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Isometry-invariant geodesics with Lipschitz obstacle[1]

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Abstract. Given a linear isometry $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ of finite order on $\mathbb{R}^n$, a general $\langle A_0 \rangle$-invariant closed subset M of $\mathbb{R}^n$ is considered with Lipschitz boundary. Under suitable topological restrictions the existence of A_0-invariant geodesics of M is proven.

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1. Introduction

The existence of closed geodesics on Riemannian manifolds was studied by many authors, see W. Klingenberg ([11]) and the reference therein. For non-smooth sets, the problem was studied by A. Canino (for p-convex sets, see [1]) and M. Degiovanni and L. Morbini (for subsets in $\mathbb{R}^n$ with Lipschitz boundary, see [7]). The existence of isometry-invariant geodesics on Riemannian manifold has been studied first by K. Grove in [8, 9, 10]. The purpose of this paper is to establish a similar existence result like that of K. Grove for isometry-invariant subsets of $\mathbb{R}^n$ with Lipschitz boundary.

Now we formulate the main result of our paper. Here only some major notions are described. For more details, see the following sections.

Let $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry on $\mathbb{R}^n$, $G = \langle A_0 \rangle$. Let $g\colon \mathbb{R}^n \to \mathbb{R}$ denote a G-invariant Lipschitz function, $M = \{x \in \mathbb{R}^n : g(x) \leq 0\}$, and let $\Lambda_{G(A_0)}(M)$ denote the subset of continuous curves γ in M with $A_0(\gamma(0)) = \gamma(1)$. The following topological hypotheses will be assumed to ensure the conclusions of the main theorem:

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(H_1) If the elements of $\mathrm{Fix}_M A_0$ are isolated, then $\mathrm{Fix}_M A_0$ is not homotopically equivalent with $\Lambda_{G(A_0)}(M)$.

(H_2) The inclusion $i : \mathrm{Fix}_M A_0 \hookrightarrow \Lambda_{G(A_0)}(M)$ does not induce an isomorphism in the Alexander–Spanier cohomology, see [13].

The main theorem of the paper states:

Theorem 1.1. *Let $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry of finite order and $g : \mathbb{R}^n \to \mathbb{R}$ a G-invariant locally Lipschitz map, where $G = \langle A_0 \rangle$. Suppose that*

$$\forall x \in \mathbb{R}^n : g(x) = 0 \Rightarrow 0 \notin \partial g(x)$$

and let

$$M = \{x \in \mathbb{R}^n : g(x) \le 0\}.$$

If M is compact and (H_1) or (H_2) holds, then there exists at least one non-trivial A_0-invariant geodesic on M.

The proof is given in Section 5. This theorem generalizes both Degiovanni and Morbini's result ([7, Theorem 4.1]) on the existence of closed geodesic in the non--smooth case, and Grove's result ([9]) on the existence of isometry-invariant geodesics of Riemannian spaces.

2. Preliminaries

First we recall some definitions and results from the papers of J.-N. Corvellec ([5]), M. Degiovanni and M. Marzocchi ([6]), M. Degiovanni and L. Morbini ([7]).

Let (M, d) be a metric space, and $B(u; \delta)$ the open ball of center u and radius δ. First we define the weak slope of a continuous function.

Definition 2.1. Let $f : M \to \mathbb{R}$ be a continuous function. For every $u \in M$ we denote by $|df|(u)$ the supremum of the numbers $\sigma \in [0, \infty[$ such that there exist a number $\delta > 0$ and a continuous map $\mathcal{H} : B(u; \delta) \times [0, \delta] \to M$ such that for every $v \in B(u; \delta)$ and $t \in [0, \delta]$ the following assertions hold

$$d\big(\mathcal{H}(v, t), v\big) \le t; \tag{1}$$

$$f\big(\mathcal{H}(v, t)\big) \le f(v) - \sigma t. \tag{2}$$

The extended real number $|df|(u)$ is called the *weak slope* of the function f in the point u.

We consider now a lower semi-continuous function $f : M \to \mathbb{R} \cup \{+\infty\}$. We set

$$\begin{aligned}
\mathcal{D}(f) &= \big\{u \in M \big| f(u) < +\infty\big\};\\
\overline{f}^{\,b} &= \big\{u \in M \big| f(u) \le b\big\};\\
f^b &= \big\{u \in M \big| f(u) < b\big\};\\
\mathrm{epi}(f) &= \big\{(u, \alpha) \in M \times \mathbb{R} \big| f(u) \le \alpha\big\}.
\end{aligned}$$

The set epi(f) will be endowed with the metric

$$d_e\big((u,\lambda),(v,\mu)\big) = \big(d^2(u,v) + (\lambda-\mu)^2\big)^{\frac{1}{2}},$$

denoted (for simplicity) again by d. We define the function $\mathcal{G}_f : \text{epi}(f) \to \mathbb{R}$ by $\mathcal{G}_f(u,\alpha) = \alpha$. The function $\mathcal{G}_f$ is Lipschitz continuous with constant 1. Through $\mathcal{G}_f$ the notion of the weak slope can be extended to lower semi-continuous functions.

Definition 2.2. Let $f : M \to \mathbb{R} \cup \{\infty\}$ be a lower semi-continuous function and $u \in \mathcal{D}(f)$ a fixed point. We set

$$|df|(u) = \begin{cases} \dfrac{|d\mathcal{G}_f|\big(u,\, f(u)\big)}{\sqrt{1-|d\mathcal{G}_f|^2\big(u,\, f(u)\big)}}, & |d\mathcal{G}_f|\big(u,\, f(u)\big) < 1 \\ +\infty, & |d\mathcal{G}_f|\big(u,\, f(u)\big) = 1. \end{cases}$$

In the sequel we shall use the following result from the paper of M. Degiovanni and L. Morbini.

Proposition 2.3 ([7], Prop. 2.3). *Let* $(u,\lambda) \in \text{epi}(f)$. *We assume that there exist* $\delta, c, \sigma > 0$ *and a continuous map* $\mathcal{H} : \{v \in B(u;\delta) \mid f(v) < \lambda+\delta\} \times [0,\delta] \to M$ *such that for any* $v \in B(u;\delta)$ *with* $f(v) < \lambda + \delta$ *and any* $t \in [0,\delta]$ *we have*

$$d\big(\mathcal{H}(v,t),v\big) \le ct,$$

$$f\big(\mathcal{H}(v,t)\big) \le f(v) - \sigma t.$$

Then we have

$$|d\mathcal{G}_f|(u,\lambda) \ge \frac{\sigma}{\sqrt{\sigma^2+c^2}}.$$

In particular, if $\lambda = f(u)$, *then we have*

$$|df|(u) \ge \frac{\sigma}{c}.$$

Remark 2.4. If M is a $\mathcal{C}^1$-Finsler manifold and $f : M \to \mathbb{R}$ is a $\mathcal{C}^1$-function, then $|df|(u) = \|df(u)\|$, cf. [6].

Definition 2.5. We say that $u \in \mathcal{D}(f)$ is a *critical point of* f if $|df|(u) = 0$. We say that $c \in \mathbb{R}$ is a *critical value* of the function f if there exists a critical point $u \in \mathcal{D}(f)$ with $f(u) = c$.

For every $c \in \mathbb{R}$ we set $K(f)_c = \{u \in \mathcal{D}(f) : |df|(u) = 0,\ f(u) = c\}$.

Definition 2.6. Let $c \in \mathbb{R}$ be a fixed number. We say that the function f satisfies the *Palais–Smale condition at level c* (shortly $(PS)_c$) if every sequence $(u_n) \subset M$ such that $f(u_n) \to c$ and $|df|(u_n) \to 0$ contains a convergent subsequence in M.

3. Geodesics under isometry action

We will work with geodesics introduced in the paper [7]. For this we need to fix some notions and notations.

Let M be a closed subset of $\mathbb{R}^n$. Each $\gamma \in W^{1,2}(a,b;\mathbb{R}^n)$ will be identified with its continuous representative $\tilde{\gamma} : [a,b] \to \mathbb{R}^n$. We will denote by $\|\cdot\|_{1,2}$ and $\|\cdot\|_p$ the usual norms in $W^{1,2}(a,b;\mathbb{R}^n)$ and $L^p(a,b;\mathbb{R}^n)$ with $1 \le p \le \infty$. We consider

$$W^{1,2}(a,b;M) = \left\{\gamma \in W^{1,2}(a,b;\mathbb{R}^n) : \gamma(s) \in M \quad \forall s \in [a,b]\right\}$$

and let the functional $E_{a,b} : W^{1,2}(a,b;M) \to \mathbb{R}$ by

$$E_{a,b}(\gamma) := \frac{1}{2}\int_a^b |\gamma'(s)|^2\, ds.$$

Definition 3.1 ([7], Def. 3.1). Let $a, b \in \mathbb{R}$, $a < b$. A curve $\gamma \in W^{1,2}(a,b;M)$ is *energy-stationary* if it is not possible to find $\delta, c, \sigma > 0$ and a map

$$\mathcal{H} : \left\{\eta \in W^{1,2}(a,b;M) : \|\eta - \gamma\|_{1,2} < \delta\right\} \times [0,\delta] \longrightarrow W^{1,2}(a,b;M)$$

such that

a) $\mathcal{H}$ is continuous from the topology of $L^2(a,b;\mathbb{R}^n) \times \mathbb{R}$ to that of $L^2(a,b;\mathbb{R}^n)$;

b) for every $\eta \in W^{1,2}(a,b;M)$ with $\|\eta - \gamma\|_{1,2} < \delta$ and $t \in [0,\delta]$, we have

$$\big(\mathcal{H}(\eta,t) - \eta\big) \in W_0^{1,2}(a,b;\mathbb{R}^n);$$

$$\big\|\mathcal{H}(\eta,t) - \eta\big\|_2 \le ct;$$

$$E_{a,b}\big(\mathcal{H}(\eta,t)\big) \le E_{a,b}(\eta) - \sigma t.$$

Remark 3.2. It is easy to show that, if the curve $\gamma \in W^{1,2}(a,b;M)$ is energy-stationary, then for every $[\alpha,\beta] \subseteq [a,b]$ the restriction $\gamma|_{[\alpha,\beta]}$ is energy-stationary too, see [7, Proposition 3.2].

Definition 3.3 ([7], Def. 3.3). Let I be an interval in $\mathbb{R}$ with $\operatorname{int}(I) \ne \emptyset$. A continuous map $\gamma : I \to M$ is a *geodesic on M*, if every $s \in \operatorname{int}(I)$ admits a neighborhood $[a,b]$ in I such that $\gamma|_{[a,b]} \in W^{1,2}(a,b;M)$ and $\gamma|_{[a,b]}$ is energy-stationary.

In [7, Theorem 3.5], it is shown that if γ is a geodesic, then it is Lipschitzian and $|\gamma'|$ is almost everywhere equal to a constant.

In the sequel, let $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry, i.e., A_0 is a linear map and $\langle A_0x, A_0y\rangle = \langle x, y\rangle$, for every $x, y \in \mathbb{R}^n$, where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product. Moreover, let us suppose that $A_0(M) = M$.

Definition 3.4. An *A_0-invariant geodesic on M* is a geodesic $\gamma : \mathbb{R} \to M$ such that

$$A_0(\gamma(s)) = \gamma(s+1), \quad \forall s \in \mathbb{R}.$$

The above curve is non-trivial if this does not reduce to a point. Otherwise, this point will be a fixed point of A_0.

Let $X_{A_0} = \{\gamma \in W^{1,2}(0,1;M) \mid A_0(\gamma(0)) = \gamma(1)\}$. We define the functional $f_{A_0} : L^2(0,1;\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ by

$$f_{A_0}(\gamma) = \begin{cases} E_{0,1}(\gamma), & \text{if } \gamma \in X_{A_0} \\ +\infty, & \text{otherwise.} \end{cases}$$

Naturally, the functional f_{A_0} is lower semi-continuous and is A_0-invariant, i.e., $f_{A_0}(A_0\gamma) = f_{A_0}(\gamma) = f_{A_0}(A_0^{-1}\gamma)$, for every $\gamma \in L^2(0,1,\mathbb{R}^n)$. First, we prove the following

Theorem 3.5. *If $\gamma \in X_{A_0}$ is a critical point of f_{A_0}, then γ is a geodesic on M.*

Proof. By contradiction, let $s_0 \in (0,1)$ such that for every neighborhood $[a,b]$ in $[0,1]$, the curve $\gamma|_{[a,b]}$ is not energy-stationary on $[a,b]$. Let $[a,b] := [0,1]$, $\delta, c, \zeta > 0$ and let $\mathcal{H}$ be a continuous function as in Definition 3.1. There exists $\delta' \in]0,\delta]$ such that for all $\eta \in X_{A_0}$

$$\|\eta - \gamma\|_2 < \delta' \quad \text{and} \quad f_{A_0}(\eta) < f_{A_0}(\gamma) + \delta' \Longrightarrow \|\eta - \gamma\|_{1,2} < \delta.$$

For every $\eta \in X_{A_0}$ with $d_2(\eta,\gamma) = \|\eta - \gamma\|_2 < \delta'$ and $f_{A_0}(\eta) < f_{A_0}(\gamma) + \delta'$ we have

$$d_2\big(\mathcal{H}(\eta,t),\eta\big) \le ct,$$
$$f_{A_0}\big(\mathcal{H}(\eta,t)\big) \le f_{A_0}(\eta) - \zeta t.$$

Applying Proposition 2.3 for f_{A_0} and $L^2(0,1,\mathbb{R}^n)$, it follows that γ is not a critical point for f_{A_0}, which is a contradiction. □

Theorem 3.6. *For every $\gamma \in X_{A_0}$ we have the relation*

$$|df_{A_0}|(A_0\gamma) = |df_{A_0}|(\gamma).$$

Proof. We prove that $|d\mathcal{G}_{f_{A_0}}|(\gamma, f_{A_0}(\gamma)) = |d\mathcal{G}_{f_{A_0}}|(A_0\gamma, f_{A_0}(\gamma))$. First, we prove that $|d\mathcal{G}_{f_{A_0}}|(\gamma, f_{A_0}(\gamma)) \le |d\mathcal{G}_{f_{A_0}}|(A_0\gamma, f_{A_0}(\gamma))$.

If $|d\mathcal{G}_{f_{A_0}}|(\gamma, f_{A_0}(\gamma)) = 0$, then the relation is obvious. Otherwise, let $0 < \zeta < |d\mathcal{G}_{f_{A_0}}|(\gamma, f_{A_0}(\gamma))$, $\delta > 0$ and the continuous function

$$\mathcal{H} : \Big(B\big((\gamma, f_{A_0}(\gamma)); \delta\big) \cap \mathrm{epi}(f_{A_0})\Big) \times [0,\delta] \to \mathrm{epi}(f_{A_0})$$

such that the following assertions hold

$$d\big(\mathcal{H}((\beta,\tau),t),(\beta,\tau)\big) \le t,$$
$$\mathcal{G}_{f_{A_0}}\big(\mathcal{H}((\beta,\tau),t)\big) \le \mathcal{G}_{f_{A_0}}(\beta,\tau) - \zeta t, \tag{3}$$

for all $(\beta,\tau) \in (B((\gamma, f_{A_0}(\gamma)); \delta) \cap \mathrm{epi}(f_{A_0}))$, $t \in [0,\delta]$.

We consider the function $\widetilde{\mathcal{H}} = (\widetilde{\mathcal{H}}_1, \widetilde{\mathcal{H}}_2) : (B((A_0\gamma, f_{A_0}(\gamma)); \delta) \cap \mathrm{epi}(f_{A_0})) \times [0,\delta] \to \mathrm{epi}(f_{A_0})$ defined by

$$\widetilde{\mathcal{H}}\big((\beta,\tau),t\big) = \Big(\big(A_0\mathcal{H}_1(A_0^{-1}\beta,\tau),t\big), \mathcal{H}_2\big((A_0^{-1}\beta,\tau),t\big)\Big).$$

We have that the function $\widetilde{\mathcal{H}}$ is well defined and is continuous. We have, for every $(\beta, \tau) \in (B((A_0\gamma,\ f_{A_0}(\gamma)); \delta) \cap \operatorname{epi}(f_{A_0}))$ and $t \in [0, \delta]$

$$\begin{aligned} & d\Big(\widetilde{\mathcal{H}}\big((\beta,\tau),t\big),(\beta,\tau)\Big) \\ & = d\Big(A_0\mathcal{H}_1\big((A_0^{-1}\beta,\tau),t\big),\mathcal{H}_2\big((A_0^{-1}\beta,\tau),t\big),(\beta,\tau)\Big) \\ & = d\Big(\Big(\mathcal{H}_1\big((A_0^{-1}\beta,\tau),t\big),\mathcal{H}_2\big((A_0^{-1}\beta,\tau),t\big)\Big),(A_0^{-1}\beta,\tau)\Big) \\ & = d\Big(\mathcal{H}\big((A_0^{-1}\beta,\tau),t\big),(A_0^{-1}\beta,\tau)\Big) \le t, \end{aligned} \tag{4}$$

and

$$\begin{aligned} \mathcal{G}_{f_{A_0}}\Big(\widetilde{\mathcal{H}}\big((\beta,\tau),t\big)\Big) &= \widetilde{\mathcal{H}}_2\big((\beta,\tau),t\big) = \mathcal{H}_2\big((A_0^{-1}\beta,\tau),t\big) \\ &\le \mathcal{G}_{f_{A_0}}(A_0^{-1}\beta,\tau) - \zeta t = \mathcal{G}_{f_{A_0}}(\beta,\tau) - \zeta t. \end{aligned} \tag{5}$$

It follows from (4) and (5) that $|d\mathcal{G}_{f_{A_0}}|(A_0\gamma,\ f_{A_0}(\gamma)) \ge \zeta$, from which we get $|d\mathcal{G}_{f_{A_0}}|(A_0\gamma,\ f_{A_0}(\gamma)) \ge |d\mathcal{G}_{f_{A_0}}|(\gamma,\ f_{A_0}(\gamma))$.

The converse inequality is proved in the same way. Therefore we have proved that

$$|d\mathcal{G}_{f_{A_0}}|\big(\gamma,\ f_{A_0}(\gamma)\big) = |d\mathcal{G}_{f_{A_0}}|\big(A_0\gamma,\ f_{A_0}(\gamma)\big).$$

Since $\gamma \in \mathcal{D}(f_{A_0})$, from Definition 2.2 we obtain the desired relation. □

We introduce the following notation

$$A_0^n = \begin{cases} A_0 \circ A_0 \circ \cdots \circ A_0, & n \in \mathbb{Z}_+ \\ \mathrm{id}_{\mathbb{R}^n}, & n = 0 \\ A_0^{-1} \circ A_0^{-1} \circ \cdots \circ A_0^{-1}, & n \in \mathbb{Z}_- \end{cases}$$

where $A_0 \circ A_0 \circ \cdots \circ A_0$ and $A_0^{-1} \circ A_0^{-1} \circ \cdots \circ A_0^{-1}$ denote the compositions of $|n|$-times.

Remark 3.7. If $\gamma \in X_{A_0}$, then $|df_{A_0}|(A_0^n\gamma) = |df_{A_0}|(\gamma)$ for every $n \in \mathbb{Z}$. Indeed, in a similar way as in Theorem 3.6, it is possible to prove that

$$|df_{A_0}|(A_0^{-1}\gamma) = |df_{A_0}|(\gamma).$$

The rest follows by induction.

The following result comes from the above remark and Theorem 3.5.

Corollary 3.8. *If $\gamma \in X_{A_0}$ is a critical point of f_{A_0}, then $A_0^n\gamma$ is a geodesic on M for every $n \in \mathbb{Z}$.*

Moreover, we need the following result to join the geodesic (segments) and to obtain A_0-invariant geodesics on M.

Proposition 3.9. *If* $\gamma \in X_{A_0}$ *is a critical point of* f_{A_0}, *then*

$$\widehat{\gamma}(s) = \begin{cases} \gamma(s+\frac{1}{2}), & 0 \le s \le \frac{1}{2} \\ A_0\gamma(s-\frac{1}{2}), & \frac{1}{2} \le s \le 1 \end{cases} \tag{6}$$

is a geodesic on M.

Proof. We observe that $\widehat{\gamma}$ is continuous and $A_0(\widehat{\gamma}(0)) = \widehat{\gamma}(1)$. Moreover, $\widehat{\gamma} \in X_{A_0}$. Using the fact that A_0 is linear isometry and the definition of f_{A_0}, we have $f_{A_0}(\widehat{\gamma}) = f_{A_0}(\gamma) < \infty$. Since γ and $A_0\gamma$ are geodesics on M, we obtain that for all point $s \in (0,1) \setminus \{\frac{1}{2}\}$ there exists a neighborhood $[a_s, b_s]$ of s in $[0,1]$ such that $\widehat{\gamma}|_{[a_s,b_s]}$ is energy-stationary. By contradiction, we assume that for $\frac{1}{2}$ there exist no neighborhood $[a,b]$ in $[0,1]$ such that the curve $\widehat{\gamma}|_{[a,b]}$ is energy-stationary. We set $[a,b] := [0,1]$. Let $\delta, c, \zeta > 0$ and a continuous function $\widehat{\mathcal{H}}$ as in Definition 3.1.

We define the function

$$\mathcal{H} : \left\{\eta \in W^{1,2}(0,1,M) : \|\eta - \gamma\|_{1,2} < \delta\right\} \times [0,\delta] \to W^{1,2}(0,1,M)$$

by

$$\mathcal{H}(\eta,t)(s) = \begin{cases} A_0^{-1}\big(\widehat{\mathcal{H}}(\tilde{\eta},t)(s+\frac{1}{2})\big), & 0 \le s \le \frac{1}{2} \\ \widehat{\mathcal{H}}(\tilde{\eta},t)(s-\frac{1}{2}), & \frac{1}{2} \le s \le 1 \end{cases} \tag{7}$$

where

$$\tilde{\eta}(s) = \begin{cases} \eta(s+\frac{1}{2}), & 0 \le s \le \frac{1}{2} \\ A_0\eta(s-\frac{1}{2}), & \frac{1}{2} \le s \le 1. \end{cases} \tag{8}$$

Obviously, $\tilde{\eta} \in X_{A_0}$. The function $\mathcal{H}$ is well defined and is continuous. Moreover, $\mathcal{H}(\eta,t) \in X_{A_0}$. For all $\eta \in X_{A_0}$ with $\|\eta - \gamma\|_{1,2} < \delta$ we have $\|\tilde{\eta} - \widehat{\gamma}\|_{1,2} < \delta$. From (7) and (8) we have

$$d_2\big(\mathcal{H}(\eta,t),\eta\big) = \big\|\mathcal{H}(\eta,t) - \eta\big\|_2 = \big\|\widehat{\mathcal{H}}(\tilde{\eta},t) - \tilde{\eta}\big\|_2 \le ct;$$

$$f_{A_0}\big(\mathcal{H}(\eta,t)\big) \le f_{A_0}(\eta) - \zeta t.$$

Applying again the machinery from Theorem 3.5 and using Proposition 2.3, we obtain that γ is not a critical point of f_{A_0}, which is in contradiction with the assumption. □

Remark 3.10. Let $\gamma \in X_{A_0}$ be a geodesic on M. We define the curve $\tilde{\gamma} : \mathbb{R} \to M$ by $\tilde{\gamma}(t) = A_0^{[t]}(\gamma(\{t\}))$, where $[t]$ is the integer part of $t \in \mathbb{R}$, and $\{t\} = t - [t]$. We observe that for every $s \in \mathbb{R}$

$$A_0\big(\tilde{\gamma}(s)\big) = A_0^{1+[s]}\Big(\gamma\big(\{s\}\big)\Big) = A_0^{[s+1]}\Big(\gamma\big(\{s+1\}\big)\Big) = \tilde{\gamma}(s+1),$$

i.e., $\tilde{\gamma}$ is A_0-invariant geodesic on M. Therefore, if we can guarantee that $\gamma \in X_{A_0}$ is a (non–trivial) curve which is a critical point of f_{A_0}, the above construction can be applied for constructing A_0-invariant geodesics on M.

Remark 3.11. We denote by $\mathrm{Fix}_M A_0$ the fixed points of the isometry A_0 on M. We observe that $\bar{\mathcal{G}}^0_{f_{A_0}}$ is homeomorphic to $\mathrm{Fix}_M A_0$. Indeed, let

$$(\gamma, \xi) \in \bar{\mathcal{G}}^0_{f_{A_0}} = \big\{(\gamma, \xi) \in \mathrm{epi}(f_{A_0}) : \xi \le 0\big\},$$

therefore $f_{A_0}(\gamma) \le \xi \le 0$. From this, we get that $\gamma \in X_{A_0}$ and $|\gamma'(s)| = 0$ almost everywhere, therefore $\gamma(s) = x_0 \in M$. Since $A_0(\gamma(0)) = \gamma(1)$ it follows that $x_0 \in \mathrm{Fix}_M A_0$.

4. Topological framework and Lipschitz obstacle

Let A_0 be a linear isometry of finite order (i.e., there exists $k \in \mathbb{N}$ such that $A_0^k = \mathrm{id}_{\mathbb{R}^n}$) and let $G = \langle A_0 \rangle$ be the group generated by A_0. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a G-invariant Lipschitz function, i.e., $g(Ax) = g(x), \forall x \in \mathbb{R}^n, A \in G$. Let

$$M = \{x \in \mathbb{R}^n : g(x) \le 0\}.$$

Of course, M is G-invariant, i.e., $AM = M$, for every $A \in G$. We suppose that

$$\forall x \in \mathbb{R}^n : g(x) = 0 \Rightarrow 0 \notin \partial g(x),$$

where ∂g is the Clarke's subdifferential, [3]. According to the paper of Krawcewicz and Marzantowicz ([12]), ∂g is G-equivariant, i.e., $\partial g(Ax) = A\partial g(x)$ for every $A \in G$. Moreover, the lower semi-continuous function $\lambda(x) = \{|\alpha| : \alpha \in \partial g(x)\}$ is also G-invariant. Using the paper of Chang ([2]), there exists a locally Lipschitz map

$$v : \{\, x \in \mathbb{R}^n \,|\, 0 \notin \partial g(x) \,\} \to \mathbb{R}^n,$$

such that

$$0 \notin \partial g(x) \Rightarrow \big\| v(x) \big\| \le 2\lambda(x); \tag{9}$$

$$0 \notin \partial g(x),\ \alpha \in \partial g(x) \Rightarrow \big\langle \alpha, v(x) \big\rangle \ge \lambda(x)^2. \tag{10}$$

Moreover, in our case, the above map can be chosen in G-equivariant way. More precisely, let

$$\widehat{v}(x) = \frac{1}{k} \sum_{A \in G} A^{-1} v(Ax).$$

Of course, $\widehat{v}(Ax) = A\widehat{v}(x), \forall A \in G, x \in \mathbb{R}^n$, i.e., $\widehat{v}$ is G-equivariant and relations (9) and (10) hold for $\widehat{v}$ (instead of v). We can choose a G-invariant neighborhood $\mathcal{O}$ of $\{x \in \mathbb{R}^n : 0 \in \partial g(x)\}$ and a G-invariant locally Lipschitz functional $\theta : \mathbb{R}^n \to [0, 1]$ such that

$$x \in \mathcal{O} \Rightarrow \theta(x) = 0,$$

$$g(x) = 0 \Rightarrow \theta(x) = 1.$$

We define

$$w(x) = \begin{cases} \theta(x)\dfrac{\widehat{v}(x)}{|\widehat{v}(x)|}, & \text{if } 0 \notin \partial g(x) \\ 0, & \text{if } x \in \mathcal{O} \end{cases}$$

w is well defined, locally Lipschitz and G-equivariant, i.e., $w(Ax) = Aw(x)$, for every $x \in \mathbb{R}^n$ and $A \in G$.

Moreover, we have

$$g(x) = 0 \Rightarrow |w(x)| = 1,$$

$$g(x) = 0, \alpha \in \partial g(x) \Rightarrow \langle \alpha, w(x) \rangle \geq \tfrac{1}{2}\lambda(x).$$

Definition 4.1. A subset L of $\mathbb{R}^n$ is *Lipschitz neighborhood retract* if there exist an open neighborhood U_L of L in $\mathbb{R}^n$ and a locally Lipschitz retraction $r : U_L \to L$.

Theorem 4.2. *The set M is Lipschitz neighborhood retract. Moreover, the retraction can be constructed such that $r \circ A_0 = A_0 \circ r$ in the neighborhood U_M of M and U_M is G-invariant.*

Proof. We consider the following Cauchy problem:

$$\begin{cases} \dfrac{\partial \eta_G}{\partial t}(x,t) = -w\big(\eta_G(x,t)\big), \\ \eta_G(x,0) = x. \end{cases}$$

Of course, $\eta_G(\cdot,t)$ is G-homeomorphism, i.e., homeomorphism and $\eta_G(Ax,t) = A\eta_G(x,t)$, $\forall A \in G$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. Therefore $U_M = \{\eta_G(x,-1) : g(x) < 0\}$ is open and is G-invariant. Further, the proof is proceeded in the same way as in ([7, Theorem 6.4]), if we replace η by η_G, obtaining a retraction r from U_M onto M which commutes with A_0. □

Let Y be a subset of $\mathbb{R}^n$ which is G-invariant. We consider the

$$\Lambda_{G(A_0)}(Y) = \Big\{\gamma \in \mathcal{C}\big([0,1];Y\big) : A_0\big(\gamma(0)\big) = \gamma(1)\Big\}$$

which is endowed with the sup-metric. If $A_0 = \text{id}$, then $\Lambda_{G(A_0)}(Y) = \Lambda(Y)$ is the well-known free loop space of Y.

Let U_M be the open neighborhood of M (from Theorem 4.2) and let

$$\Lambda^1_{G(A_0)}(U_M) := \Big\{\gamma \in W^{1,2}(0,1;U_M) : A_0\big(\gamma(0)\big) = \gamma(1)\Big\}$$

be endowed with the $W^{1,2}$-metric. Then there exists a continuous mapping

$$\mathcal{K}_{A_0} : \Lambda_{G(A_0)}(U_M) \times [0,1] \to \Lambda_{G(A_0)}(U_M),$$

which satisfies the assertions from [7, Lemma 5.2] replacing $\Lambda(U)$ by $\Lambda_{GA_0}(U_M)$ and $\Lambda^1(U)$ by $\Lambda^1_{GA_0}(U_M)$.

The next result is the equivariant form of [7, Theorem 5.3].

Theorem 4.3. *The map* $\pi : \mathrm{epi}(f_{A_0}) \to \Lambda_{G(A_0)}(M)$ *defined by* $\pi(\gamma, \lambda) = \gamma$ *is a homotopy equivalence.*

Proof. Let $r : U_M \to M$ be as in Theorem 4.2 and $\mathcal{K}_{A_0} : \Lambda_{G(A_0)}(U_M) \times [0, 1] \to \Lambda_{G(A_0)}(U_M)$ as above. Let $\gamma \in \Lambda_{G(A_0)}(M)$. Then $\gamma_r = r \circ \mathcal{K}_{A_0}(\gamma, 1)$ is also in $\Lambda_{G(A_0)}(M)$ since γ_r is continuous and

$$A_0\big(\gamma_r(0)\big) = \gamma_r(1).$$

Further, the proof is similar to that of [7, Theorem 5.3]. □

Theorem 4.4. *For every* $(\gamma, \lambda) \in \mathrm{epi}(f_{A_0})$ *with* $\lambda > f_{A_0}(\gamma)$ *we have*

$$|d\mathcal{G}_{f_{A_0}}|(\gamma, \lambda) = 1.$$

Proof. We can follow the proof of the [7, Theorem 6.4] with the corresponding modification, i.e., instead of f, X, ν we use f_{A_0}, X_{A_0}, w respectively. □

5. The existence of A_0-invariant geodesics

Now, we are in the position to prove the main result of this paper:

Proof of Theorem 1.1. The set $\mathrm{epi}(f_{A_0})$ endowed with the metric $L^2 \times \mathbb{R}$ is complete. Moreover, since M is compact, $\overline{(\mathcal{G}_{f_{A_0}})^b}$ is compact for any $b \in \mathbb{R}$, therefore $\mathcal{G}_{f_{A_0}}$ satisfies the $(PS)_c$ condition, for any $c \in \mathbb{R}$. Suppose that $\mathcal{G}_{f_{A_0}}$ has no critical point $(\gamma, \xi) \in \mathrm{epi}(f_{A_0})$ with

$$a = 0 < \mathcal{G}_{f_{A_0}}(\gamma, \xi) \leq b = +\infty.$$

The case of (H_1). From [5, Theorem 2.10], it follows that $K(\mathcal{G}_{f_{A_0}})_0$ is a weak deformation retract of $\mathrm{epi}(f_{A_0})$. Since $K(\mathcal{G}_{f_{A_0}})_0$ is homeomorphic to $\mathrm{Fix}_M A_0$ and $\mathrm{epi}(f_{A_0})$ is homotopically equivalent with $\mathcal{G}_{f_{A_0}}$, this is in contradiction with the assumption.

The case of (H_2). From [7, Theorem 2.7], it follows that the inclusion

$$i : (\overline{\mathcal{G}_{f_{A_0}}})^0 \hookrightarrow \mathrm{epi}(f_{A_0})$$

induces an isomorphism between the Alexander–Spanier cohomology groups of the above sets, which is a contradiction with the assumption.

Therefore, in the both cases, there exists $(\gamma,\lambda) \in \mathrm{epi}(f_{A_0})$ with $\lambda > 0$ and $|d\mathcal{G}_{f_{A_0}}|(\gamma, \lambda) = 0$. From Theorem 4.4 it follows that $\lambda = f_{A_0}(\gamma)$. Therefore, $\gamma \in X_{A_0}$ and $|df_{A_0}|(\gamma) = 0$. So, γ is a critical point for f_{A_0} with $0 < f_{A_0} < +\infty$. Applying the machinery described in Remark 3.10, we obtain that the curve $\tilde{\gamma} : \mathbb{R} \to M$ defined by

$$\tilde{\gamma}(t) = A_0^{[t]}\big(\gamma(\{t\})\big)$$

is a non-trivial A_0-invariant geodesic on M. The proof is complete. □

Remark 5.1. In fact, since A_0 is of finite order, the above A_0-invariant geodesic on M is a closed geodesic on M. Therefore, if we want to guarantee the existence of a closed geodesic on M (in this way), it is enough to establish the existence of an A_0-invariant geodesic.

Finally, we give some examples.

Example 5.2. If $A_0 = \mathrm{id}_{\mathbb{R}^n}$, then we obtain the notion of the closed geodesics, introduced in [7]. Of course, $\mathrm{Fix}_M A_0 = M$ and $\Lambda_{G(A_0)}(M) = \Lambda(M)$ is the free loop space. Moreover, if M is defined as above and we suppose that it is compact, simply connected and non-contractible in itself, then, according to a result of Vigué-Poirrier and Sullivan [14], the hypothesis (H_2) holds. Therefore, there exists a non-trivial closed geodesics on M.

Example 5.3. If $\mathrm{Fix}_M A_0 = \emptyset$, then both hypotheses hold, therefore there exists at least one A_0-invariant geodesic.

Remark 5.4. The existence of the closed geodesics, in the sense of [7], is a difficult problem. If the set M is "symmetric" enough and if we can construct a linear isometry, which leaves M invariant, the existence of a closed geodesic is simpler, as shown in the next example.

Example 5.5. Let

$$M = \left\{ x \in \mathbb{R}^3 : \sum_{i=1}^{3} x_i^2 \leq 1 \leq \sum_{i=1}^{3} |x_i| \right\}.$$

Let $A_0(x_1, x_2, x_3) = (x_1, -x_2, -x_3)$. The fixed points of A_0 are

$$\mathrm{Fix}_M A_0 = \{(1,0,0), (-1,0,0)\}.$$

Of course, $A_0^2 = \mathrm{id}$ and $\Lambda_{G(A_0)} M$ is 0-connected, therefore the two sets above are not homotopically equivalent, i.e., the hypothesis (H_1) holds. Therefore, there exists an A_0-invariant geodesic on M which is at same time a closed geodesic.

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