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On the geometry of fiber product preserving bundle functors[1]

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Abstract. Using the description of a fiber product preserving bundle functor F in terms of Weil algebras, we deduce several geometric properties of the F-prolongations of principal and associated bundles. Then we clarify that the flow prolongation with respect to F of a projectable vector field can be constructed by using a natural morphism.

Keywords. Fiber product preserving bundle functor, Weil algebra, principal bundle, associated bundle.

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Our starting point is the classical result reading that the product preserving bundle functors on the category $\mathcal{M}f$ of all smooth manifolds and all smooth maps coincide with the Weil functors and their natural transformations are in bijection with the algebra homomorphisms, see [7] for a survey. Hence the Weil algebras can be viewed as a unified technique for investigating the geometric properties of all these functors. Some recent geometric results deduced in this way can be found in ([2, 3, 5]).

Let $\mathcal{FM}_m$ be the category of fibered manifolds with m-dimensional bases and fiber preserving maps with local diffeomorphisms as base maps, [7]. A fiber product preserving bundle functor F on $\mathcal{FM}_m$ was characterized in terms of Weil algebras (in [6]), which is summarized in Section 1 of the present paper. We point out that the most important examples are

(i) the r-th jet prolongation $J^r Y$ of a fibered manifold $p : Y \to M, m = \dim M$,
(ii) the r-th vertical jet prolongation of Y

$$J^r_v Y = \bigcup_{x \in M} J^r_x(M, Y_x)\,,$$

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where $J^r_x(M, Y_x)$ is the space of all r-jets of M into the fiber Y_x with source x,

(iii) the vertical A-prolongation of Y

$$V^AY = \bigcup_{x\in M} T^A(Y_x)\,,$$

where A is a Weil algebra and T^A is the corresponding Weil functor,

(iv) iterations and some subbundles, for instance the second non-holonomic prolongation $\widetilde{J}^2Y = J^1(J^1Y \to M)$ and the second semiholonomic prolongation $\bar{J}^2Y \subset \widetilde{J}^2Y$.

In [4], the iteration of two fiber product preserving bundle functors on $\mathcal{FM}_m$ is described. This description is heavily based on the fact that F induces a principal bundle $W^FP(M, W^A_HG)$ for every principal bundle $P(M, G)$. Moreover, if $P[S, l]$ is a fiber bundle associated to P, then $F(P[S, l])$ is a fiber bundle associated to W^FP.

The present paper is devoted to some geometric constructions that are applicable to each F. In Section 2 we deduce a Weilian formula for the Lie algebra of the structure group W^A_HG of W^FP. We also characterize geometrically the prolongation of the infinitesimal action of the left action $l : G \times S \to S$. The main result of Section 3 is Proposition 4 reading that the flow prolongation $\mathcal{F}\eta$ of a projectable vector field $\eta : Y \to TY$ over a vector field $\xi : M \to TM$ can be constructed by applying a natural morphism μ^F_Y to the functorial prolongation $\mathcal{F}\eta : FY \to FTY$ and the r-th jet prolongation $j^r\xi$, where r is the base order of F.

Unless otherwise specified, we use the terminology and notation from ([7]). All manifolds and maps are assumed to be infinitely differentiable.

1. The foundations

Let F be a bundle functor on $\mathcal{FM}_m$. We say that F preserves fiber products, if $F_x(Y \times_M \bar{Y}) = F_xY \times F_x\bar{Y}$ for every pair of fibered manifolds $Y, \bar{Y}$ over the same base M and each $x \in M$. The construction of product fibered manifolds and product morphisms defines an injection $\iota : \mathcal{M}f_m \times \mathcal{M}f \to \mathcal{FM}_m$, $\iota(M, N) = M \times N \to M$, $\iota(f, g) = f \times g$, where $\mathcal{M}f_m$ means the category of m-dimensional manifolds and local diffeomorphisms. Then the restriction $\widetilde{F} = F \circ \iota$ is a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$. Clearly, $\widetilde{F}$ induces a bundle functor G on $\mathcal{M}f$ as follows. For every manifold N we set $GN = \widetilde{F}_0(\mathbb{R}^m, N)$, $0 \in \mathbb{R}^m$ and for every $f : N \to \bar{N}$ we define $Gf = \widetilde{F}_0(\mathrm{id}_{\mathbb{R}^m}, f) : GN \to G\bar{N}$. Since F preserves fiber products, G preserves products. By the classical theory, [7], there exists a Weil algebra A such that $G = T^A$.

An integer r is called the base order of F, if $j^r_x f_1 = j^r_x f_2$ implies

$$\widetilde{F}_x(f_1, g) = \widetilde{F}_x(f_2, g) \tag{1}$$

for every pair of local diffeomorphisms $f_1, f_2 : M \to \bar{M}$ and every $g : N \to \bar{N}$, [6]. In this case, consider $X = j^r_0\varphi \in G^r_m$, where $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ is a diffeomorphism, $\varphi(0) = 0$. Then the maps $F_0(\varphi \times \mathrm{id}_N) : T^AN \to T^AN$ determine a natural equivalence $T^A \to T^A$ depending on X only. By the classical result, this defines a map $H : G^r_m \to \mathrm{Aut}\ A$, where $\mathrm{Aut}\ A$ is the group of all algebra automorphisms of A. By [6], H is a group homomorphism. Denote by H_N the induced left action of G^r_m on $T^A N$. Then $\widetilde{F}(M, N)$ coincides with the associated bundle

$$\widetilde{F}(M, N) = P^r M[T^A N, H_N]\,, \tag{2}$$

where $P^r M$ is r-th order frame bundle of M. Consider a local diffeomorphism $f : M \to \overline{M}$ and a map $g : N \to \overline{N}$. By naturality, $T^A g : T^A N \to T^A \overline{N}$ is a G^r_m-equivariant map. Since $P^r f : P^r M \to P^r \overline{M}$ is a local isomorphism of principal bundles, we can construct the induced map

$$P^r f[T^A g] : P^r M[T^A N] \to P^r \overline{M}[T^A \overline{N}]$$

and we have

$$\widetilde{F}(f, g) = P^r f[T^A g] : \widetilde{F}(M, N) \to \widetilde{F}(\overline{M}, \overline{N})\,, \tag{3}$$

see, [6]. Conversely, given a Weil algebra A and a group homomorphism $H : G^r_m \to \mathrm{Aut}\ A$, then (2) and (3) define a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$, [6].

Moreover, F induces a natural transformation $t_Y : J^r Y \to FY$ as follows. Every section $s : M \to Y$ can be interpreted as a morphism $\widetilde{s}$ of the trivial fibered manifold $\mathrm{id}_M : M \to M$ into Y and we set

$$t_Y(j^r_x s) = (F\widetilde{s})(x), \quad x \in M\,. \tag{4}$$

The restriction of this natural transformation to the products $\mathbb{R}^m \times N$ yields a G^r_m-equivariant algebra homomorphism $t : \mathbb{D}^r_m \to A$, where $\mathbb{D}^r_m = J^r_0(\mathbb{R}^m, \mathbb{R})$, [6]. Then FY is identified with the set of all equivalence classes $\{u, Z\} \in \widetilde{F}(M, Y) = P^r M[T^A Y]$, $u \in P^r M$, $Z \in T^A Y$ satisfying

$$t_M(u) = T^A p(Z)\,, \tag{5}$$

where $t_M : T^r_m M \to T^A M$ denotes the map induced by t and the inclusion $P^r M \subset T^r_m M$ is taken into account. Conversely, given A, H and an equivariant algebra homomorphism $t : \mathbb{D}^r_m \to A$, then (5) determines a subspace $FY \subset \widetilde{F}(M, Y)$. If $\underline{f} : M \to \overline{M}$ is the underlying base map of an $\mathcal{FM}_m$-morphism $f : Y \to \bar{Y}$, then $\widetilde{F}(\underline{f}, f)$ maps FY into $F\bar{Y}$. This defines a fiber product preserving bundle functor $F = (A, H, t)$ on $\mathcal{FM}_m$, [6].

2. On the prolongation of principal and associated bundles

In general, the canonical injection $\nu_M : M \to T^A M$ can be constructed as follows. Consider a singleton pt and write $i_x : pt \to M$ for the map $i_x(pt) = x$. Then $T^A i_x : pt \to T^A M$ and we have

$$\nu_M(x) = (T^A i_x)(pt)\,. \tag{6}$$

If V is a vector space, then $T^A V$ is also a vector space. The scalar multiplication $m_V : \mathbb{R} \times V \to V$ induces $T^A m_V : A \times T^A V \to T^A V$.

We shall write

$$az = T^A m_V(a, z), \quad a \in A,\ z \in T^A V. \tag{7}$$

The rule

$$v \otimes a \approx T^A m_V\big(a, \nu_V(v)\big), \quad a \in A,\ v \in V, \tag{8}$$

identifies $T^A V$ with the tensor product $V \otimes A$.

If $\gamma : G \times G \to G$ is the multiplication of a Lie group G, then $T^A G$ with the multiplication $T^A\gamma$ is also a Lie group. The Lie algebra Lie $G = \mathfrak{g}$ is a subset in TG, so that $T^A\mathfrak{g} \subset T^A TG$. In [7, p. 328], it is proved that the canonical exchange map $\varkappa_G : T^A TG \to TT^A G$ identifies $T^A\mathfrak{g} = \mathfrak{g} \otimes A$ with Lie $(T^A G)$. The bracket in Lie $(T^A G)$ satisfies

$$[v_1 \otimes a_1,\ v_2 \otimes a_2] = [v_1, v_2] \otimes a_1 a_2\,, \tag{9}$$

$v_i \in \mathfrak{g}$, $a_i \in A$, $i = 1, 2$, where the bracket $[v_1, v_2]$ is in $\mathfrak{g}$ and the product $a_1 a_2$ is in A.

Remark 1. The Weil algebra corresponding to the functor T^r_k of (k,r)-velocities is $\mathbb{D}^r_k$. Even in this classical situation, (9) gives a very suitable description of the bracket of Lie$(T^r_k G)$.

Let A or H be the Weil algebra or the group homomorphism $H : G^r_m \to \operatorname{Aut} A$ determined by F. In [4], it is deduced that every principal bundle $P(M, G)$ induces a principal bundle $W^F P(M, W^A_H G)$, where

$$W^F P = P^r M \times_M FP \tag{10}$$

and the structure group $W^A_H G$ is the semi-direct product $W^A_H G = G^r_m \rtimes_H T^A G$ with the multiplication

$$(X_1, Z_1)(X_2, Z_2) = \Big(X_1 \circ X_2,\ T^A\gamma\big(H(X_2)_G(Z_1), Z_2\big)\Big), \tag{11}$$

$X_i \in G^r_m$, $Z_i \in T^A G$.

The right action $\varrho : P \times_M (M \times G) \to P$ of G on P induces

$$F\varrho : FP \times_M P^r M[T^A G] \to FP$$

and the right action of $W^A_H G$ on $W^F P$ is given by

$$(u, U)(X, Z) = \Big(u \circ X,\ F\varrho\big(U, \{u \circ X, Z\}\big)\Big)\,,$$

$u \in P^r M$, $U \in FP$.

The group homomorphism $H : G^r_m \to \operatorname{Aut} A$ determines a Lie algebra homomorphism $h : \mathfrak{g}^r_m \to \operatorname{Lie}(\operatorname{Aut} A) \approx \operatorname{Der} A$. Using the theory of semi-direct products of Lie algebras, [1], we obtain

Proposition 1. *The multiplication in Lie* $W^A_H G = \mathfrak{g}^r_m \rtimes (\mathfrak{g} \otimes A)$ *is of the form*

$$\begin{aligned}&\big[(X_1, v_1 \otimes a_1), (X_2, v_2 \otimes a_2)\big] = \\ &\big([X_1, X_2], v_2 \otimes h(X_1)(a_2) - v_1 \otimes h(X_2)(a_1) + [v_1, v_2] \otimes a_1 a_2\big),\end{aligned} \tag{13}$$

where the bracket $[X_1, X_2]$ *or* $[v_1, v_2]$ *is in* $\mathfrak{g}^r_m$ *or* $\mathfrak{g}$, *respectively.*

Remark 2. This formula seems to be new even in the case $F = J^r$, in which $W^{J^r} P = W^r P$ is the r-th principal prolongation of P, [7, p. 150]. This situation appears in the gauge theories of mathematical physics.

In many geometrical problems related with the associated bundle $P[S, l]$, one uses the infinitesimal action of G on S. We shall interpret it as a map $I(l) : \mathfrak{g} \times S \to TS$,

$$I(l) = Tl \circ (\iota \times 0_S)\,, \tag{14}$$

where $\iota : \mathfrak{g} \to TG$ is the canonical injection and 0_S is the zero section of TS. We can construct $T^A(I(l)) : T^A\mathfrak{g} \times T^A S \to T^A TS$.

Consider the induced action $T^A l : T^A G \times T^A S \to T^A S$.

Proposition 2. *We have*

$$I(T^A l) = \varkappa_S \circ T^A\big(I(l)\big). \tag{15}$$

Proof. Applying T^A to (14), we obtain

$$T^A\big(I(l)\big) = T^A Tl \circ (T^A\iota \times T^A 0_S)\,.$$

The naturality of $\varkappa$ yields

$$\varkappa_S \circ T^A\big(I(l)\big) = TT^A l \circ (\varkappa_G \circ T^A\iota \times \varkappa_S \circ T^A 0_S).$$

But $\varkappa_G \circ T^A\iota$ is the injection of Lie $(T^A G)$ into $TT^A G$ and one verifies immediately $\varkappa_S \circ T^A 0_S = 0_{T^A S}$. □

We are going to characterize $I(T^A l)$ in another way. For every vector field $\zeta : S \to TS$, we write $\mathcal{T}^A\zeta : T^A S \to TT^A S$ for its flow prolongation. We have $\mathcal{T}^A\zeta = \varkappa_S \circ T^A\zeta$, [7]. Consider the injection $i_v : pt \to \mathfrak{g}$, $v \in \mathfrak{g}$. Then

$$I(l)(v) := I(l) \circ (i_v \times \mathrm{id}_S)$$

is a vector field on S. If $1 \in A$ is the unit, then $v \otimes 1 \in V \otimes A$ and $I(T^A l)\ (v \otimes 1)$ is a vector field on $T^A S$.

Lemma 1. $I(T^A l)(v \otimes 1)$ *coincides with* $\mathcal{T}^A(I(l)(v))$.

Proof. We have $\mathcal{T}^A(I(l)(v)) = \varkappa_S \circ T^A(I(l) \circ (i_v \times \mathrm{id}_S)) = I(T^A l)(T^A i_v \times 0_{T^A S})$. But (7) and (8) imply $T^A i_v = i_{v \otimes 1}$. □

We recall that every $a \in A$ induces a natural map $L(a)_S : TT^A S \to TT^A S$, [7].

Lemma 2. *For every $a \in A$ and $z \in T^A\mathfrak{g}$, we have*

$$I(T^A l)(az) = L(a)_S \circ I(T^A l)(z)\,. \tag{16}$$

Proof. In general, the naturality of $L(a)$ yields

$$TT^A l\big(L(a)_G(Z), L(a)_S(X)\big) = L(a)_S\big(TT^A l(Z, S)\big)$$

for every $Z \in TT^A G$ and $X \in TT^A S$. We put i_z on the place of Z and $0_{T^A S}$ on the place of X. Then $L_G(a) \circ i_z = i_{az}$ by (7), $TT^A l(i_z, 0_{T^A S}) = I(T^A l)(z)$ by definition and $L(a)_S \circ 0_{T^A S} = 0_{T^A S}$ by linearity of $L(a)_S$. □

Combining Lemmas 1 and 2, we obtain immediately

Proposition 3. *For every $v \in \mathfrak{g}$ and every $a \in A$, we have*

$$I(T^A l)(v \otimes a) = L(a)_S \circ \mathcal{T}^A\big(I(l)(v)\big). \tag{17}$$

Consider the associated bundle $P[S, l]$. By [4], $F(P[S, l])$ is an associated bundle $W^F P[T^A S, W^A_H l]$, where the action $W^A_H l : W^A_H G \times T^A S \to T^A S$ is given by

$$W^A_H l\big((X, Z), Q\big) = H(X)_S\big(T^A l(Z, Q)\big),$$

where $X \in G^r_m,\ Z \in T^A G,\ Q \in T^A S$.

In order to characterize the infinitesimal action $I(W^A_H l)$, we use the semi-direct decomposition $W^A_H G = G^r_m \rtimes_H T^A G$. The infinitesimal action of G^r_m on $T^A S$ is given by the natural transformations $H_S(X)$, so its infinitesimal action is $I(H_S)$. An element $(X, v \otimes a) \in \mathrm{Lie}(W^A_H G) = \mathfrak{g}^r_m \rtimes (\mathfrak{g} \otimes \mathcal{A})$ is the sum $X + v \otimes a$. This implies

$$I(W^A_H l)(X, v \otimes a) = I(H_S)(X) + L(a)_S \circ \mathcal{T}^A(I(l)(v)). \tag{18}$$

3. Prolongation of vector fields

If ξ is a vector field on M, then its flow prolongation $\mathcal{P}^r\xi$ is a right-invariant vector field on $P^r M$, whose value at every $u \in P^r_x M$ depends on $j^r_x\xi =: X$ only, [7]. Write $i(u, X)\mathcal{P} =^r \xi(u)$, so that

$$i : P^r M \times_M J^r TM \to TP^r M\,. \tag{19}$$

By Section 1, for every fibered manifold $p : Y \to M$ we have

$$FTY = F(TY \to M) \subset P^r M[T^A TY]\,.$$

Consider $\{u, Z\} \in FTY$, $u \in P^r M$, $Z \in T^A TY$. Then $\varkappa_Y Z \in TT^A Y$, $\varkappa_Y Z = (\partial/\partial t)|_0 \zeta(t)$, $\zeta : \mathbb{R} \to T^A Y$. Taking into account the natural transformation $t_{TM} : J^r TM \to FTM$ and the projection $FTp : FTY \to FTM$, we can construct the fiber product $J^r TM \times_{FTM} FTY$. Consider

$$(X, Z) \in J^r TM \times_{FTM} FTY \tag{20}$$

and any $u \in P^r M$ over the same base point. Write

$$i(u, X) = \frac{\partial}{\partial t}\Big|_0 \gamma(t), \qquad \gamma : \mathbb{R} \to P^r M.$$

Then (20) implies $t_M(\gamma(t)) = T^A p(\zeta(t))$. So $\{\gamma(t), \zeta(t)\}$ is a curve on FY. Then we define

$$\mu^F_Y\big(X, \{u, Z\}\big) = \frac{\partial}{\partial t}\Big|_0 \big\{\gamma(t), \zeta(t)\big\}. \tag{21}$$

By right-invariancy, this is independent of the choice of u. Hence we obtain a map

$$\mu^F_Y : J^r TM \times_{FTM} FTY \to TFY\,.$$

Moreover, it is useful to introduce

$$\begin{aligned} &\widetilde{\mu}^F_Y : J^r TM \times_{FTM} FTY \to J^r TM \times_{TM} TFY, \\ &\widetilde{\mu}^F_Y\big(X, \{u, Z\}\big) = \Big(X, \mu^F_Y\big(X, \{u, Z\}\big)\Big). \end{aligned} \tag{22}$$

If $V \in TFY$ and $X \in J^r TM$ are over the same element of TM, we can write

$$V = \frac{\partial}{\partial t}\Big|_0 \big\{\gamma(t), \chi(t)\big\}, \qquad \chi : \mathbb{R} \to T^A Y,$$

and construct $\{u, (\partial/\partial t)|_0 \chi(t)\} \in FTY$. This implies that $\widetilde{\mu}^F_Y$ is a diffeomorphism.

If $\eta : Y \to TY$ is a projectable vector field over a vector field $\xi : M \to TM$, we can construct the functorial prolongation $F\eta : FY \to FTY$ and the rth jet prolongation $j^r\xi : M \to J^r TM$. Then the values of $j^r\xi \times_{\mathrm{id}_M} F\eta$ lie in $J^r TM \times_{FTM} FTY$.

Proposition 4. *The flow prolongation $\mathcal{F}\eta$ of η satisfies*

$$\mathcal{F}\eta = \mu^F_Y \circ (j^r\xi \times_{\mathrm{id}_M} F\eta)\,. \tag{23}$$

Proof. In general, for a principal bundle $P(M, G)$ and an associated bundle $P[S]$, one verifies easily that a right-invariant vector field ζ on P and a left-invariant vector field σ on S induce a vector field $\{\zeta, \sigma\}$ on $P[S]$. In our case, the flow construction implies that the vector field

$$\{\mathcal{P}^r\xi, \mathcal{T}^A\eta\} = \{\varkappa_M \circ T^r_m\xi, \varkappa_G \circ T^A\eta\} \tag{24}$$

on $P^r M[T^A Y]$ is restrictible to the submanifold FY and this restriction coincides with $\mathcal{F}\eta$. According to Section 1 and [6], the functorial prolongation $F\eta : FY \to FTY$ of $\eta : Y \to TY$ is of the form $\mathrm{id}_{P^r M}[T^A\eta]$. Since (19) identifies $j^r\xi$ with $\mathcal{P}^r\xi$, we have

$$j^r\xi \times_{\mathrm{id}_M} F\eta : FY \to J^r TM \times_{FTM} FTY\,.$$

Then (23) follows directly from (24). $\square$

Remark 3. If $\eta : Y \to VY$ is a vertical vector field, then $F\eta : FY \to FVY$. In [4], M. Doupovec and the author established a natural exchange isomorphism $\mathrm{ex}_Y : FVY \to VFY$. Analysing the proof of Proposition 4, we deduce

$$\mathcal{F}\eta = \mathrm{ex}_Y \circ F\eta \tag{25}$$

as a special case of (23).

Remark 4. In the case $F = J^r$, we have $FTM = J^r TM$, so that

$$\mu_Y^{J^r} : J^r TY \to TJ^r Y\,. \tag{26}$$

This map was constructed in a quite different way by L. Mangiarotti and M. Modugno, [8].

Remark 5. We shall show in a next paper how the isomorphism $\widetilde{\mu}_Y^F$ can be used for determining the F-prolongation of the canonical action of the Lie algebroid of a principal bundle P on a vector bundle associated to P.

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