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On modules with differential[1]

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Abstract. The paper contains some constructions in the categories of modules with differential, as defined by the first author in a previous paper. Actions and preactions on morphisms and objects, quotients and ideals in these categories are studied.

Keywords. Module with differential, anchored module, (pre)infinitesimal module, Lie pseudoalgebra, (pre)action, ideal.

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1. Introduction

The aim of this paper is to continue the ideas from the previous work [12] of the first author, where the modules with differential are defined. A more abstract algebraic treatment of the category of Lie algebroids, as performed in [3], is also given in the paper. Some algebraic properties and constructions from [3] are extended to the category of Lie pseudoalgebras in [8]. In this paper we make systematic and natural extensions of constructions performed in [3] — actions and preaction on morphisms and objects, quotients and ideals — not only in the category Lie pseudoalgebras, but in the categories of modules with differential, as pure algebraic theory.

Actions and preactions on morphisms of modules with differential are studied in Section 2. Actions and preactions on modules with differential are studied in Section 3, considering also the curvature of a splitting. Quotients and ideals of modules with differential are studied in Section 4.

We recall some basic definitions. Let **k** be a commutative ring. A *module* is a couple (A, L), where A is a commutative and associative **k**-algebra and L is an A-module. An *anchored module* (or a *module with arrows* in [13, 12]) is a module

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(A, L) which has an *anchor* (or an *arrow*), i.e., an A-linear mapping $D : L \longrightarrow \mathcal{D}er(A)$; we denote $D(X)(a) = [X, a]_L$. A *linear connection* ∇ (or a *linear L-connection*) on a module (A, M), related to an anchored module (A, L), is a map

$$\nabla : L \times M \longrightarrow M, \quad \nabla(X, u) \overset{\text{not.}}{=} \nabla_X u,$$

such that the Koszul conditions hold. A *preinfinitesimal module* is an anchored module (A, L) with the anchor D and a *bracket* $[\cdot, \cdot]$, i.e., a $\mathbf{k}$-bilinear map $[\cdot, \cdot]_L : L \times L \longrightarrow L$ which is skew-symmetric and enjoys the property $[X, a \cdot Y]_L = [X, a]_L \cdot Y + a \cdot [X, Y]_L$, $(\forall) X, Y \in L,\ a \in A$. A preinfinitesimal module (A, L) is an *infinitesimal module* if the condition $[D(X), D(Y)] = D([X, Y]_L)$ holds. An infinitesimal module (A, L) is a *Lie pseudoalgebra* if $\mathcal{J} = 0$, where $\mathcal{J}(X, Y, Z) = [[X, Y]_L, Z]_L + [[Y, Z]_L, X]_L + [Z, X]_L, Y]_L$ is the Jacobiator of the bracket. According to [12], the anchored modules, (pre)infinitesimal modules and Lie pseudoalgebras are called *modules with differential.*

2. Actions and preactions on morphisms

In this section we extend to modules with differential the correspondence of [3] between actions of Lie algebroids and action morphisms.

Definition 2.1. Let (A, L) be an anchored module or a preinfinitesimal module and $\varphi : A \to A'$ a morphism of commutative and unitary algebras. A *preaction* of L on A' (or on φ) is a map $[\cdot, \cdot] : L \times A' \to A'$ such that, for every $u \in A$, $u', v' \in A'$ and $X \in L$ the following conditions are fulfilled:

1. $[\cdot, \cdot]$ is k-bilinear;
2. $[uX, v'] = \varphi(u)[X, v']$;
3. $[X, \varphi(u)] = \varphi([X, u]_L)$;
4. $[X, u'v'] = u'[X, v'] + [X, u']v'$.

If (A, L) is a preinfinitesimal module and moreover

5. $[[X, Y]_L, u'] = [X, [Y, u']_L] - [Y, [X, u']_L]$, $(\forall) X, Y \in L,\ u' \in A'$,

then we say that $[\cdot, \cdot]$ is an *action* of L on A' (or on φ).

Condition 4 implies that for every $X \in L$ there is $X^* \in \mathcal{D}er\ A'$ such that $[X, u] = X^*(u)$. Conditions 1 and 2 imply that $X \to X^*$ is an A-module morphism which induces an A'-module morphism $\widetilde{\psi} : A' \otimes_A L \to \mathcal{D}er\ A'$,

$$\sum_i a_i \otimes_A X_i \to \sum_i a_i X_i^*.$$

In this way $(A', A' \otimes_A L)$ becomes an anchored module.

Definition 2.2. If $(A', L') \xrightarrow{(\varphi, \psi)} (A, L)$ is a module with differential morphism such that $\psi : A' \otimes_A L \to L'$ is an A'-module isomorphism, then we say that (φ, ψ) is a *preaction morphism* in the case of anchored modules and preinfinitesimal modules and an *action morphism* in the case of infinitesimal modules and Lie pseudoalgebras.

A (pre)action morphism induces a module with differential structure on the module $(A', A' \otimes_A L)$ which is isomorphic with (A', L'). We can extend now ([3, Theorem 2.6]) to modules with differential:

Theorem 2.3. 1) *Let (A, L) be a module with differential, $\varphi : A \to A'$ be an algebra morphism and denote $L' = A' \otimes_A L$. Then we have:*

1.a) *If (A, L) is an anchored module and $[\cdot, \cdot]$ is a preaction of L on A', then the formula*

$$\Big[\sum_i a'_i \otimes_A X_i, v'\Big]_{L'} \overset{\text{def.}}{=} \sum_i a'_i[X_i, v'], \tag{1}$$

$\forall \sum_i a'_i \otimes_A X_i \in L'$, $v' \in A'$ defines an anchored module on (A', L').

1.b) *If (A, L) is a preinfinitesimal module and $[\cdot, \cdot]$ is a preaction of L on A', then the formulas* (1) *and*

$$\begin{aligned}\Big[\sum_i u'_i \otimes_A X_i, \sum_j v'_j \otimes_A Y_j\Big]_{L'} &\overset{\text{def.}}{=} \sum_{i,j} u'_i v'_j \otimes_A [X_i, Y_j]_L \\ &+ \sum_{i,j}(u'_i[X_i, v'_j]) \otimes_A Y_j - \sum_{i,j}(v'_j[Y_j, u'_i]) \otimes_A X_i\end{aligned} \tag{2}$$

define a preinfinitesimal module structure (A', L'). Moreover, $(A', L') \xrightarrow{(\varphi, \mathrm{id}_{L'})} (A, L)$ is a preaction morphism in both cases (a) *and* (b).

1.c) *If (A, L) is an infinitesimal module or a Lie pseudoalgebra and $[\cdot, \cdot]$ is an action of L on A', then the formulas* (1) *and* (2) *yield on the module (A', L') an infinitesimal module (Lie pseudoalgebra respectively) structure such that*

$$(A', L') \xrightarrow{(\varphi, \mathrm{id}_{L'})} (A, L)$$

is a morphism of infinitesimal module (Lie pseudoalgebra, respectively) which is an action morphism.

2.a) *If $(A', L') \xrightarrow{(\varphi, \psi)} (A, L)$ is a preaction morphism of anchored module or preinfinitesimal module, then the formula*

$$[X, u'] \overset{\text{def.}}{=} [X', u'], \tag{3}$$

where $u' \in A'$, $X \in L$ and $X' = \psi^{-1}(1 \otimes_A X) \in L'$ defines a preaction of L on A'.

2.b) *If $(A', L') \xrightarrow{(\varphi, \psi)} (A, L)$ is an action morphism of infinitesimal module or Lie pseudoalgebra, the formula* (3) *defines an action of L on A'.*

3) *The constructions* 1) *and* 2) *are mutually inverse, up to natural isomorphisms of modules with differential.*

Proof. As in the proof of [12, Lemma 3.1], it can be proved that the definitions do not depend on the tensor decompositions. A straightforward computation shows that (1) defines a derivation on A'. In the same way it can be shown that (2) defines a bracket on L'; the assertion that $(\varphi, \mathrm{id}_{L'})$ is a preaction morphism follows easy.

Assertion 1.c) results using the last relation in Definition 2.1. The implication "if (A, L) is an infinitesimal module and $[\cdot, \cdot]$ is an action, then (A', L') is an infinitesimal module" follows by a straightforward computation. In the Lie pseudoalgebra case, it can be shown that the Jacobi map vanishes. Assertion 2.a) follows by a straightforward verification of the first four conditions in Definition 2.1. It suffices to check the assertion 2.b) in the case of infinitesimal modules. In order to prove 3), first observe that taking a module with differential (A, L) and a (pre)action of L on A', one can define the module with differential $(A', L' = A' \otimes_A L)$ using 1.a) or 1.b) and the (pre)action of L on A' using 2.a) or 2.b), then the initial (pre)action is obtained. Conversely, taking a (pre)action morphism $(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$ and one consider a (pre)action using 2.a) or 2.b), then using 1.a) or 1.b), a module with differential, isomorphic with the initial one, is obtained. □

It follows that the (pre)actions are in one to one correspondence with the structures of modules with differential on $(A', A' \otimes_A L)$, in the following sense:

Corollary 2.4. *Let (A, L) be a module with differential, $\varphi : A \to A'$ be an algebra morphism and $L' = A' \otimes_A L$. Then the (pre)actions of L on φ are in one to one correspondence to the structures of modules with differential on (A', L'), such that*

$$(A', L') \xrightarrow{(\varphi,\mathrm{id}_{L'})} (A, L)$$

is a morphism of modules with differential of the same kind.

The constructions performed in this subsection can be easily translated on wide or strict categories of modules with differential defined in [12].

3. Actions and preactions on modules with differential

In this section we extend the study of the actions on Lie algebroids performed in [3, Section 3] to modules with differential.

Definition 3.1. 1. Let (A', L') and (A, L) be anchored modules, $\varphi : A \to A'$ be a morphism of algebras such that

$$[X', \varphi(u)] = 0, \quad (\forall) X' \in L', u \in A \tag{4}$$

and $[\cdot, \cdot] : L \times A' \to A'$ be a preaction of L on φ. Then we say that $[\cdot, \cdot]$ defines a *preaction* (of anchored modules) of L on L' (via φ).

2. Let (A', L') and (A, L) be preinfinitesimal modules, $[\cdot, \cdot] : L \times A' \to A'$ be a preaction of anchored modules (as above) of L on L' via $\varphi : A \to A'$, and $\nabla : L \times L' \to L'$ be a linear L-connection on L', considered as an A-module throw φ, such that

$$\nabla_X(u'X') = [X, u']X' + u'\nabla_X X',$$

$(\forall) X \in L,\ X' \in L',\ u' \in A'$. Then we say that $[\cdot, \cdot]$ and ∇ define a *preaction* (of preinfinitesimal modules) of L on L' (via φ).

3. Let (A', L') and (A, L) be infinitesimal modules, $[\cdot,\cdot]$ and ∇ be as above, defining a preaction of preinfinitesimal modules of L on L' via $\varphi : A \to A'$, such that $[\cdot,\cdot] : L \times A' \to A'$ is an action of L on A' and the following relation holds true:

$$[X, [X', u']_{L'}] - [X', [X, u']]_{L'} = [\nabla_X X', u']_{L'},$$

$(\forall) X \in L$, $X' \in L'$, $u' \in A'$. Then we say that $[\cdot,\cdot]$ and ∇ define an *action* (of infinitesimal modules) of L on L' (via φ).

4. Let (A', L') and (A, L) be Lie pseudoalgebras, $[\cdot,\cdot]$ be an action of infinitesimal modules of L on L' via $\varphi : A \to A'$ and ∇ be an L-connection which has a vanishing curvature, such that $\nabla_X[X', Y']_{L'} = [\nabla_X X', Y']_{L'} + [X', \nabla_X Y']_{L'}$, $(\forall) X \in L$, $X', Y' \in L'$. Then we say that $[\cdot,\cdot]$ and ∇ define an *action* (of Lie pseudoalgebras) of L on L' (via φ).

This Definition can be adapted for a strict or wide category of modules with differential, defined in [12].

Theorem 3.2. *Let (A', L') and (A, L) be modules with differential of the same kind and $[\cdot,\cdot]$ be a preaction (in the case of anchored modules or preinfinitesimal modules) or $[\cdot,\cdot]$ and ∇ be an action (in the case of infinitesimal modules or Lie pseudoalgebras) of L on L' via $\varphi : A \to A'$. Then the formula*

$$\Big[\sum_i (u'_i \otimes X_i) \oplus X', u'\Big]_{L''} \overset{\text{def.}}{=} \sum_i u'_i [X_i, u'] + [X', u']_{L'}$$

defines an anchor, and the formula

$$\begin{aligned}&\Big[\sum_i (u'_i \otimes X_i) \oplus X', \sum_j (v'_j \otimes Y_j) \oplus Y'\Big]_{L''} \overset{\text{def.}}{=} \\ &\quad\Big\{\sum_{i,j} u'_i v'_j \otimes_A [X_i, Y_j]_L + \sum_{i,j} u'_i [X_i, v'_j] \otimes_A Y_j - \sum_{i,j} v'_j [Y_j, u'_i] \otimes_A X_i \\ &\qquad + \sum_j [X', v'_j] \otimes_A Y_j\Big\} \\ &\quad - \sum_i [Y', u'_i] \otimes_A X_i \oplus \Big\{[X', Y']_{L'} + \sum_i u'_i \nabla_{X_i} Y' - \sum_j v'_j \nabla_{Y_j} X'\Big\}\end{aligned}$$

defines a bracket (if it exists), such that the A'-module $L'' = (A' \otimes_A L) \oplus L'$ becomes a module with differential structure of the same kind as L'.

This theorem can be adapted for a strict or wide category of modules with differential.

We study now the fibrations, i.e., the morphisms of modules with differential which have within an injective morphism of algebras and a surjective morphism of modules, and finally we study split fibrations. Defining the curvature of a splitting of split modules, then the split fibrations correspond to a vanishing curvature. We give the definitions of fibration and split fibration starting from [3, Section 3].

Definition 3.3. A contravariant morphism of modules or a morphism of modules with differential $(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$ is a *fibration* if φ is an injection and ψ is a surjection. A module split fibration is an A'-modules morphism ψ having a right inverse. The module with differential fibration *splits* if there is a preaction (in the case of anchored modules or preinfinitesimal modules) or an action (in the case of infinitesimal modules or Lie pseudoalgebras) such that, considering the structure of module with differential on $(A', (A' \otimes_A L))$ given by Theorem 2.3, there is an A'-morphism of modules with differential $S : A' \otimes_A L \longrightarrow L'$ which is a right inverse for ψ in the category of A'-modules with differential.

Notice that for a (pre)action the condition (4) ensures that $L' \xrightarrow{\psi} A' \otimes_A L$ is a morphism in a category of A'-modules with differential.

Proposition 3.4. *Consider a preinfinitesimal module* (A, L), $Q \in a_1(L, L) = \mathrm{End}_A L$ *and*

$$N_Q(X, Y) \stackrel{\text{def.}}{=} [QX, QY]_L + Q^2[X, Y]_L - Q[QX, Y]_L - Q[X, QY]_L,$$

$\forall X, Y \in L$. *Then we have:*

(i) $N_Q \in a_2(L, L)$;

(ii) *If Q is a projector (i.e., $Q^2 = Q$) then, denoting as $P = \mathrm{id}_L - Q$ the supplementary projector, we have $N_Q = N_P = Q[PX, PY]_L + P[QX, QY]_L$. We call N_Q the* Nijenhuis tensor *associated with Q.*

Definition 3.5. Let (A', L') be a preinfinitesimal module,

$$(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$$

be a fibration of modules and $S : A' \otimes_A L \longrightarrow L'$ be a splitting of module. It follows the direct sum decomposition of the module $L' : L' = \mathrm{Im}\, S \oplus \ker \psi$. Denote as $h\ (= S \circ \psi)$ and $v\ (= \mathrm{id}_{L'} - S \circ \psi)$ the projectors associated with this decomposition. Denote as

$$\Omega_S(X', Y') \stackrel{\text{def.}}{=} -v[hX, hY]_{L'}$$

$\forall X', Y' \in L'$, called the *curvature* of S.

Proposition 3.6. *We have:*

1. $\Omega_S \in a_2(L', \ker \psi)$.
2. *If $(A', \ker \psi)$ is a preinfinitesimal submodule of (A', L'), i.e., $\forall X', Y' \in \ker \psi \Rightarrow [X', Y']_{L'} \in \ker \psi$, then*

$$\Omega_S = -N_v = -N_h. \tag{5}$$

3. *If $(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$ is a fibration of preinfinitesimal modules, then $(A', \ker \psi)$ is a preinfinitesimal submodule of (A', L') and for every splitting $S : A' \otimes_A L \longrightarrow L'$ of modules, the relation* (5) *holds true.*

We deal now with the splitings of modules with differential.

Theorem 3.7. 1. *Let* (A', L') *and* (A, L) *be modules with differential and* $\varphi : A \to A'$ *be a morphism of algebras. Consider a preaction (in the case of the anchored modules or the preinfinitesimal modules), or an action (in the case of the infinitesimal modules or the Lie pseudoalgebras) of* L *on* L' *(via* φ*). Consider also the structure of module with differential on* $(A' \otimes_A L) \oplus L'$ *given by Theorem* 3.2 *and denote as*

$$\Psi : (A' \otimes_A L) \oplus L' \longrightarrow A' \otimes_A L$$

the canonical projection on the first term. Then the morphism of modules with differential of the same kind

$$(A' \otimes_A L) \oplus L' \xrightarrow{(\mathrm{id}_{A'}, \Psi)} A' \otimes_A L$$

is a split fibration.

2. *Conversely, for every split fibration of modules with differential*

$$(A', L') \xrightarrow{(\varphi, \psi)} (A, L),$$

there is a canonical (pre)action of L *on* $\ker \psi$*, such that, considering on the* A'*-module* $(A' \otimes_A L) \oplus \ker \psi$ *the structure of module with differential given by Theorem* 3.2*, then there exists a canonical isomorphism of* A'*-modules with differential*

$$\Phi : L' \longrightarrow (A' \otimes_A L) \oplus \ker \psi.$$

3. *The constructions* 1 *and* 2 *are mutually inverse, up to natural isomorphisms.*

A way to construct new modules with differential is the following result.

Corollary 3.8. *Let* (A, L) *be a module with differential and* $(A', L') \xrightarrow{(\varphi, \psi)} (A, L)$ *be a split fibration of modules, which splits by* S*. Let us suppose that* $\ker \psi$ *is a module with differential of the same kind as* (A, L) *and a (pre)action of* L *on* $\ker \psi$ *is given. Then there is a structure of module with differential on* (A', L') *such that*

$$(A', L') \xrightarrow{(\varphi, \psi)} (A, L)$$

is a split fibration of modules with differential which splits by S*.*

A connection between the splitting and the curvature is given by the following result.

Corollary 3.9. *Let* $(A', L') \xrightarrow{(\varphi, \psi)} (A, L)$ *be a fibration of (pre)infinitesimal modules or Lie pseudoalgebras, which splits as anchored modules by the morphism of anchored* A'*-modules*

$$S : A' \otimes_A L \longrightarrow L'.$$

Then the given fibration splits by S *as (pre)infinitesimal modules, or Lie pseudoalgebras respectively, iff* S *has a null curvature,* $\Omega_S = 0$*.*

4. Quotients and ideals

We start with the case of anchored modules. For the kernel of a fibration of anchored modules we have:

Proposition 4.1. *Let* $(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$ *be a fibration of anchored modules. Then* $(A', \ker\psi)$ *is an anchored submodule of* (A', L') *and the following properties are fulfilled:*

I1) *Every* $X' \in L'$ *such that there is* $X \in L$ *such that* $\psi(X') = 1 \otimes_A X$*, we say that* X' projects on L*, has the property that for every* $a \in A$ *there is* $b \in A$ *such that*

$$[X', \varphi(a)]_{L'} = \varphi(b).$$

I2) *For every* $Z' \in \ker\psi$ *and* $a \in A$*, we have*

$$[Z', \varphi(a)]_{L'} = 0.$$

Proposition 4.2. *Let* (A', L') *be an anchored module,* $(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$ *be a fibration of modules, such that the properties* I1), I2) *from Proposition* 4.1 *are fulfilled and denote as*

$$\widetilde{L} = \{1 \otimes_A X | X \in L\}.$$

Then there exist a structure of anchored module on $(A, \widetilde{L})$ *and a fibration of anchored modules*

$$(A', L') \xrightarrow{(\varphi,\widetilde{\psi})} (A, \widetilde{L})$$

such that $\ker\psi = \ker\widetilde{\psi}$.

Definition 4.3. An *anchored ideal* of an anchored module (A', L') is every anchored submodule (A', K), $K \subset L'$ such that there is a fibration of modules $(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$, $K = \ker\psi$ satisfying the conditions I1) and I2) of Proposition 4.1.

Using Proposition 4.2 we have:

Corollary 4.4. *Let* (A', L') *be an anchored module and* $K \subset L'$ *be an anchored ideal.*

Then there is a fibration of anchored modules $(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$ *such that* $K = \ker\psi$.

Notice that for every module (A, L), the A-morphism η, considered in the proof of Proposition 4.2, is surjective.

Corollary 4.5. *Let us suppose that under the hypothesis of Proposition* 4.2*, the* A*-morphism* η *has a right inverse* δ*. Then an anchored module structure can be defined on the module* (A, L)*, such that* $(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$ *becomes a fibration of anchored modules.*

Notice that in the particular case when the module (A, M) is projective, η is an isomorphism of A-modules. We have the following universal property:

Proposition 4.6. *Let* $(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$ *be a fibration of anchored modules,*

$$(A', L') \xrightarrow{(\varphi'',\psi'')} (A'', L'')$$

be a morphism of anchored modules and

$$(A, L) \xrightarrow{(\varphi',\psi')} (A'', L'')$$

be a morphism of modules, such that the diagram

$$\text{(6)}\qquad \begin{array}{ccc} (A', L') & \xrightarrow{(\varphi,\psi)} & (A, L) \\ & {\scriptstyle (\varphi'',\psi'')} \searrow \quad \swarrow {\scriptstyle (\varphi',\psi')} & \\ & (A'', L'') & \end{array}$$

commutes in the category of modules. Then (φ', ψ') *is a morphism of anchored modules and the diagram* (6) *is commutative in the category of anchored modules.*

We study now the fibrations of (pre)infinitesimal modules and Lie pseudoalgebras.

Proposition 4.7. *Let* $(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$ *be a fibration of (pre)infinitesimal modules, or Lie pseudoalgebras. Then* $(A', \ker\psi)$ *is a submodule with differential of the same kind and the following relations are fulfilled:*

I3) *For every* $X' \in L'$ *which projects on* L*, i.e.,* $\psi(X') = 1 \otimes_A X$*, for an* $X \in L$*, and* $Z' \in \ker\psi$ *we have* $[X', Z'] \in \ker\psi$.

I4) *If* $X', Y' \in L'$ *project on* L*, then* $[X', Y']_{L'}$ *projects on* L.

Proposition 4.2 can be extended for all differential modules:

Proposition 4.8. *Let* (A', L') *be a (pre)infinitesimal module, or a Lie pseudoalgebra,* $(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$ *be a fibration of modules, such that the properties* I1–I4 *considered in Propositions* 4.1 *and* 4.7 *are fulfilled and denote as*

$$\widetilde{L} = \{1 \otimes_A X | X \in L\}.$$

Then there exist a module with differential structure on $(A, \widetilde{L})$ *and a fibration of modules with differential*

$$(A', L') \xrightarrow{(\varphi,\widetilde{\psi})} (A, \widetilde{L}),$$

such that $\ker\psi = \ker\widetilde{\psi}$.

Definition 4.9. An *ideal* of a (pre)infinitesimal module, or of a Lie pseudoalgebra (A', L') is a submodule with differential structure (A', K), $K \subset L'$ such that there is a fibration of modules $(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$, $K = \ker\psi$, such that the conditions I1)–I4) are fulfilled.

Using Proposition 4.2 we can extend Corollaries 4.4 and 4.5 to all the modules with differential:

Corollary 4.10. *Let (A', L') be a module with differential and $K \subset L'$ be an ideal (of a module with differential). Then there is a fibration of modules with differential $(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$ such that $K = \ker\psi$.*

Notice that for a fibration of modules with differential $(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$, then $(A', L') \xrightarrow{(\varphi,\psi)} (A, \widetilde{L})$ is also a fibration of modules with differential, of the same kind, but it is possible that the modules (A, L) and $(A, \widetilde{L})$ be not isomorphic. It follows that the first theorem of isomorphism does not hold true, in the general case of modules with differential.

For every module (A, L), the A-morphism $\eta : L \to \widetilde{L}$, $\eta(x) = 1 \otimes_A X$, is surjective.

Corollary 4.11. *Let us suppose that under the hypothesis of Proposition* 4.8*, the A-morphism η has a right inverse δ. Then a structure of module with differential can be defined on the module (A, L), such that $(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$ is a fibration of modules with differential.*

The above Corollary can be used for example in the case where (A, L) is a projective module, when η is an isomorphism. Proposition 4.6 has an analogous one for the other modules with differential, but the hypothesis is more strong:

Proposition 4.12. *Let $(A', L') \xrightarrow{(\varphi,\psi)} (A, L)$ be a fibration of (pre)infinitesimal modules, or Lie pseudoalgebras, $(A', L') \xrightarrow{(\varphi'',\psi'')} (A'', L'')$ be a morphism of modules with differential of the same kind, and $(A, L) \xrightarrow{(\varphi',\psi')} (A'', L'')$ be a morphism of modules, such that the diagram* (6) *is commutative in the category of modules. Let us suppose that the morphism of the A''-modules*

$$\varphi' \otimes_{A''} \mathrm{id}_{L''} : A \otimes_{A''} L'' \longrightarrow A' \otimes_{A''} L''$$

is an injective map. Then (φ', ψ') is a morphism of modules with differential, of the same kind, and the diagram (6) *is commutative in the considered category of modules with differential.*

If the module (A'', L'') is flat (in particular projective), then the morphism of A''-modules $\varphi' \otimes_{A''} id_{L''}$ is an injective map and the above result holds true.

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