correspondence $\bigwedge^p(J^rT^*) : \mathcal{M}f_n \to \mathcal{VB}$ is a vector bundle functor.

Example 6. Let $C_p^{r+1} = \{i = (i_1, \dots, i_p) \mid i_1, \dots, i_p = 0, 1, \dots, r,\ 0 \le i_1 < i_2 < \cdots < i_p \le r\}$. Consider $i = \{i_1, \dots, i_p\} \in C_p^{r+1}$. Let X be a vector field on an n-manifold M. We define $A^{[i]}(X) = A^{\langle i_1\rangle}(X) \wedge \cdots \wedge A^{\langle i_p\rangle}(X) : \bigwedge^p(J^rT^*M) \to \mathbb{R}$, where $A^{\langle j\rangle}$ for $j = 0, \dots, r$, are the natural operators from Example 1. (We observe that $A^{\langle j\rangle}(X) : J^rT^*M \to \mathbb{R}$ is fiber linear, i.e., a section of $(J^rT^*M)^*$. So, we can apply the skew-symmetric tensor product.) The correspondence

$$A^{[i]} : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}\Big(\bigwedge^p(J^rT^*)\Big)$$

is a natural operator.

Proposition 21. *For $p \ge 2$ every natural operator*

$$A : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}\Big(\bigwedge^p(J^rT^*)\Big)$$

lifting a vector field on an n-manifold M into a function $A(X) : \bigwedge^p(J^rT^*M) \to \mathbb{R}$ is of the form

$$A = H\big((A^{[i]})_{i\in C_p^{r+1}}\big)$$

for some uniquely determined smooth map $H : \mathbb{R}^{C_p^{r+1}} \to \mathbb{R}$.

Proposition 22 (Decomposition Proposition). *Let $p \ge 2$. Consider a natural operator $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(\bigwedge^p(J^rT^*))$ lifting a 1-form ω on an n-manifold M into a 1-form $B(\omega)$ on $\bigwedge^p(J^rT^*M)$. Then there exists the uniquely determined real number a such that $B = aB^V + \theta$ for some canonical 1-form θ on $\bigwedge^p(J^rT^*)$.*

Proposition 23. *For $p \ge 2$ every canonical 1-form on $\bigwedge^p(J^rT^*)$ is zero.*

Theorem 5. *For $p \ge 2$ every natural operator $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(\bigwedge^p(J^rT^*))$ is a constant multiple of the vertical lifting.*

Proposition 24. *For $p \ge 2$ every natural operator*

$$C : T|_{\mathcal{M}f_n} \rightsquigarrow T^*\Big(\bigwedge^p(J^rT^*)\Big)$$

of closed type is zero.

Proposition 25. *For $p \geq 2$ every canonical closed 2-form on $\bigwedge^p(J^rT^*)$ is zero. In particular there is no canonical symplectic structure on $\bigwedge^p(J^rT^*)$.*

7.5. The case of $\bigodot^p(J^rT^*)$

For every n-manifold M we have $\bigodot^p(J^rT^*M)$, the symmetric tensor p-power of J^rT^*M. Any embedding $\varphi : M \to N$ induces

$$\bigodot^p(J^rT^*\varphi) : \bigodot^p(J^rT^*M) \to \bigodot^p(J^rT^*N).$$

The correspondence $\bigodot^p(J^rT^*) : \mathcal{M}f_n \to \mathcal{VB}$ is a vector bundle functor.

Example 7. Let $S_p^{r+1} = \{i = (i_1, \dots, i_p) \mid i_1, \dots, i_p = 0, \dots, r,\ 0 \leq i_1 \leq i_2 \leq \dots \leq i_p \leq r\}$. Consider $i = (i_1, \dots, i_p) \in S_p^{r+1}$. Let X be a vector field on an n-manifold M. We define $A^{[\![i]\!]}(X) = A^{\langle i_1 \rangle}(X) \odot \dots \odot A^{\langle i_p \rangle}(X) : \bigodot^p(J^rT^*M) \to \mathbb{R}$, where $A^{\langle j \rangle}$ for $j = 0, \dots, r$ are the natural operators from Example 1. The correspondence $A^{[\![i]\!]} : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(\bigodot^p(J^rT^*))$ is a natural operator.

Proposition 26. *For $p \geq 2$ every natural operator*

$$A : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}\Big(\bigodot^p(J^rT^*)\Big)$$

lifting a vector field on an n-manifold M into a function $A(X) : \odot^p(J^rT^*M) \to \mathbb{R}$ is of the form

$$A = H\big((A^{[\![i]\!]})_{i \in S_p^{r+1}}\big)$$

for some uniquely determined smooth map $H : \mathbb{R}^{S_p^{r+1}} \to \mathbb{R}$.

Proposition 27 (Decomposition Proposition). *Let $p \geq 2$. Consider a natural operator $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(\bigodot^p(J^rT^*))$ lifting a 1-form ω on an n-manifold M into a 1-form $B(\omega)$ on $\odot^p(J^rT^*M)$. Then there exists the uniquely determined real number a such that $B = aB^V + \theta$ for some canonical 1-form θ on $\bigodot^p(J^rT^*)$.*

Proposition 28. *For $p \geq 2$ every canonical 1-form on $\bigodot^p(J^rT^*)$ is zero.*

Theorem 6. *For $p \geq 2$ every natural operator $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(\bigodot^p(J^rT^*))$ is a constant multiple of the vertical lifting.*

Proposition 29. *For $p \geq 2$ every natural operator*

$$C : T|_{\mathcal{M}f_n} \rightsquigarrow T^*\Big(\bigodot^p(J^rT^*)\Big)$$

of closed type is zero.

Proposition 30. *For $p \geq 2$ any canonical closed 2-form on $\bigodot^p(J^rT^*)$ is zero. In particular there is no canonical symplectic structure on $\bigodot^p(J^rT^*)$.*

7.6. The case of $\bigotimes^p(J^rT^*)$

For every n-manifold M we have $\bigotimes^p(J^rT^*M)$, the tensor p-power of J^rT^*M. Every embedding $\varphi : M \to N$ induces

$$\bigotimes^p(J^rT^*\varphi) : \bigotimes^p(J^rT^*M) \to \bigotimes^p(J^rT^*N).$$

The correspondence

$$\bigotimes^p(J^rT^*) : \mathcal{M}f_n \to \mathcal{VB}$$

is a vector bundle functor.

Example 8. Consider $i = (i_1, \dots, i_p) \in \{0, \dots, r\}^p$. Let X be a vector field on an n-manifold M. We define

$$A^{[\![i]\!]}(X) = A^{\langle i_1\rangle}(X) \otimes \cdots \otimes A^{\langle i_p\rangle}(X) : \bigotimes^p(J^rT^*M) \to \mathbb{R},$$

where $A^{\langle j\rangle}$ for $j = 0, \dots, r$ are the natural operators from Example 1. The correspondence $A^{[\![i]\!]} : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(\bigotimes^p(J^rT^*))$ is a natural operator.

Proposition 31. *For $p \geq 2$ every natural operator*

$$A : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}\Big(\bigotimes^p(J^rT^*)\Big)$$

lifting a vector field on an n-manifold M into a function $A(X) : \bigotimes^p(J^rT^*M) \to \mathbb{R}$ is of the form

$$A = H\big((A^{[\![i]\!]})_{i\in\{0,\dots,r\}^p}\big)$$

for some uniquely determined smooth map $H : \mathbb{R}^{\{0,\dots,r\}^p} \to \mathbb{R}$.

Proposition 32 (Decomposition Proposition). *Let $p \geq 2$. Consider a natural operator $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(\bigotimes^p(J^rT^*))$ lifting a 1-form ω on an n-manifold M into a 1-form $B(\omega)$ on $\bigotimes^p(J^rT^*M)$. Then there exists the uniquely determined real number a such that $B = aB^V + \theta$ for some canonical 1-form θ on $\bigotimes^p(J^rT^*)$.*

Proposition 33. *For $p \geq 2$ every canonical 1-form on $\bigotimes^p(J^rT^*)$ is zero.*

Theorem 7. *For $p \geq 2$ every natural operator $B : T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(\bigotimes^p(J^rT^*))$ is a constant multiple of the vertical lifting.*

Proposition 34. *For $p \geq 2$ every natural operator*

$$C : T|_{\mathcal{M}f_n} \rightsquigarrow T^*\Big(\bigotimes^p(J^rT^*)\Big)$$

of closed type is zero.

Proposition 35. *For $p \geq 2$ any canonical closed 2-form on $\bigotimes^p(J^rT^*)$ is zero. In particular there is no canonical symplectic structure on $\bigotimes^p(J^rT^*)$.*

8. Final remarks

Remark 2. Of course, the list of generalizations in Section 7 presenting the results from Sections 1–6 is not complete. For example we can consider

$$J^r(J^{r_1}T^* \otimes \cdots \otimes J^{r_K}T^*).$$

Remark 3. The cotangent bundle functor T^* can be considered as $(T|_{\mathcal{M}f_n})^*$. So we can formulate the following most general problems. Let $F : \mathcal{M}f \to \mathcal{VB}$ be a vector bundle functor. Find all natural operators

$$T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(F|_{\mathcal{M}f_n})^*$$

and

$$T^*|_{\mathcal{M}f_n} \rightsquigarrow T^*(F|_{\mathcal{M}f_n})^*.$$

In Sections 1–6 we solved these problems for $F = (J^rT^*)^*$ and extended (in Section 7) for $F = (J^r(\bigotimes^p T^*))^*$, $F = (\bigotimes^p(J^rT^*))^*$, etc.

Now, we have (almost) solved the above problems and we will present them in some next paper. We will apply them (for example) for $F = (J^r(\cdot, \mathbb{R}^p))^*$, the extended linear (p, r)-velocities bundle.

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