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On the passage from orthogonal (conformal) to symplectic (contact) from the point of view of spinor theory[1]

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Abstract. The aim of this review article is twofold. On the one hand, it reviews basics of class of infinite-dimensional representation theory applied to basic example of symplectic Dirac operator in the framework of parabolic geometry. On the other hand, it offers to pure representation theorists a natural geometrical context for realization of class of infinite-dimensional representations.

We extend the notion of symplectic spinors based on (infinite-dimensional) Segal–Shale–Weil representation, defined by B. Kostant, in the framework of contact parabolic geometry, and introduce parallel vocabulary with conformal Spin-Riemannian geometry.

Keywords. Discrete series representations, Enright resolutions, invariant differential operators, symplectic spinors, contact geometry.

MS classification. 22E47, 18G10, 32L10.

1. Introduction

There are two parallel motivations for writing this article. The first one is to enrich D^b (the bounded derived category) of sheaves of natural vector bundles, i.e., sheaves attached to homogeneous parabolic geometry for (non-compact) real forms of Lie groups (algebras), by injective resolutions of infinite-dimensional (irreducible, unitary) modules (Harish–Chandra $(\mathfrak{g}, K)$-modules with highest weight). In terms of (generalized) Verma modules and their homomorphisms, they are called

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Enright resolutions, and are the substitute of Bernstein–Gelfand–Gelfand resolutions of irreducible finite-dimensional representations by (generalized) Verma modules.

The second motivation comes from explicit example of Segal–Shale–Weil module (SSW-mod in what follows). The SSW-mod valued sections over symplectic manifold with metaplectic structure were called by Kostant *symplectic spinors*, as a symplectic analogue of usual spinor fields in Spin-Riemannian geometry. One explicit resolution coming from the first item will be one of the sources of apparently much more wider analogy.

In this article we shall pay special attention to contact (symplectic) analogues of usual Dirac and Twistor operators in conformal Spin-Riemannian geometry as examples of conformally invariant differential operators. A uniform construction (and so classification) of such conformally invariant differential operators can be described by following scheme. Let the operator act on sections of irreducible module $\mathbb{E}$ of conformal group $\mathfrak{co}$, and let π be $\mathfrak{co}(n)$-invariant projection of $\mathbb{E} \otimes \mathbb{R}^n$ on one of its irreducible submodules. Then the (invariant differential) operator is given by covariant derivative of any Weyl connection associated to conformal structure composed with projection onto irreducible submodule. In the classification procedure one usually starts from the homogeneous case, $M = G/P$ for parabolic subgroup $P \subset G$. There are methods allowing extension of subclass of invariant operators to curved cases (in the language of E. Cartan), see [3], and these methods recover all parabolic geometries (the case of conformal geometry being just one typical example). The choice of the parabolic subgroup for the homogeneous model can be equivalently characterized by choice of grading on the Lie algebra $\mathfrak{g}$ of the Lie group G. The conformal case then corresponds to the $|1|$-graded orthogonal (even and odd) Lie algebra.

In summary, parabolic geometry (parabolic invariant theory) supplements us with suitable language allowing plain formulation of invariance (invariant differential operators) and bounded derived category (i.e., sequences by invariant differential operators, which are resolutions in the homogeneous case).

Definition 1.1. Let $\mathcal{G}$ be a principal P-bundle on a manifold M, and let for $X \in \mathfrak{p}$ be $X^\sharp$ the corresponding fundamental vector field. The left (right) action of $p \in P$ on $\mathcal{G}$ will be denoted $L_p : P \times \mathcal{G} \to \mathcal{G}$ ($R_p : \mathcal{G} \times P \to \mathcal{G}$).

A Cartan connection on $\mathcal{G}$ is a 1-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ satisfying:

- $\omega(X^\sharp) = X$ for all $X \in \mathfrak{p}$,
- $(R_p)^\star \omega = \mathrm{Ad}_p^{-1} \omega$ for all $p \in P$,
- $\omega|_{T_u\mathcal{G}} : T_u\mathcal{G} \xrightarrow{\sim} \mathfrak{g}$ is isomorphism for all $u \in \mathcal{G}$.

A parabolic structure (parabolic geometry) on M is given by data $(\mathcal{G}, P, M, \omega)$.

Let $\mathfrak{g}$ be a $|k|$-graded (real, complex) semisimple Lie algebra, i.e.,

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k,$$

such that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$$

and $\mathfrak{g}_1$ generates the subalgebra $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$. We denote the subalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ of $\mathfrak{g}$ by $\mathfrak{p}$. Let us define the grading element $E \in Z(\mathfrak{g}_0)$ to be the unique element in $\mathfrak{g}$ whose adjoint action is given by $[E, X] = jX$ for any $X \in \mathfrak{g}_j$, such that $j \in \{-k, \ldots, k\}$. In particular, if $\mathfrak{g}$ is complex, there always exists a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ containing E, and one can choose a system Δ_+ of positive roots for $\mathfrak{h}$ in such a way, that all root spaces corresponding to the simple roots are contained in $\mathfrak{g}_0 \oplus \mathfrak{g}_1$. The grading on $\mathfrak{g}$ is then given by the $\mathfrak{g}_1$-lengths of roots, i.e., if α is a root, then the root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{g}_i$, where i is the sum of all coefficients of simple roots in the expansion of α. Then $\mathfrak{p}$ is the parabolic subalgebra of $\mathfrak{g}$ and the decomposition $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ is the Levi decomposition of $\mathfrak{p}$ onto its reductive part and nil-radical.

Note also, that for a $|k|$-graded real Lie algebra $\mathfrak{g}$, its complexification $\mathfrak{g}_{\mathbb{C}}$ is also $|k|$-graded, so in general, we will deal with certain real forms of $(\mathfrak{g}, \mathfrak{p})$, where $\mathfrak{g}$ is complex semisimple Lie algebra and $\mathfrak{p}$ is its (complex) parabolic subalgebra.

We use the standard notation of crossed Dynkin diagrams in order to describe the pair $(\mathfrak{g}, \mathfrak{p})$. In particular, in the Dynkin diagram of $\mathfrak{g}$ we cross out the simple roots whose root spaces are contained in $\mathfrak{g}_1$.

In this article we treat $|2|$-graded case corresponding to the contact geometry.

$$
\begin{aligned}
&\mathfrak{g} = \mathrm{sp}(n+1, \mathbb{R}), \text{ the split real form of } C_{n+1} = \mathrm{Sp}(n+1, \mathbb{C}),\\
&\mathfrak{g}_0 = \mathrm{sp}(n, \mathbb{R}) \oplus \mathbb{R}E, \text{ where } E \text{ is the grading element (center of } \mathfrak{g}_0),\\
&\mathfrak{g}_0^s = \mathrm{sp}(n, \mathbb{R}), \text{ the split real form of } C_n = \mathrm{Sp}(n, \mathbb{C}),\\
&\mathfrak{g}_1 = \mathbb{R}^{2n},\\
&\mathfrak{g}_2 = \mathbb{R},\\
&\mathfrak{p} = \mathrm{sp}(n, \mathbb{R}) \oplus \mathbb{R}E \oplus \mathbb{R}^{2n} \oplus \mathbb{R},\\
&\mathfrak{p}_+ = \mathbb{R}^{2n} \oplus \mathbb{R}.
\end{aligned}
\tag{1}
$$

We will present purely representational-theoretical approach, i.e., no analytical aspects or considerations connected with globalization representation functors will appear. In particular, all infinite-dimensional modules will be $(\mathfrak{g}, K)$-modules with highest weight, i.e., the Verma modules.

For Verma module V_λ, we denote its maximal subquotient by L_λ. The relation between submodules of V_λ (by Harish–Chandra property is the composition series of finite length) and their corresponding subquotients are given (in terms of character identities) by Kazhdan–Lusztig polynomials.

2. Infinite-dimensional representations in parabolic geometries

2.1. Discrete series representation

We shall consider the following set of representations, characterized for the first time by Harish–Chandra, see for example [10].

Theorem 2.1. *Let (π, V_π) be an irreducible unitary representation of $G_\mathbb{R}$ (real Lie group, V_π is a module of Harish–Chandra class). Then the following conditions are equivalent*:

- *for some $u, v \in V_\pi$, $u \neq 0 \neq v$, $g \to (\pi(g)u, v) \in L_2(G_\mathbb{R})$;*
- *for any $u, v \in V_\pi$, $u \neq 0 \neq v$, $g \to (\pi(g)u, v) \in L_2(G_\mathbb{R})$;*
- *there is $G_\mathbb{R}$-equivariant embedding $V_\pi \hookrightarrow L_2(G_\mathbb{R})$;*
- *the Plancherel measure $d\mu(\pi)$ for the decomposition of $L_2(G_\mathbb{R})$ is strictly positive, i.e.,*

$$\int_{\pi' \in \hat{G}_\mathbb{R}} \delta(\pi - \pi') d\mu(\pi) =: \deg(\pi) > 0.$$

The representation π fulfilling one of the equivalent conditions above is called discrete series representation.

Definition 2.2. The set of isomorphism classes of discrete series representations (of $G_\mathbb{R}$) is called *discrete representation series* (of $G_\mathbb{R}$).

This set of representations enters in twofold way. The first one is as the representational (valuation) module of sections on which invariant differential operators act, the second is as modules whose resolutions are Enright resolutions.

2.2. Tensor product of finite-dimensional with infinite-dimensional representations

Following the work of Dixmier, Zuckermann, Kostant and many others, we shall describe basic facts of the decomposition of finite-dimensional with infinite--dimensional representations, see [9].

Let $\mathfrak{g}$ be a (complexified) Lie algebra, $U_\mathfrak{g}$ its universal enveloping algebra, and

$$\pi_\lambda : \mathfrak{g} \longrightarrow \mathrm{End}(V_\lambda), \quad \pi : \mathfrak{g} \longrightarrow \mathrm{End}(V), \tag{2}$$

where $\lambda \in \Lambda^+(\mathfrak{g})$ and $\Lambda^+(\mathfrak{g})$ is positive Weyl chamber of $\mathfrak{g}$ for finite-dimensional $\mathfrak{g}$-mod V_λ. We assume that infinite-dimensional module V is Harish–Chandra module with infinitesimal character $\chi_\pi : Z(U_\mathfrak{g}) \to \mathbb{C}$, i.e., $\pi(z) = \chi_\pi(z)\,\mathrm{Id}_V \in \mathrm{End}(V)$ for all $z \in Z(U_\mathfrak{g}) \subset U_\mathfrak{g}$. Moreover there is $\nu \in \mathfrak{h}^\star$ (dual of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$), such that $\chi_\pi = \chi_\nu$. The basic question is the decomposition of tensor product

$$\pi_\lambda \otimes \pi : \mathfrak{g} \longrightarrow \mathrm{End}(V_\lambda \otimes V). \tag{3}$$

Let us denote $S_\lambda := \{\lambda_1, \dots, \lambda_k\} \subset \mathfrak{h}^\star$ the set of weights of V_λ, $\dim V_\lambda = k$.

Theorem 2.3. *Let $z \in Z(U_\mathfrak{g})$ be arbitrary, and $\tilde{z} = (\pi_\lambda \otimes \pi)(z)$. Then $\tilde{z}$ satisfies algebraic equation*

$$\Pi_{i=1}^k \big(\tilde{z} - \chi_{\nu+\lambda_i}(z)\big)(V_\lambda \otimes V) = 0.$$

In particular for quadratic element $z = \mathrm{Cas}$, called Casimir operator, its possible values are (on $V_\lambda \otimes V$) $|\rho + \nu + \lambda_i|^2 - |\rho|^2$, ρ being half sum of positive roots and $|\cdot|$ being norm induced by Killing–Cartan form.

Theorem 2.4. *Assume $X_1 \subseteq X_2 \subseteq V_\lambda \otimes V$ are $U_{\mathfrak{g}}$ -submodules, and the representation of $U_{\mathfrak{g}}$ on X_2/X_1 admits an infinitesimal character χ. Then χ is necessarily of the form $\chi_{\nu+\lambda_i}$ for some $i \in \{1, \dots, k\}$.*

Let us denote $P_0 = \emptyset$ and

$$P_i := \Big\{ v \in V_\lambda \otimes V \Big| \big(\tilde{z} - \chi_{\nu+\lambda_i}(z)\big)(V_\lambda \otimes V) = 0 \ \ \forall z \in Z(U_{\mathfrak{g}}) \Big\},$$

$i = 1, \dots, k$, so that $\emptyset = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_k = V_\lambda \otimes V$ is filtration of $V_\lambda \otimes V$ by $U_{\mathfrak{g}}$-modules. Then if $P_i/P_{i-1} \neq 0$, it admits an infinitesimal character $\chi_{\nu+\lambda_i}$.

Corollary 2.5. *If the characters $\chi_{\nu+\lambda_i}$, $i = 1, \dots, k$ are distinct, let us put*

$$Y_i := \big\{ v \in V_\lambda \otimes V \big| \tilde{z} v = \chi_{\nu+\lambda_i}(z) y \quad \forall z \in Z(U_{\mathfrak{g}}) \big\}.$$

Then if non-zero, Y_i is maximal submodule with infinitesimal character $\chi_{\nu+\lambda_i}$ and the tensor product representation is fully reducible

$$V_\lambda \otimes V \simeq \bigoplus_{i=1}^{k} Y_i \,. \tag{4}$$

Finiteness of Jordan–Hőlder composition series is assured by Harish–Chandra assumption on module V.

The tensor product of Harish–Chandra modules with highest weight (in fact, Verma modules) with finite-dimensional modules (in fact, the fundamental vector representation) is the main input in the study of symplectic Dirac operator and symplectic Twistor operator (as first-order invariant differential operators). Implicitly, these tensor products are contained in the B-G-G and Enright resolutions.

2.3. Enright resolutions

In this section we review the construction of projective resolutions of fundamental series of representations (e.g., including discrete series) in the spirit of generalization of Bernstein–Gelfand–Gelfand resolutions and their geometric realizations. Further results incorporating Lie algebra homology with values in infinite--dimensional modules will appear elsewhere. The main source is [4].

Let $\mathfrak{g}_{\mathbb{R}}$ be a real Lie algebra, and $\mathfrak{k}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ its maximal compact subalgebra (Lie algebra of maximal compact subgroup of adjoint group of $\mathfrak{g}_{\mathbb{R}}$). Let $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{k}_{\mathbb{C}}$ respectively, be their complexifications, $\mathfrak{h}^1_{\mathbb{C}}$ Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}} := Z_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{h}^1_{\mathbb{C}})$ Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We have for the couple $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}^1_{\mathbb{C}})$ respectively, the set of roots a Weyl group $(\Delta, W_{\mathfrak{g}_{\mathbb{C}}})$, resp. $(\Delta_{\mathfrak{k}_{\mathbb{C}}}, W_{\mathfrak{k}_{\mathbb{C}}})$. The Weyl group $W_{\mathfrak{k}_{\mathbb{C}}}$ is isomorphic to the Weyl group $W_{\mathfrak{g}_{\mathbb{R}}}$. Fix the set of positive roots $\Delta^+ \subset \Delta$; the corresponding restriction to $\mathfrak{k}_{\mathbb{C}}$ induces the positive system $\Delta^+_{\mathfrak{k}_{\mathbb{C}}} \subset \Delta^+ \subset \Delta$. The notion of positivity determines ordering $\leq$ and length function on $W_{\mathfrak{k}_{\mathbb{C}}}$. Let us define

$$\$:= \{\lambda \in \mathfrak{h}^\star_{\mathbb{C}} \mid \lambda|_{(\mathfrak{h}^1_{\mathbb{C}})^\star} \text{ is } \Delta_{\mathfrak{k}_{\mathbb{C}}}\text{-integral}\} \,. \tag{5}$$

In what follows, we shall use only the language of Verma modules, [4]; the reference to further results on generalized Verma modules can be found in the same source.

The basic strategy in the case of fundamental series representations is similar to the case of realization of finite-dimensional representations as subquotients of Verma modules. We attach to basic $\mathfrak{g}_{\mathbb{C}}$-module $A\,(= V_{s_0\cdot\lambda})$ the lattice of modules $\{A_s\}_{s\in W}$ parametrized by length function on W, such that $F_\lambda = A_1/\sum_{s\neq 1} A_s$ is irreducible $\mathfrak{g}_{\mathbb{C}}$-mod with highest weight λ. All modules A_s have the properties to be

- torsion free $U_{\mathfrak{n}}^-$-modules,
- for every simple $\alpha \in \Delta$, $s \in W$, $l(s_\alpha \cdot s) = l(s) + 1$, $A_s/A_{s_\alpha\cdot s}$ is $U(\mathrm{sl}_2(\alpha))$-finite.

For $\lambda \in \$$, the Verma $\mathfrak{g}_{\mathbb{C}}$-module $V_{\mathfrak{g}_{\mathbb{C}},\Delta^+,\lambda}$ admits a lattice of $\mathfrak{g}_{\mathbb{C}}$-mod $\{A_s\}_{s\in W_{\mathfrak{k}_{\mathbb{C}}}}$ above it. The subquotient $L_{\Delta,\lambda}$,

$$L_{\Delta,\lambda} := A_1/\sum_{s\neq 1} A_s\,, \tag{6}$$

is $U_{\mathfrak{k}_{\mathbb{C}}}$-finite $\mathfrak{g}_{\mathbb{C}}$-mod fundamental series module with parameters (Δ,λ), i.e., fundamental series of $\mathfrak{g}_{\mathbb{R}}$ or $(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})$.

Let us further introduce the standard notation

$$\delta = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha, \qquad \delta_{\mathfrak{k}_{\mathbb{C}}} = \frac{1}{2}\sum_{\alpha\in\Delta^+_{\mathfrak{k}_{\mathbb{C}}}}\alpha$$

and $\langle\cdot\,,\cdot\rangle$ for Killing–Cartan form.

Theorem 2.6. *Let $\lambda \in \$$ such that $2\langle\lambda+\delta,\alpha\rangle/\alpha,\alpha \notin \{1,2,\dots\}\ \forall\alpha \in \Delta^+$. Then*

- *if $\lambda+\delta$ is regular, $L_{\Delta,\lambda}$ is non-zero and irreducible (fundamental series representation);*
- *if $\lambda|_{(\mathfrak{h}^1_{\mathbb{C}})^\star} + 2\delta_{\mathfrak{k}_{\mathbb{C}}}$ is $(-\Delta^+_{\mathfrak{k}_{\mathbb{C}}})$-dominant, $L_{\Delta,\lambda}$ is non-zero and irreducible (fundamental series representation).*

In particular, every member of fundamental series representation is realized as a subquotient in the category of (generalized) Verma modules. This realization of discrete series is called Enright–Varadarajan.

This is an analogue of results of Harish–Chandra and Chevalley on realization of finite-dimensional representations for any complex Lie group. Similar to the results of B-G-G, there is resolution of fundamental series representation based on realization (6).

Theorem 2.7. *Let us assume the same conditions on $\lambda \in \$$ as in the previous theorem, Theorem 2.6. For $0 \le i \le r = \mathrm{card}(\Delta^+_{\mathfrak{k}_{\mathbb{C}}})$ let us write $C_i := \bigoplus_{l(s)=i} A_s$. Then there exists a projective resolution of $L_{\Delta,\lambda}$ by $\mathfrak{g}_{\mathbb{C}}$-modules (Verma modules)*

$$0 \to C_r \to C_{r-1} \to \cdots \to C_0 \to L_{\Delta,\lambda} \to 0. \tag{7}$$

The construction of the family of Verma modules is more complicated than in the case of B-G-G. Additional to the action of $W_{\mathfrak{k}_{\mathbb{C}}}$ on highest weights is the completion functor C amounting suitable lifting property (extending given Verma module by finite-dimensional vector space).

Let $s_0 \in W_{\mathfrak{k}_{\mathbb{C}}}$ be the unique element such that

$$s_0 \cdot \Delta^+_{\mathfrak{k}_{\mathbb{C}}} = -\Delta^+_{\mathfrak{k}_{\mathbb{C}}};$$

then $A = A_{s_0}$. The construction proceeds inductively on the length: for $s, s_\alpha \in W_{\mathfrak{k}_{\mathbb{C}}}$ (α is simple root of $\Delta_{\mathfrak{k}_{\mathbb{C}}}$), $l(s_\alpha \cdot s) = l(s) + 1$, A_s is the completion of $A_{s_\alpha \cdot s}$, $A_s = C_\alpha(A_{s_\alpha \cdot s})$.

The main properties of the functor $C : A \to C(A)$ are

(i) commutativity (stability) with the functor of tensor product with any finite-dimensional representation F, $C(F \otimes A) \simeq F \otimes C(A)$;

(ii) C is left exact functor.

Applying the same dualizing procedure as for finite-dimensional modules using infinite jet (differential) operator j^∞ translating homomorphisms of (generalized) Verma modules into invariant differential operators, see [1, Chapter 11.] from p. 158 on, we acquire — for one particular discrete series representation and first two places of resolution — the geometric realization as (injective) resolution of discrete series representation in terms of invariant differential operators. It reads (the operators and resolving module will be described explicitly in the next section)

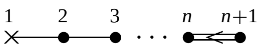
$$0 \to L_{\mathfrak{g}} \to \Gamma(M, L_{\mathfrak{g}_0}) \xrightarrow{T^s} \Gamma\big(M, L_{\mathfrak{g}_0}(\tfrac{1}{2})\big) \to \cdots \tag{8}$$

In differential geometrical realization it is usual to consider the spaces of C^∞-sections as a completion of the space of sections Γ regarded as $(\mathfrak{g}, K)$-module.

2.4. Contact geometry of C_{n+1}-type

In this section we introduce symplectic Dirac and Twistor operators as first-order invariant differential operators for contact geometry. The underlying language is invariant parabolic theory, for more see [6].

The homogeneous model of contact parabolic geometry of C_{n+1}-type is described by crossed Dynkin diagram:



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The Lie algebra of the Lie group of automorphisms of invariant differential operators is the split real form $\mathfrak{g} = \mathrm{sp}(n+1, \mathbb{R})$ of complex Lie algebra C_{n+1}. The maximal parabolic subalgebra is $|2|$-graded and decomposes $\mathfrak{g} \simeq \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_1 \simeq \mathbb{R}^{2n}$, $\mathfrak{g}_2 \simeq \mathbb{R}^{2n}$ respectively, are $\mathfrak{g}_0 \simeq \mathrm{sp}(n, \mathbb{R}) \oplus \mathbb{R}E$ modules (E being the generator of the center of reductive Lie algebra $\mathfrak{g}_0$) isomorphic to fundamental vector representation, respectively trivial representation. Lie group with (graded) Lie algebra $(\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2) \subset \mathfrak{g}$ will be denoted by P (it is parabolic subgroup).

As a particular module of $\mathfrak{g}_0$ (in fact $\mathfrak{p}$-module with trivial action of nilpotent subalgebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$) we take SSW-module

$$L^2(\mathbb{R}^{2n}) \simeq L(-\tfrac{1}{2}\,\omega_n) \oplus L(\omega_{n-1} - \tfrac{3}{2}\,\omega_n) =: L(\overline{\nu_0}) \oplus L(\overline{\nu_1})\,.$$

The general machinery concerning the decomposition of finite with infinite-dimensional representations can be considerably refined in our special case (see [2]).

Theorem 2.8. *Let* $\lambda = \sum_i \lambda_i \omega_i$ *be integral dominant weight of* C_n *and* F_λ *the corresponding finite-dimensional irreducible representation. Let us denote*

$$\tau_\lambda = \Big\{\lambda - \sum_i d_i \epsilon_i \Big| d_i \in \mathbb{Z}_{\geq 0},\ \sum_i d_i = 0 \text{ mod } 2, \qquad (9)$$
$$0 \leq d_i \leq \lambda_i,\ i = 1, \ldots, n-1;\, 0 \leq d_n \leq 2\lambda_{n+1} + 1\Big\}.$$

Then (taking one irreducible component) $L(-\frac{1}{2}\omega_n) \otimes F_\lambda$ *is completely reducible for any finite-dimensional module* F_λ *and*

$$L(-\tfrac{1}{2}\,\omega_n) \otimes F_\lambda \simeq \bigoplus_{\mu \in \tau_\lambda} L(-\tfrac{1}{2}\,\omega_n + \mu)\,. \qquad (10)$$

In particular,

$$\Pi_{\mu \in \tau_\lambda} \big(\tilde{z}_\mu - \chi_{-\frac{1}{2}\omega_n + \mu}(z_\mu)\big)\big(L(-\tfrac{1}{2}\,\omega_n) \otimes F_\lambda\big) = 0.$$

for all $z_\mu \in Z(U(\mathfrak{g}_0))$.

As a special case of finite-dimensional representation we consider (complexified) fundamental vector representation. Then this decomposition is of the form:

$$\begin{array}{ccc}
L(\overline{\nu_i}) & & L(\overline{\nu_i}) \otimes \mathbb{C}^{2n} \\
(-\frac{1}{2}, \ldots, -\frac{1}{2} - \delta_{i,1}) & \xrightarrow{D^s} & (-\frac{1}{2}, \ldots, -\frac{1}{2}, -\frac{1}{2} - \delta_{i,0}) \\
& \searrow^{T^s} & \\
& & (\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2} - \delta_{i,1}).
\end{array}$$

These results of representation theory are used in the following theorem, see [6]. Let us denote by

$$\pi_1 \,:\, L(\overline{\nu_i}) \otimes \mathbb{C}^{2n} \to L(\bar{\nu}_{1-i}) \subset L(\overline{\nu_i}) \otimes \mathbb{C}^{2n}, \quad i = 0, 1,$$

the projection on one irreducible component in the tensor product, and by π_2 the projection on the complement (second irreducible component).

Theorem 2.9. *Let M be a manifold with a given contact symplectic structure, i.e., we have the multiplet $(\mathcal{G}, P, M, \omega)$ determining parabolic geometry. Let us consider Weyl connection ∇^ω in the given parabolic geometry.*

Let π_1 be the projection described on the previous picture, j^{-1} the isomorphism

$$(\mathrm{Id} - \mathrm{Ker}\,\pi_1) \simeq L(\bar{\nu}_{1-i}),$$

$i = 0, 1$. *We define the contact symplectic Dirac operator as the composition of operations on module of values of sections*:

$$D^s = j^{-1} \circ \pi_1 \circ (1 \otimes \omega) \circ \nabla^\omega : \Gamma\big(M, L(\bar{\nu}_i)\big) \to \Gamma\big(M, L(\bar{\nu}_i) \otimes (\mathbb{C}^{2n})^\star\big) \to \Gamma\big(M, L(\bar{\nu}_i) \otimes (\mathbb{C}^{2n+1})\big) \to \Gamma\big(M, L(\bar{\nu}_{\delta_{i+1,1}})\big), \tag{11}$$

where the action of $1 \otimes \omega$ denotes dualization (by ω) on the second part of the tensor product.

Let $i = 0, 1$ and $\pi_2 := \mathrm{Id} - \mathrm{Im}\,\pi_1$ be the projector on complementary irreducible module to the one discussed in the first item. Then we define the contact symplectic Twistor operator as the composition

$$T^s = k^{-1} \circ \pi_2 \circ (1 \otimes \omega) \circ \nabla^\omega : \Gamma\big(M, L(\bar{\nu}_i)\big) \to \Gamma\big(M, L(\bar{\nu}_i) \otimes (\mathbb{C}^{2n})^\star\big) \to \Gamma\big(M, L(\bar{\nu}_i) \otimes (\mathbb{C}^{2n})\big) \to \Gamma\big(M, L(\omega_1 + \bar{\nu}_i)\big), \quad i = 0, 1. \tag{12}$$

where k^{-1} has quite analogous rôle as j^{-1} in the previous item.

Then D^s, T^s respectively, are invariant differential operators of first-order for parabolic geometry of C-type for generalized conformal weights

$$w_{D^s} = \frac{(2n+1)(n+1)}{2(n+1)} \qquad w_{T^s} = \frac{(n+2)}{2(n+1)}. \tag{13}$$

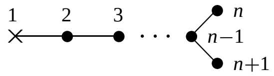
The symplectic Dirac operator has been first time studied on symplectic manifolds with metaplectic structure by K. Habermann, see [5], but only from the differential geometry point of view and no invariance or representation theory involved. The relation between these two operators is still not understood, but the operator of K. Habermann seems to be the restriction of our on symplectic part of contact geometry.

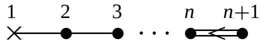
3. The unification — vocabulary

In this last section, we shall summarize all facts, and show a remarkable unified vocabulary.

In terms of Lie algebras, the orthogonal Lie algebra of the (Lie) group of automorphisms of invariant differential operators on the one side corresponds to symplectic Lie algebra on the other side; in terms of fibers over base manifolds, orthogonal side corresponds to conformal Lie algebra, symplectic side to the Lie algebra of endomorphisms of contact domain.

The crossed Dynkin diagrams are

orthogonal side: 

symplectic side: 

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The (half) spinor representations $S^{\pm}$ on the orthogonal side (the sections valued in this module are called spinor fields) correspond to symplectic (half) spinor representations $L^{\pm}$ (the sections are called symplectic spinors).

This notion has been proposed by B. Kostant, see [8]. We shall go on and considerably extend, in the framework of *parabolic invariant theory*, his proposal. We shall write down all results only for one half spinor representation, the second being quite analogous.

(i) To the orthogonal side

$$S \otimes_{\mathbb{C}} \mathbb{C}^{2n} \simeq S \oplus S_{\frac{3}{2}}$$

there corresponds symplectic side

$$L \otimes_{\mathbb{C}} \mathbb{C}^{2n} \simeq L \oplus L_{\frac{1}{2}}$$

(with obvious meaning of notation $S_{\frac{3}{2}}$ and $L_{\frac{1}{2}}$ respectively).

(ii) On the orthogonal side there is Clifford multiplication $c : S \otimes_{\mathbb{C}} \mathbb{C}^{2n} \to S$, which is inverse to Spin-equivariant embedding $S \hookrightarrow S \otimes_{\mathbb{C}} \mathbb{C}^{2n}$.

It has been proved in [6] that on the symplectic side there is symplectic Clifford multiplication $c_s : L \otimes_{\mathbb{C}} \mathbb{C}^{2n} \to L$, which is inverse to Sp-equivariant embedding $L \hookrightarrow L \otimes_{\mathbb{C}} \mathbb{C}^{2n}$. In this way, we are able to geometrically realize the projectors on two irreducible modules in the decomposition of $L \otimes_{\mathbb{C}} \mathbb{C}^{2n}$ — the module L is in the image of symplectic Clifford multiplication ($\mathrm{Im}(c_s) \simeq L$), the second module is in its kernel ($\mathrm{Ker}(c_s) \simeq L_{\frac{3}{2}}$).

(iii) On the orthogonal side, for the projection $\pi : \mathrm{Spin} \to SO$ there is relation in the Clifford algebra connecting Spin-representation with SO-representation, $gvg^{-1} = \pi(g)v$ for all $v \in \mathbb{C}^{2n}$, $g \in \mathrm{Spin}(2n)$.

On the symplectic side we have similar relation inside $\mathrm{End}(L)$ (due to infinite-dimensionality of symplectic Clifford algebra), see [7]. For $\pi' : Mp \to Sp$ and for all $v \in \mathbb{C}^{2n}$, $g \in Mp$, we have $gc_s(v)g^{-1} = c_s(\pi'(g)v)$.

(iv) On the orthogonal side (conformal Riemannian Spin-geometry), there are two first-order invariant differential operators on spinor module S, Dirac operator $D : S \to S$ and Twistor operator $T : S \to S_{\frac{3}{2}}$.

On the symplectic side (contact Mp-geometry) there are due to the two target representation spaces also two first-order invariant differential operators on SSW-module, symplectic Dirac operator $D^s : L \to L$ and symplectic Twistor operator $T^s : L \to L_{\frac{1}{2}}$.

(v) Let us assume to work with homogeneous model of the geometry, i.e., the (base) manifold is the quotient of Lie groups.

In the case of conformal geometry (orthogonal side) we have B-G-G resolution for every finite-dimensional irreducible module of the Lie group of conformal diffeomorphisms; in the case of spinor module, the invariant differential operator acting between first and second place of resolution corresponds to the basic Twistor operator. Thus the resolution controls the kernel of Twistor operator on the sphere, $\mathrm{Ker}(T) \simeq S$ (S is $\mathfrak{g}$-mod for $\mathfrak{g} = \mathrm{spin}(n+2)$), see [12].

In the case of contact geometry (symplectic side) we have Enright resolution for every discrete series representation. The geometric (injective) realization gives the kernel of symplectic Twistor operator $\mathrm{Ker}(T^s) \simeq L$, where L is $\mathfrak{g}$-mod for $\mathfrak{g} = \mathrm{sp}(n+1)$.

The previous analysis raises obvious question — to which extend and how far this analogy works and can be extended. We have no complete answer to this.

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